REGULARITY OF DEGENERATE MONGE-AMPÈRE AND PRESCRIBED GAUSSIAN CURVATURE EQUATIONS IN TWO DIMENSIONS

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Abstract. We use *a priori* inequalities for quasilinear equations to obtain a regularity theorem for the Dirichlet problem for the Monge-Ampère equation,

$$u_{xx}u_{yy}-(u_{xy})^2=k(x,y),$$

and the prescribed Gaussian curvature equation,

$$u_{xx}u_{yy}-(u_{xy})^2=k(x,y)\left(1+u_x^2+u_y^2\right)^2,$$

where $k(x,y)$ is close to a function of one variable alone when k is small, but permitted to vanish to *infinite* order.

1. Introduction

In this paper we consider regularity of solutions to the Dirichlet problem for the generalized Monge-Ampère equation in two dimensions,

$$\left\{\begin{array}{rcll} u_{xx}u_{yy}-(u_{xy})^2 & = & k(x,y,u,u_x,u_y), & (x,y)\in\Omega \\ u & = & \phi(x,y), & (x,y)\in\partial\Omega \end{array}\right. , \tag{1.1}$$

where $k(x,y,r,p,q)\geq 0$ and $\phi(x,y)$ are smooth (infinitely differentiable) on $\overline{\Omega}\times\mathbb{R}^3$ and $\overline{\Omega}$ respectively, and Ω is a bounded convex planar domain with smooth positively curved boundary $\partial\Omega$. We will be primarily interested in the classical Monge-Ampère and prescribed Gaussian curvature $\mathcal{K}$ equations in two dimensions (where $k=k(x,y)$ and $k=\mathcal{K}(x,y)\left(1+p^2+q^2\right)^2$ respectively), and moreover, in the case where $k(x,y,r,p,q)$ may vanish to infinite order, thus rendering the problem both nonlinear and non-subelliptic. Much is known in both the linear and subelliptic cases of boundary value problems, but very few results exist in the case where both linearity and subellipticity fail, and the problem we consider here represents a first attempt to understand the simplest of such cases. The reason we restrict attention to dimension two is in order to exploit the remarkable partial Legendre transform T associated with the Monge-Ampère equation in two dimensions: if u solves (1.1) and we define $(s,t)=T(x,y)$,

$$\left\{\begin{array}{rcl} s & = & x \\ t & = & u_y(x,y) \end{array}\right. ,$$

then $w=y(s,t)$ is a weak solution to the quasilinear second order equation

$$\partial_s^2 w+\partial_t f(s,w(s,t),r(s,t),z(s,t),t)\,\partial_t w=0,\qquad (s,t)\in T\Omega, \tag{1.2}$$

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where $w = y(s,t)$ arises from inverting the partial Legendre transform T, which is invertible when $k > 0$, and $r = u$, $z = u_x$ satisfy

$$\begin{cases} r_s = z + tw_s, & r_t = tw_t \\ z_s = kw_t, & z_t = -w_s \end{cases} .$$

Thus the question of interior regularity of solutions to the fully nonlinear equation (1.1) is reduced to the regularity of solutions to the quasilinear equation (1.2). Note that in the case $k = 1$, we then have $z + iw$ is an analytic function of $s + it$. Finally, we point out that in collaboration with C. Rios [23], we have generalized the partial Legendre transform to a quasilinear system in higher dimensions, and used this to successfully investigate a subellipitic case of the Monge-Ampere equation in higher dimensions. However, this generalized partial Legendre system transform does not seem to accomodate the non-subelliptic case treated here. Finally, we refer the reader to an earlier preprint [27] with additional detail.

1.1. **Classical Monge-Ampère equation.** Here we discuss the classical Monge-Ampère equation, the special case $k = k(x,y)$ in (1.1). In [4], Caffarelli, Nirenberg and Spruck have shown that if k is positive on $\overline{\Omega}$, and thus the problem is elliptic, then there is a unique convex solution u of (1.1) in $C^\infty\left(\overline{\Omega}\right)$ (earlier results in two dimensions are in E. Heinz [13], [14], H. Lewy [18], [19], L. Nirenberg [21] and A. V. Pogorelov [22]). In [11] and [12], Guan, Trudinger and Wang have shown that for k merely nonnegative and smooth, there is a unique convex solution u to (1.1) in the generalized sense of Alexandrov (see Appendix B) in $C^{1,1}\left(\overline{\Omega}\right)$ with norm depending only on $\|k\|_{C^{1,1}\left(\overline{\Omega}\right)}$, $\|\phi\|_{C^{3,1}(\partial\Omega)}$ and Ω. An example of Sibony reported in [10] and [12] shows that $C^{1,1}$ cannot be improved (see Subsection 1.4 below for a refinement of this example). Moreover, as the example

$$\begin{aligned} u(x,y) &= |(x,y)|^3 = \left(x^2+y^2\right)^{\frac{3}{2}}, \\ u_{xx}u_{yy} - \left(u_{xy}\right)^2 &= 18\,|(x,y)|^2 = 18\left(x^2+y^2\right), \end{aligned}$$

shows, we may have failure of smoothness even when k vanishes to second order at an isolated point. In fact, radial solutions to (1.1) in the unit disk Ω are easily characterized - see Appendix B at the end of the paper. Here we only remark that if $k(x,y) = r^{2N}$ (where $r^2 = x^2 + y^2$), then $u \approx C + r^{N+2}$, which fails to be smooth for N a positive odd integer. The case $N = 1$ is the example cited above.

In [10], Guan has shown that the $C^{1,1}$ solution u to (1.1) is smooth if k vanishes to finite order in the sense that

$$k(x,y) \approx x^{2\ell} + By^{2m}, \qquad (x,y) \in \Omega,$$

for some $B \geq 0$ and positive integers $\ell \leq m$, and if in addition,

$$u_{yy} \geq c > 0. \tag{1.3}$$

This suggests that the difficulty with the radial examples above may lie in the degeneracy of the rank of the Hessian (which vanishes at the origin for the radial examples, but not in Guan's theorem).

Here we will give conditions *on the data k and ϕ alone* (and *not* on the solution u) that guarantee the smoothness of solutions to (1.1) - namely that $k > 0$ for $x \neq 0$ and k is close to a function of x alone when k is small. In the special case

$k = k(x,y)$ is a function only of the spatial variables, the precise notion we need is

$$|\partial_y k(x,y)| \leq C_L k(x,y)^{\frac{3}{2}}, \quad (x,y) \in L, \tag{1.4}$$

for all compact subsets L of Ω (see Appendix B for a discussion of how (1.4) relates to k being almost independent of y). Of course the radial examples $u \approx C + \left(x^2+y^2\right)^{\frac{N+2}{2}}$ mentioned above fail to be C^{N+2} for N odd, while the corresponding smooth $k = \left(x^2+y^2\right)^N$ satisfies $|\partial_y k| \leq C k^{1-\frac{1}{2N}}$, thus showing that the exponent $\frac{3}{2}$ in (1.4) cannot be replaced by any number less than one[1]. By the rotation invariance of the Hessian, we may instead assume the analogous condition requiring k to be close to a function of $ax+by$ alone when k is small (where (a,b) denotes a unit vector).

As motivation for considering the condition (1.4), we observe that if $k(x,y) = k(x)$ is smooth, positive for $x \neq 0$ and independent of y, then the quasilinear equation arising from the partial Legendre transform (see subsubsection 1.3.1 below for details),

$$\mathcal{L}w = \left[\partial_x^2 + \partial_y k(x)\,\partial_y\right] w = 0, \tag{1.5}$$

is a linear equation and the known linear hypoelliptic theory implies that all solutions of (1.5) are smooth (see Fediĭ [7] and also [6] and [16]).

We prove that if (1.4) holds, then u is smooth in all of Ω except possibly on at most two line segments of the y-axis, each of which has at least one endpoint on the boundary of Ω - and if there are two such segments, they do not meet, so that there is always a nontrivial portion of the y-axis across which the solution u is smooth. We also discuss below extra conditions *on the solution* u which prohibit the existence of such singular segments. In order to make these statements more precise, we introduce the quantities

$$\begin{aligned} \omega_-(x) &= \inf_{y:(x,y)\in\Omega} u_y(x,y), \\ \omega^+(x) &= \sup_{y:(x,y)\in\Omega} u_y(x,y), \end{aligned} \tag{1.6}$$

for $x \in P\Omega$, the projection of Ω onto the x-axis, along with the following subdomain of Ω:

$$\begin{aligned} \Omega^* &= \left\{(x,y)\in\Omega : \omega_-(x) < u_y(x,y) < \omega^+(x)\right\} \\ &= \Omega - \left\{(0,y) : u_y(0,y) = \omega_-(0) \ \textit{or}\ \omega^+(0)\right\}. \end{aligned} \tag{1.7}$$

Note that the infimums in (1.6) are not attained when $x \neq 0$, since u is strictly convex when $k > 0$. Moreover, if $u \in C^1\left(\overline{\Omega}\right)$, then

$$\omega_-(x) = u_y(x,\alpha(x)), \omega^+(x) = u_y(x,\beta(x))$$

where the lower and upper boundaries of Ω are given as the graphs of α and β respectively.

The subdomain Ω^* is thus obtained from Ω by removing any segments from the y-axis that are projections onto the x,y-plane of maximal straight line segments in the graph of u that have at least one endpoint lying above the boundary $\partial\Omega$. See [22], [5] and below for a discussion of such segments in Ω, which we will refer to as Pogorelov segments. Note that we ignore any maximal line segments in the

[1] In a paper [24] in preparation with C. Rios, it is shown that the exponent $\frac{3}{2}$ can be replaced with the near optimal exponent $1+\varepsilon$, $\varepsilon > 0$.

graph of u both of whose endpoints lie above points in Ω - such segments are ruled out by the conclusion in both Theorems 1.1 and 1.3 that u_{yy} is positive in Ω^*. We emphasize again that in the theorem below, k may vanish to infinite order on the line $x = 0$, and it is this lack of subellipticity in a nonlinear problem that is the novel difficulty here.

Theorem 1.1. *Suppose $k(x,y)$ is smooth and nonnegative in Ω, is positive for $x \neq 0$, and satisfies (1.4), namely*

$$|\partial_y k(x,y)| \leq C_L k(x,y)^{\frac{3}{2}}, \quad (x,y) \in L \text{ compact} \subset \Omega.$$

Suppose u is a $C^{1,1}(\overline{\Omega})$ convex solution to the Monge-Ampère boundary value problem

$$\begin{cases} u_{xx}u_{yy} - (u_{xy})^2 &= k(x,y), \quad (x,y) \in \Omega \\ u &= \phi(x,y), \quad (x,y) \in \partial\Omega \end{cases}, \tag{1.8}$$

where ϕ is smooth and $\partial\Omega$ is smooth with positive curvature. Then u is smooth in the subdomain Ω^* as given in (1.7), and $u_{yy} > 0$ in Ω^*. Moreover, Ω^* is connected provided

$$\omega_-(0) < \omega^+(0). \tag{1.9}$$

1.1.1. *Pogorelov segments.* In many cases, conditions on the boundary data ϕ prohibit line segments in the graph of the solution u from extending all the way across Ω, and thus force condition (1.9). Note that it is impossible for two Pogorelov segments to meet in Ω if u is $C^{1,1}$. An important example occurs when $\phi \equiv 0$ (but k is not identically zero). Indeed, a nontrivial convex function on $\overline{\Omega}$ that vanishes on $\partial\Omega$ is strictly negative in Ω. Thus Ω^* is connected for convex $C^{1,1}$ solutions to the homogeneous boundary value problem for the Monge-Ampère equation when $k > 0$ for $x \neq 0$. Alternatively, Ω^* is connected if the convex hull of the space curve

$$\Gamma_\phi = \{(x,y,\phi(x,y)) : (x,y) \in \partial\Omega\} \tag{1.10}$$

contains a point lying strictly below the line segment L_ϕ that joins the two points of intersection of the curve Γ_ϕ and the plane $x = 0$. Indeed, this prevents any supporting plane of a convex solution to (1.1) from containing L_ϕ, and so L_ϕ cannot lie in the graph of u.

We close our discussion of Pogorelov segments here by indicating that if we impose additional conditions on our solutions u, we can prohibit the existence of Pogorelov segments. For example, if $u_y(0,\cdot)$ is not constant near the two points in $\partial\Omega \cap \{x = 0\}$, then Pogorelov segments cannot exist and u is smooth in all of Ω. Less trivially, if $u \in C^2$ (rather than just $C^{1,1}$) solves (1.1) with data symmetric about the y-axis, and its Hessian has rank at least one everywhere, then again Pogorelov segments cannot exist and u is smooth in all of Ω (see Appendix A for a proof). This lends support to the notion that a solution u should be smooth if the rank of its Hessian is positive everywhere. The solution u in our theorem does in fact satisfy $u_{yy} > 0$ away from any exceptional segments, thus lending even further support. Finally, we mention the following result, proved in Appendix A, relating the existence of Pogorelov segments to regularity of u.

Proposition 1.2. *If $\phi \equiv 0$, k vanishes on a line, and a solution u to (1.8) has a Pogorelov segment on the same line, then u must fail to be in $C^2(\overline{\Omega}) \cap C^3(\Omega)$. Moreover, if in addition both Ω and k are symmetric about the given line, then the homogeneous Monge-Ampere boundary value problem (1.8) has* no *nontrivial*

$C^2\left(\overline{\Omega}\right)$ solutions, thus forcing all solutions to have only the minimal regularity $C^{1,1}\left(\overline{\Omega}\right)$.

1.2. **Prescribed Gaussian curvature.** We also consider more general Monge-Ampère equations and show that for $\mathcal{K}$ as above - smooth, positive for $x \neq 0$, and close to a function of x alone when $\mathcal{K}$ is small - the homogeneous boundary value problem for the equation of prescribed Gaussian curvature

$$\frac{u_{xx}u_{yy} - (u_{xy})^2}{\left(1 + u_x^2 + u_y^2\right)^2} = \mathcal{K}(x, y),$$

has smooth solutions in Ω except possibly on the at most two Pogorelov segments mentioned above (they are separated and have one endpoint each on $\partial\Omega$ if they exist). Here we use the *a priori* estimates of Guan [11] for the equation of prescribed Gaussian curvature as well as imposing the additional condition of Ivochkina [15],

$$0 \le \mathcal{K}(x, y) < \lambda, \qquad (x, y) \in \overline{\Omega}, \tag{1.11}$$

where λ is the minimal curvature of $\partial\Omega$, along with the condition $|\partial_y \mathcal{K}| \le C\mathcal{K}^{\frac{3}{2}}$.

Theorem 1.3. *Suppose $\mathcal{K}(x, y)$ is smooth, nonnegative and satisfies (1.11) in $\overline{\Omega}$. Moreover, we suppose $\mathcal{K}(x, y)$ is positive for $x \neq 0$ and satisfies (1.4) with $\mathcal{K}$ in place of k, namely*

$$|\partial_y \mathcal{K}(x, y)| \le C_L \mathcal{K}(x, y)^{\frac{3}{2}}, \quad (x, y) \in L \text{ compact} \subset \Omega.$$

Suppose u is the convex $C^{1,1}$ solution to the prescribed Gaussian curvature homogeneous boundary value problem

$$\begin{cases} u_{xx}u_{yy} - (u_{xy})^2 = \mathcal{K}(x, y)\left(1 + u_x^2 + u_y^2\right)^2, & (x, y) \in \Omega \\ u = 0, & (x, y) \in \partial\Omega \end{cases}, \tag{1.12}$$

where $\partial\Omega$ is smooth with positive curvature (as given in [11]). Then u is smooth in the connected subdomain Ω^* given by (1.7), and $u_{yy} > 0$ in Ω^*.

Remark 1.1. *Theorem 1.3 applies in particular when the Gaussian curvature $\mathcal{K} = \mathcal{K}(x)$ is a function of x alone. In this case, unlike that of the Monge-Ampère equation, the partial Legendre transform results in a* nonlinear *system of equations (see (1.16) below):*

$$\left[\partial_s^2 + \partial_t \mathcal{K}(s)\left(1 + z(s, t)^2 + t^2\right)^2 \partial_t\right] w(s, t) = 0, \tag{1.13}$$

$$\begin{cases} \partial_s z(s, t) = \mathcal{K}(s)\left(1 + z(s, t)^2 + t^2\right)^2 \partial_t w(s, t) \\ \partial_t z(s, t) = -\partial_s w(s, t) \end{cases}. \tag{1.14}$$

The second order equation (1.13) is the compatibility condition $\partial_s\partial_t z = \partial_t\partial_s z$ for the first order Cauchy-Riemann system (1.14).

Analogues of the above theorems for the general boundary value problem (1.1) are discussed briefly in Remark 2.2 below.

1.3. **Quasilinear equations.** In this subsection we consider the quasilinear equation that results from an application of the partial Legendre transform to the generalized Monge-Ampère equation (1.1). However, we will only give applications to the classical Monge-Ampère and prescribed Gaussian curvature equations.

1.3.1. *Partial Legendre transform.* As in [10] and [28], we use the partial Legendre transformation $(s,t) = T(x,y)$ given by

$$\left\{ \begin{array}{lcl} s & = & x \\ t & = & u_y(x,y) \end{array} \right. , \tag{1.15}$$

where u is a smooth convex solution of (1.1) and $k(x,y,v,p,q)$ is smooth and positive on $\overline{\Omega} \times \mathbb{R}^3$ (although we will permit k to be arbitrarily small). We first note that $k > 0$ implies $u_{yy} > 0$ and so $u_y(x,y)$ is strictly increasing in y, making the partial Legendre transform T one-to-one. We must also consider the functions

$$\left\{ \begin{array}{lclcl} w & = & y & = & y(s,t) \\ z & = & u_x(x,y) & = & u_x(s,y(s,t)) \\ r & = & u(x,y) & = & u(s,y(s,t)) \end{array} \right. ,$$

where $(x,y) = (s,y(s,t))$ is the inverse partial Legendre transform. We claim that the functions w, z and r satisfy the quasilinear divergence form equation

$$\partial_s^2 w + \partial_t k(s,w(s,t),r(s,t),z(s,t),t)\,\partial_t w = 0, \qquad (s,t) \in \Omega', \tag{1.16}$$

in the weak sense, where Ω' is a domain satisfying $(s,w(s,t),r(s,t),z(s,t),t) \in \Omega$ for all $(s,t) \in \Omega'$. As in [10], the Jacobian of T is $\begin{bmatrix} 1 & 0 \\ u_{xy} & u_{yy} \end{bmatrix}$, and that of $S = T^{-1}$ is $\begin{bmatrix} 1 & 0 \\ -\frac{u_{xy}}{u_{yy}} & \frac{1}{u_{yy}} \end{bmatrix}$. Note that $u_{yy} \geq c u_{xx} u_{yy} \geq k > 0$. Thus

$$\left\{ \begin{array}{lclcl} \partial_s & = & x_s \partial_x + y_s \partial_y & = & \partial_x - \frac{u_{xy}}{u_{yy}} \partial_y, \\ \partial_t & = & x_t \partial_x + y_t \partial_y & = & \frac{1}{u_{yy}} \partial_y, \end{array} \right.$$

and for $\eta \in C_c^\infty(T(\Omega))$, we have

$$\begin{aligned} & \int_{T(\Omega)} (y_s \eta_s + k y_t \eta_t)\, ds dt \\ = & \int_\Omega \left\{ \left(-\frac{u_{xy}}{u_{yy}} \right) \left(\eta_x - \frac{u_{xy}}{u_{yy}} \eta_y \right) + k \left(\frac{1}{u_{yy}} \right) \left(\frac{1}{u_{yy}} \eta_y \right) \right\} u_{yy} dx dy \\ = & \int_\Omega \left\{ -\frac{u_{xy}}{u_{yy}} \eta_x + \left(\frac{u_{xy}}{u_{yy}} \right)^2 \eta_y + k \left(\frac{1}{u_{yy}} \right)^2 \eta_y \right\} u_{yy} dx dy \\ = & \int_\Omega \left\{ -u_{xy} \eta_x + \frac{(u_{xy})^2 + k}{u_{yy}} \eta_y \right\} dx dy \\ = & \int_\Omega \{ -u_{xy} \eta_x + u_{xx} \eta_y \} dx dy = \int_\Omega \{ -u_{xy} \eta_x + u_{xy} \eta_x \} dx dy = 0, \end{aligned}$$

since

$$u_{xx} = \frac{k + (u_{xy})^2}{u_{yy}} \tag{1.17}$$

by equation (1.1).

In order to obtain *a priori* estimates for this equation we need to derive the corresponding equations for r and z. To calculate of the s and t derivatives of r

and z, recall that $\partial_s = \partial_x - \frac{u_{xy}}{u_{yy}}\partial_y$ and $\partial_t = \frac{1}{u_{yy}}\partial_y$ so that $w_s = -\frac{u_{xy}}{u_{yy}}$ and $w_t = \frac{1}{u_{yy}}$. We thus have

$$\begin{cases} r_s &= \left(\partial_x - \frac{u_{xy}}{u_{yy}}\partial_y\right) u \\ r_t &= \frac{1}{u_{yy}}\partial_y u \\ z_s &= \left(\partial_x - \frac{u_{xy}}{u_{yy}}\partial_y\right) u_x \\ z_t &= \frac{1}{u_{yy}}\partial_y u_x \end{cases},$$

which give the compatibility conditions

$$\begin{cases} r_s &= z + t w_s \\ r_t &= t w_t \\ z_s &= k w_t \\ z_t &= -w_s \end{cases}, \tag{1.18}$$

where k is evaluated at $(s, w(s,t), r(s,t), z(s,t), t)$. The equations (1.18) show that the (s,t) derivatives of z and r satisfy the same or better size estimates as do those of w, provided the sup norms of w, z and r are all *a priori* bounded (of course, only the bound on z is needed for this purpose). This is indeed the case for solutions arising from the partial Legendre transform by the *a priori* estimates for first order derivatives in [4] (which require only $k \geq 0$).

In estimating the higher derivatives of solutions to (1.16), we must differentiate the equation $\mathcal{L}w = 0$, where

$$\mathcal{L}w = \partial_s^2 w + \partial_t \widetilde{k}(s,t)\, \partial_t w$$

and $\widetilde{k}(s,t) = k(s, w(s,t), r(s,t), z(s,t), t)$. We obtain

$$\mathcal{L}(\partial w) = -\partial_t \left(\partial \widetilde{k}\right) \partial_t w$$

and by using (1.18), we have

$$\begin{aligned} \partial_s \widetilde{k} &= k_1 + k_2 w_s + k_3 (z + t w_s) + k_4 k w_t, \\ \partial_t \widetilde{k} &= k_2 w_t + k_3 t w_t - k_4 w_s + k_5, \end{aligned} \tag{1.19}$$

where the partial derivatives k_j are evaluated at the point $(s, w(s,t), r(s,t), z(s,t), t)$. Moreover, (1.19) shows that the general case has the same form as the Monge-Ampère equation itself - namely, that the gradient of $\widetilde{k}$ is linear in the gradient of w with coefficients k_j, except that in the classical Monge-Ampère equation, $k_3 = k_4 = k_5 = 0$.

We close this section by informally indicating how *a priori* bounds on solutions to (1.16) with $k = k(x,y) > 0$ translate into *a priori* bounds on solutions of (1.1). In the case $k = k(x,y)$ there are of course no compatibility conditions (1.18). The *a priori* bounds we refer to have the form that for solutions $y(s,t)$ of (1.16) in $T\Omega$, the quantity $\|D^\alpha y(s,t)\|_{L^\infty(K)}$ is bounded by a constant depending only on α, $\|y\|_{L^\infty(K)}$, the size of derivatives of k, and the distance from the compact set K to $\partial T\Omega$. Then we claim that u is smooth in Ω with similar *a priori* bounds on *its* derivatives on sets of the form $T^{-1}K$. Indeed, if $y(s,t) = u_y(s,\cdot)^{-1}(t)$ is smooth with *a priori* bounds, then since $x(s,t) = s$, we conclude that $S = T^{-1}$ and hence also T is smooth with *a priori* bounds since the Jacobian of S is $y_t = \frac{1}{u_{yy}}$ which is bounded away from 0 (by [12]) and ∞ (by the estimates above). Then u_y is smooth with *a priori* bounds, and hence also u_{yy} and u_{xy}. From (1.17) we then obtain that u_{xx} is also smooth with *a priori* bounds. Thus all the second order partial derivatives of u are smooth with *a priori* bounds, and so then u is also smooth with *a priori* bounds on its derivatives. In our application to the Monge-Ampère equation, we will apply this type of argument with $k+\delta$ in place of k, where k is now permitted to vanish in Ω and $\delta>0$. We can then conclude that *a priori* bounds hold for solutions corresponding to $k+\delta$ with constants independent of $\delta>0$. However these *a priori* bounds turn out to be independent of $0<\delta\leq 1$ only on compact subsets of the subdomain Ω^* given in (1.7), namely

$$\Omega^*=\left\{(x,y)\in\Omega:\omega_-(x)<u_y(x,y)<\omega^+(x)\right\},$$

since in the image

$$T\Omega^*=\left\{(x,y):\omega_-(x)<y<\omega^+(x),x\in P\Omega\right\},$$

the infima and suprema in (1.6) may now be attained inside Ω when $x=0$ (since k vanishes there). This accounts for the possible presence of Pogorelov segments as discussed above.

1.3.2. *A priori estimates.* Here we consider the (system of) degenerate quasilinear equations (1.16) and (1.18):

$$\begin{aligned}\mathcal{L}w &= \left[\partial_x^2+\partial_y k(x,w(x,y),r(x,y),z(x,y),y)\partial_y\right]w=0,\\ \partial_x r &= z+t\partial_x w\\ \partial_y r &= y\partial_y w\\ \partial_x z &= k\partial_y w\\ \partial_y z &= -\partial_x w\end{aligned}\tag{1.20}$$

for $(x,y)\in\Omega'$, where $k(x,y,v,p,q)$ is smooth in a domain $\Omega\times\mathbb{R}^3$, and $w(x,y)$ and Ω' are such that

$$(x,w(x,y))\in\Omega\ for\ all\ (x,y)\in\Omega',\tag{1.21}$$

and where k is positive for $x\neq 0$. Note that we have departed from the notation used in the previous section - $y(s,t)$ has been replaced by $w(x,y)$ and $T(\Omega)$ by Ω'. However, as we indicated above, in order to implement an approximation scheme in our application to the Monge-Ampere and prescribed Gaussian curvature equations, we will need to consider more generally the family of quasilinear equations obtained from (1.20) by replacing k with $k+\delta$:

$$\begin{aligned}\mathcal{L}_\delta w &= \left[\partial_x^2+\partial_y\left[k(x,w(x,y),r(x,y),z(x,y),y)+\delta\right]\partial_y\right]w=0,\\ \partial_x r &= z+t\partial_x w\\ \partial_y r &= y\partial_y w\\ \partial_x z &= [k+\delta]\partial_y w\\ \partial_y z &= -\partial_x w\end{aligned}\tag{1.22}$$

for $0\leq\delta<1$, and obtain estimates on the solutions *uniformly* in δ. For this we will employ an *a priori* estimate from our companion paper [26].

Before stating this *a priori* estimate, it will be convenient to recall the classical inequality

$$|\nabla k(x,y)|\leq B\sqrt{k(x,y)},\qquad (x,y)\in L,\tag{1.23}$$

for a compact subset L of Ω, and its more general form,

(1.24)
$$|\nabla k(x,y)| \leq C\left\{\left\|\nabla^2 k\right\|_\infty^{\frac{1}{2}} + (dist((x,y),\partial\Omega))^{-\frac{1}{2}}\right\}\sqrt{k(x,y)}, \qquad (x,y)\in\Omega,$$

if k is merely nonnegative with bounded first and second derivatives on a domain Ω. See the subsection on interpolation inequalities in Appendix B. We will often apply these inequalities to the functions $k(\cdot,\cdot,v,p,q)$ for fixed v, p and q. We will also need some notation. Let $\mathcal{P}_c\left(\Omega\times\mathbb{R}^3\right)$ denote the collection of all compact subsets of $\Omega\times\mathbb{R}^3$. We will say that a positive function f defined on $\mathcal{P}_c\left(\Omega\times\mathbb{R}^3\right)$ is increasing if $f(L_1)\leq f(L_2)$ whenever $L_1,L_2\in\mathcal{P}_c\left(\Omega\times\mathbb{R}^3\right)$ with $L_1\subset L_2$. We will also need the *a priori* estimates $u_{xx},u_{yy}\leq C$ in [11] and [12] for convex solutions u of (1.1) in the special cases of the classical Monge-Ampère and prescribed Gaussian curvature equations. These estimates translate into the following estimates for w under the partial Legendre transform:

$$(1.25)\qquad \begin{cases} w_y(x,y) & \geq \frac{1}{C} \\ 1+w_x(x,y)^2 & \leq Cw_y(x,y), \\ k(x,w(x,y),r(x,y),z(x,y),y)\,w_y(x,y) & \leq C. \end{cases}$$

See Remark 2.1 below for a proof, and note also that $kw_x^2\leq C^2$ is a consequence of the latter two lines. In the general case $k=k(x,y,v,p,q)$, we impose conditions in addition to (1.4), namely

$$(1.26)\qquad \begin{array}{rcl} |k_i| & \leq & Ck^{d(i)}, \qquad 2\leq i\leq 4, \\ |k_{55}| & \leq & Ck^{\frac{1}{2}}, \end{array}$$

on compact subsets of $\Omega\times\mathbb{R}^3$, where $d(i)=\begin{cases}\frac{3}{2}, & i=2,3\\ 1, & i=4\end{cases}$.

Note that on compact subsets of $\Omega\times\mathbb{R}^3$ we always have $|k_i|\leq Ck^{\frac{1}{2}}$ by (1.23) and $|k_{ij}|\leq C$. In the case of the classical Monge-Ampère equation when $k=k(x,y)$, the general conditions (1.26) reduce to the single condition (1.4). Moreover, in the case of the equation of prescribed Gaussian curvature $\mathcal{K}$, when $k=\mathcal{K}(x,y)\left(1+p^2+q^2\right)^2$, (1.26) again reduces to $|k_2|\leq Ck^{\frac{3}{2}}$, or equivalently $|\partial_y\mathcal{K}|\leq C\mathcal{K}^{\frac{3}{2}}$, since $k_3\equiv 0$ and

$$\begin{array}{rcl} k_4 & = & 4\mathcal{K}(x,y)\left(1+p^2+q^2\right)p\leq 4k, \\ k_{55} & = & 4\mathcal{K}(x,y)\left(1+p^2+3q^2\right)p\leq 12k. \end{array}$$

Remark 1.2. *The conditions (1.26) on k_i are precisely those, which together with (1.25), imply (1.23) for*

$$\widetilde{k}(s,t)=k(s,w(s,t),r(s,t),z(s,t),t),$$

namely $\left|\nabla_{(s,t)}\widetilde{k}\right|\leq C\sqrt{\widetilde{k}}$. Note that $d(4)=1$ is less than $d(2)=d(3)=\frac{3}{2}$ since (1.18) yields

$$|z_s|=kw_t\leq C \text{ and } |z_t|=|w_s|\leq Ck^{-\frac{1}{2}},$$

by the a priori estimates (1.25), and thus the term

$$k_4(s,w(s,t),r(s,t),z(s,t),t)\left|\nabla_{(s,t)}z\right|$$

is bounded by $\sqrt{k}$ *since* $|k_4| \leq Ck$ *by (1.26). This is important in our application to prescribed Gaussian curvature where* $|k_4| \leq Ck$ *is automatic. The reason for the special hypothesis on the second derivative* k_{55}*, and not the others, is that the strong hypotheses on* k_2*,* k_3 *and* k_4 *actually turn out to imply that* $|\nabla k_i| \leq C\sqrt{k}$ *for* $2 \leq i \leq 4$*.*

We can now state the *a priori* estimate we need from [26].

Theorem 1.4. *Suppose* $k(x,y,v,p,q)$ *is smooth and nonnegative in a domain* $\Omega \times \mathbb{R}^3$*, is positive for* $x \neq 0$ *and satisfies (1.26). Let* ζ *and* $\varkappa$ *be smooth cutoff functions supported in* Ω' *with* $\varkappa = 1$ *on the support of* ζ*. Then for every multi-index* α*, there is an increasing positive function* $\mathcal{C}_\alpha(L)$*, defined for* $L \in \mathcal{P}_c(\Omega \times \mathbb{R}^3)$*, depending only on* Ω*,* Ω'*,* ζ*,* $\varkappa$ *and* k*,* and independent of $0 \leq \delta \leq 1$*, such that*

$$\|\zeta D^\alpha w\|_\infty \leq \mathcal{C}_\alpha(L)$$

for all smooth solutions (w,r,z) *of the quasilinear system of equations (1.22) with* $0 < \delta \leq 1$*, satisfying (1.25), and such that* $(x, w(x,y), r(x,y), z(x,y), y) \in L$ *for all* (x,y) *in the support of* $\varkappa$*.*

1.4. **Examples.** Our first example shows that in Theorem 1.1, we cannot simultaneously fatten the zero set of k and relax the nondegeneracy condition corresponding to (1.9). Note that (1.9) is equivalent to requiring that the restriction of u to the vertical line $x=0$ not be linear. The following function u, that vanishes identically on two vertical lines, is a $C^{1,1}$ solution, and no better, of the Monge-Ampère boundary value problem (1.8) with smooth ϕ, and smooth nonnegative k satisfying all of the hypotheses of Theorem 1.1 except that k vanishes on $[-\delta,\delta]\times\mathbb{R}$ and (1.9) fails with $x=\pm\delta$ in place of $x=0$ - all the other hypotheses are uniform in $\delta>0$. Let

$$\varphi(x,y) = \frac{y^2}{2}\psi(x) + \frac{x^2}{2}$$

for a smooth function ψ to be chosen later. We observe that

$$\begin{aligned} k &= \varphi_{xx}\varphi_{yy} - (\varphi_{xy})^2 = \left(\frac{y^2}{2}\psi''(x)+1\right)(\psi(x)) - (y\psi'(x))^2 \\ &= \psi(x) + \frac{y^2}{2}\left(\psi(x)\psi''(x) - 2\psi'(x)^2\right), \end{aligned}$$

and

$$k_2 = y\left(\psi(x)\psi''(x) - 2\psi'(x)^2\right).$$

Now it is possible to achieve

$$\left|\psi(x)\psi''(x) - 2\psi'(x)^2\right| \leq Ck^{2-\varepsilon},$$

for example with $\psi(x) = \begin{cases} e^{-\frac{2}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. In fact, we then have $\psi'(x) = e^{-\frac{2}{x^2}}\left(\frac{4}{x^3}\right)$ and $\psi''(x) = e^{-\frac{2}{x^2}}\left(\frac{16}{x^6} - \frac{12}{x^4}\right)$ so that (1.4) holds with $|k_2| \leq Ck^{2-\varepsilon}$ for any $\varepsilon > 0$. We now employ a modification of an example of Sibony reported in

[10]. Let Ω be the unit disk and for $\delta > 0$ small, define u on $\overline{\Omega}$ by

$$u(x,y) = \begin{cases} \varphi\left(x^2-\delta^2, y\right), & |x| \geq \delta \\ \left(y^2 - \left(1-\delta^2\right)\right)^2, & |y| \geq \sqrt{1-\delta^2} \\ 0 & |x| < \delta, |y| < \sqrt{1-\delta^2} \end{cases}.$$

Then in Ω, $k = u_{xx}u_{yy} - (u_{xy})^2$ is smooth, nonnegative, positive for $|x| > \delta$, and satisfies (1.4). The boundary values ϕ are also smooth (uniformly in δ) since $\varphi\left(\cos^2\theta - \delta^2, \sin\theta\right)$ and $\left(\sin^2\theta - \left(1-\delta^2\right)\right)^2$ together with all their derivatives match up at the four angles θ_0 where $\tan\theta_0 = \frac{\sqrt{1-\delta^2}}{\delta}$ (since $\psi(x)$ vanishes to infinite order at $x = 0$). Also, the nondegeneracy condition holds except on the vertical lines $x = \pm\delta$. Finally, u is no better than $C^{1,1}$ across these vertical lines since u_{xx} has jump discontinuities there.

We now comment on the existence of line segments in the graphs of solutions to the Dirichlet problem (1.1). Of course, under the hypotheses of Theorem 1.1, all solutions u satisfy $u_{yy} > 0$ in Ω^*, and so a maximal line segment cannot exist in the graph of a solution u unless at least one of its endpoints lies above the intersection of the y-axis with $\partial\Omega$. In the case exactly one of the endpoints lies above $\partial\Omega$, we call the projection of such a maximal line segment onto the x, y-plane, a Pogorelov segment (see [22] and [5]). We first point out that, under the hypotheses of Theorem 1.1, it is possible to have a line segment in the graph of u provided both endpoints lie above $\partial\Omega$, but we are unable to find any nonsmooth such solutions. For example, $u(x,y) = \frac{1}{2}\left(x^2 + x^4y^2\right)$ vanishes on the y-axis, and $k = u_{xx}u_{yy} - u_{xy}^2 = x^4 - 10x^6y^2$ is positive for $x \neq 0$ in a small neighbourhood of the origin, with $|\partial_y k| \leq Ck^{\frac{3}{2}}$ there. An example with k depending only on x is given by $u(x,y) = \frac{x^2}{\rho(y)}$ where ρ satisfies the differential equation $\rho''(y) + \rho(y)^3 = 0$. Then

$$k = u_{xx}u_{yy} - u_{xy}^2 = -\frac{2x^2\rho''(y)}{\rho(y)^3} = 2x^2.$$

An example with k vanishing to infinite order at the y-axis is given by

$$u(x,y) = e^{-\frac{1}{x^2}} + \frac{y^2}{2}e^{-\frac{3}{x^2}}$$

Then $u(0,y) = 0$ for all y, and

$$k = u_{xx}u_{yy} - u_{xy}^2 = e^{-\frac{4}{x^2}}\left(\frac{4}{x^6} - \frac{6}{x^4}\right) - 18y^2e^{-\frac{6}{x^2}}\left(\frac{1}{x^6} - \frac{1}{2x^4}\right).$$

Thus in a small disc about the origin, k is positive for $x \neq 0$ and $|\partial_y k| \leq Ck^{\frac{3}{2}}$.

Next we observe that Pogorelov segments can exist for the solutions u that arise in Theorem 1.1 if we omit the hypothesis (1.4), and moreover the conclusion $u_{yy} > 0$ in Ω^* may fail. To see this, consider $v(x,y) = u(x,y) + g(y)$ where u is one of the three examples above and g is smooth and convex. Then the determinant of the Hessian of v is

$$v_{xx}v_{yy} - v_{xy}^2 = u_{xx}u_{yy} - u_{xy}^2 + g''(y)\,u_{xx}$$

which in a small disc about the origin, is smooth and positive for $x \neq 0$. However, $v_{yy}(0,y) = g''(y)$ need not be positive in Ω^*, and of course v may have Pogorelov segments lying above the y-axis. The problem with the above example is that k no longer vanishes on the y-axis. Indeed, as we show in Corollary 3.4 of Appendix

A, if $\phi \equiv 0$, k vanishes on a line and u has a Pogorelov segment on the same line, then u must fail to be in $C^2\left(\overline{\Omega}\right) \cap C^3(\Omega)$. Moreover, if in addition both Ω and k are symmetric about the given line, then the homogeneous Monge-Ampere boundary value problem has *no* nontrivial $C^2\left(\overline{\Omega}\right)$ solutions, whether or not there is a Pogorelov segment! The homogeneous boundary condition may be relaxed to a more general condition relating ϕ, k and $\partial\Omega$ (see (3.15) in Theorem 3.6 of Appendix A).

We emphasize that under the hypotheses of Theorem 1.1, we are unable to find any solutions, smooth or not, having a Pogorelov segment. We suspect they do not exist, and that consequently the solutions in Theorem 1.1 are smooth in all of Ω when Ω^* is connected. Perhaps the solutions are always smooth, even when Ω^* is not connected due to a line segment in the graph of u extending all the way across Ω. On the other hand, Appendix A shows that in many cases, if Pogorelov segments exist, then the solutions are as irregular as can be.

Finally we point out that the conclusion of Theorem 1.4 may fail even in the case $k = k(x,y)$ without the assumption (1.4). Indeed, while the maximum principle (see Theorem 9.1 in [9]) yields an *a priori* bound for functions w that solve the quasilinear Dirichlet problem

$$\left\{ \begin{array}{rcll} \left\{\partial_s^2 + \partial_t k(s, w(s,t))\,\partial_t\right\} w & = & 0 & (s,t) \in \Omega' \\ w & = & \phi(s,t) & (s,t) \in \partial\Omega' \end{array} \right. \tag{1.27}$$

(w is already *a priori* bounded if Ω is bounded), the t-derivatives may not be bounded. For example, if $k = 18\left(s^2+t^2\right)$ and

$$w = \frac{1}{3\sqrt{2}} \sqrt{\sqrt{\left(9s^2\right)^2 + (6t)^2} - 9s^2},$$

then w solves the first equation in (1.27), but w_t fails to belong to L^3. Indeed if we apply the partial Legendre transform $s = x$, $t = u_y(x,y)$ to the solution $u = \left(x^2+y^2\right)^{\frac{3}{2}}$ of the Monge-Ampère equation

$$u_{xx}u_{yy} - \left(u_{xy}\right)^2 = k(x,y) = 18\left(x^2+y^2\right),$$

the above function w results. We remark that even though k vanishes here, the partial Legendre transform is one-to-one and a simple limiting argument justifies its use. Now $u_{yy} \approx r = \left(x^2+y^2\right)^{\frac{1}{2}}$ and so using $\partial_t w = \frac{1}{u_{yy}}$ and $\frac{\partial(s,t)}{\partial(x,y)} = u_{yy}$, we obtain

$$\int\int \left|\partial_t w(s,t)\right|^p dsdt = \int\int \left|\frac{1}{u_{yy}}\right|^p u_{yy} dxdy \approx \int_0^1 r^{2-p} dr < \infty$$

if and only if $p < 3$. We can of course come to the same conclusions by direct calculation using

$$\left|w_t(s,t)\right| \approx \left\{ \begin{array}{ll} |t|^{-\frac{1}{2}}, & |t| \geq |s|^2 \\ |s|^{-1}, & |t| \leq |s|^2 \end{array} \right.,$$

so that

$$\int\int \left|w_t\right|^p dsdt \approx \int_0^1 |t|^{-\frac{p}{2}} |t|^{\frac{1}{2}} dt + \int_0^1 |s|^{-p} |s|^2 ds < \infty$$

if and only if $p < 3$.

2. Proofs of the main theorems

2.1. **Monge-Ampère equation.** Here we prove Theorem 1.1 on the classical Monge-Ampère equation with $k = k(x,y)$.

Remark 2.1. *In our application to the Monge-Ampère equation, we will need (1.25), i.e. the inequalities* $w_y \geq c > 0$, $kw_y \leq C$, $|w_x| \leq C\sqrt{w_y}$ *and* $\sqrt{k}\,|w_x| \leq C$ *for smooth solutions* w *of (1.20) arising from the partial Legendre transform. In this case, reverting to the original variables* (s,t)*, the inequalities follow immediately from the* a priori *estimates* $u_{xx} \leq C$, $u_{yy} \leq C$ *in* [12] *since* $\max\left\{k, u_{xy}^2\right\} \leq k + u_{xy}^2 = u_{xx}u_{yy}$*:*

$$\begin{aligned}
y_t &= \frac{1}{u_{yy}} \geq \frac{1}{C}, \\
k(s, y(s,t))\, y_t(s,t) &= k(x,y)\frac{1}{u_{yy}(x,y)} \leq u_{xx}(x,y) \leq C, \\
y_s(s,t)^2 &= \frac{u_{xy}(x,y)^2}{u_{yy}(x,y)^2} \leq \frac{u_{xx}(x,y)}{u_{yy}(x,y)} \leq C y_t(s,t), \\
k(s,y(s,t))\, y_s(s,t)^2 &= k(x,y)\frac{u_{xy}(x,y)^2}{u_{yy}(x,y)^2} \leq u_{xx}(x,y)^2 \leq C^2.
\end{aligned}$$

Note that the fourth inequality also follows by combining the second and third inequalities.

Proof. (of Theorem 1.1) For $\delta \geq 0$, let u^δ be the convex solution of the Monge-Ampère boundary problem

$$\left\{\begin{array}{rcll} u^\delta_{xx}u^\delta_{yy} - \left(u^\delta_{xy}\right)^2 &=& k(x,y) + \delta, & (x,y) \in \Omega \\ u^\delta &=& \phi(x,y), & (x,y) \in \partial\Omega \end{array}\right. .$$

For $\delta > 0$, u^δ is smooth in $\overline{\Omega}$ by [4] and for $\delta = 0$, $u^0 = u \in C^{1,1}\left(\overline{\Omega}\right)$ by [12]. A simple and well known calculation shows that $U = u^{\delta_1} - u^{\delta_2}$ solves the linear equation

$$\begin{aligned}
\mathbf{L}U &= \delta_1 - \delta_2, \quad (x,y) \in \Omega, \\
U &= 0, \quad (x,y) \in \partial\Omega,
\end{aligned}$$

where

$$\mathbf{L}U = \left(\frac{u^{\delta_1}_{yy} + u^{\delta_2}_{yy}}{2}\right)U_{xx} + \left(\frac{u^{\delta_1}_{xx} + u^{\delta_2}_{xx}}{2}\right)U_{yy} - \left(u^{\delta_1}_{xy} + u^{\delta_2}_{xy}\right)U_{xy}. \tag{2.1}$$

When $\delta_1 > \delta_2 \geq 0$, $\mathbf{L}$ is elliptic (since it is the average of two nonnegative operators, one of which is positive) with $C^{1,1}$ coefficients and $U \in C^{1,1} \subset W^{2,2} = H^2$ (the Sobolev space of functions with distributional derivatives in L^2). The maximum principle for such equations (see e.g. Alexandrov's maximum principle, Theorem 9.1 in [9]) shows that $U \leq 0$ or $u^{\delta_1} \leq u^{\delta_2}$ in $\overline{\Omega}$. It follows that if we write

$$\partial\Omega = \{(x, \alpha(x)) : x \in P\Omega\} \cup \{(x, \beta(x)) : x \in P\Omega\}$$

where $\alpha(x) < \beta(x)$ and both α and $-\beta$ are strictly convex functions on the projection $P\Omega$ of Ω onto the x-axis, then

$$(2.2)\qquad u_y^{\delta_1}(x,\beta(x)) = \lim_{y\to\beta(x)^-}\frac{\phi(x,\beta(x)) - u^{\delta_1}(x,y)}{\beta(x)-y} \geq \lim_{y\to\beta(x)^-}\frac{\phi(x,\beta(x)) - u^{\delta_2}(x,y)}{\beta(x)-y} = u_y^{\delta_2}(x,\beta(x)),$$

and similarly,

$$(2.3)\qquad u_y^{\delta_1}(x,\alpha(x)) \leq u_y^{\delta_2}(x,\alpha(x)),$$

for $\delta_1 > \delta_2$.

Now apply the partial Legendre transform T^δ as in (1.15) with u^δ in place of u,

$$\begin{cases} s &= x \\ t &= u_y^\delta(x,y) \end{cases},$$

and use (2.2) and (2.3) to obtain that the transformed regions

$$T^\delta\Omega = \left\{\left(x, u_y^\delta(x,y)\right) : (x,y)\in\Omega\right\}$$

satisfy $T^{\delta_2}\Omega \subset T^{\delta_1}\Omega$ for $\delta_1 > \delta_2$. Note also that (1.9) implies that the interior of $T^0\Omega$ is the domain

$$T^0\Omega^* = \{(s,t) : u_y(s,\alpha(s)) < t < u_y(s,\beta(s)), s\in P\Omega\},$$

where Ω^* is as in (1.7). Now $u_{yy}^\delta > 0$ for $\delta > 0$, and we let $w^\delta(s,t)$ denote the inverse function $u_y^\delta(s,\cdot)^{-1}(t)$. We claim that Theorem 1.4 applies to the transformed functions w^δ to show that all their derivatives are uniformly bounded on compact subsets of $T^0\Omega^*$ (which is contained in $T^\delta\Omega$). Here we are taking $\Omega' = T^0\Omega^*$. To apply Theorem 1.4 we must show that for every compact subset K of $T^0\Omega^*$, there is a compact subset L of Ω^* and $c > 0$ such that

$$(2.4)\qquad \left(s, w^\delta(s,t)\right) \in L, \quad \textit{for all } (s,t)\in K, 0<\delta<c.$$

To see (2.4) we first prove that $u_y^\delta \to u_y$ uniformly on $\overline{\Omega}$ as $\delta\to 0$. By the C^2 *a priori* estimates in [12], the functions u^δ and their gradients are uniformly Lipschitz, and hence equicontinuous, on the compact set $\overline{\Omega}$. Thus given a sequence $\left\{u^{\delta_n}\right\}$ with $\delta_n \to 0$, there is a subsequence converging in $C^{0,1}\left(\overline{\Omega}\right)$ to a $C^{1,1}$ function, say v. By a result of Alexandrov (see [5] or Theorem 4.1 in Appendix B), the absolutely continuous measures with density $k + \delta_n = \det\nabla^2 u^{\delta_n}$ converge weakly to $\det\nabla^2 v$, the representing measure (see the appendix for the definition) for the convex function v. Since $k+\delta_n$ also converges weakly to k, we conclude that $\det\nabla^2 v = k$, and thus that v is a generalized solution of (1.8). By the uniqueness of generalized solutions (see Proposition 3 in [6]), $v = u$. Thus we've proved that every sequence $\left\{u^{\delta_n}\right\}$ with $\delta_n\to 0$ has a subsequence converging to u in $C^{0,1}\left(\overline{\Omega}\right)$, and it follows that $u^\delta \to u$ in $C^{0,1}\left(\overline{\Omega}\right)$ as $\delta\to 0$. In particular, $u_y^\delta \to u_y$ uniformly on $\overline{\Omega}$. Now since K is compact, there is $\varepsilon > 0$ such that

$$u_y(s,\alpha(s)) + \varepsilon < t < u_y(s,\beta(s)) - \varepsilon$$

for $(s,t)\in K$, and by the uniform convergence of u^{δ_n} to u, there is $c > 0$ such that

$$(2.5)\qquad u_y^\delta(s,\alpha(s)) + \frac{\varepsilon}{2} < t < u_y^\delta(s,\beta(s)) - \frac{\varepsilon}{2}$$

for $(s,t) \in K$ and $0 < \delta < c$. Now $u^\delta_{yy} \le C$ by the C^2 *a priori* estimates in [12], and it follows that

$$u^\delta_y\left(s, \alpha(s) + \frac{\varepsilon}{2C}\right) \le u^\delta_y(s, \alpha(s)) + \frac{\varepsilon}{2},$$

which implies upon taking inverses in the second variable that

$$\alpha(s) + \frac{\varepsilon}{2C} \le w^\delta\left(s, u^\delta_y(s, \alpha(s)) + \frac{\varepsilon}{2}\right). \tag{2.6}$$

Similarly, we obtain

$$w^\delta\left(s, u^\delta_y(s, \beta(s)) - \frac{\varepsilon}{2}\right) \le \beta(s) - \frac{\varepsilon}{2C}. \tag{2.7}$$

Combining (2.5), (2.6) and (2.7) we obtain that

$$\alpha(s) + \frac{\varepsilon}{2C} \le w^\delta\left(s, u^\delta_y(s, \alpha(s)) + \frac{\varepsilon}{2}\right) < w^\delta(s,t) < w^\delta\left(s, u^\delta_y(s, \beta(s)) - \frac{\varepsilon}{2}\right) \le \beta(s) - \frac{\varepsilon}{2C}$$

for $(s,t) \in K$ and $0 < \delta < c$, which proves (2.4) with

$$L = \left\{(x,y) \in \Omega : \alpha(x) + \frac{\varepsilon}{2C} \le w^\delta(x,y) \le \beta(x) - \frac{\varepsilon}{2C}\right\}.$$

The only other hypothesis of Theorem 1.4 that we need to verify is (1.25), which holds by Remark 2.1 and the fact that the C^2 *a priori* estimates in [12] are independent of $0 < \delta \le 1$. Thus for each α, the derivatives $D^\alpha w^\delta$ are bounded uniformly in $0 < \delta \le 1$ on compact subsets of $T^0\Omega^*$.

Next, we apply the inverse function theorem to the inverse partial Legendre transform $\left(T^\delta\right)^{-1}(s,t) = \left(s, w^\delta(s,t)\right)$, whose Jacobian is $w^\delta_t = \frac{1}{u^\delta_{yy}}$. We claim that this implies the functions u^δ_y, together with all their derivatives, are uniformly bounded, and hence equicontinuous, on compact subsets of Ω^*. Indeed, if we denote by S^δ the inverse transform $\left(T^\delta\right)^{-1}$, then differentiating the equation $S^\delta \circ T^\delta(x,y) = (x,y)$ yields $\left(DS^\delta \circ T^\delta\right)\left(DT^\delta\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and so

$$DT^\delta = \left(DS^\delta \circ T^\delta\right)^{-1} = \frac{\left(cofactor DS^\delta\right) \circ T^\delta}{\left(\det DS^\delta\right) \circ T^\delta}.$$

Using the quotient and chain rules, induction now shows that every partial derivative of a component of DT^δ is a sum of products of derivatives of S^δ and nonnegative powers of $\frac{1}{\det DS^\delta} = \frac{1}{w^\delta_t} = u^\delta_{yy}$. Since the functions u^δ_{yy} are uniformly bounded in δ by the *a priori* estimates in [12], and since $T^\delta(x,y) = (x, u_y(x,y))$, our claim is justified. There is however a point left open in this argument. If L is a compact subset of Ω^*, we must show that $T^\delta L$ lies in a fixed compact subset K of $T^0\Omega^*$ for δ small. However, by compactness there is $\varepsilon > 0$ such that

$$T^0 L \subset \{(s,t) : u_y(s, \alpha(s)) + \varepsilon < t < u_y(s, \beta(s)) - \varepsilon, s \in P\Omega\}.$$

We showed above that $u^\delta_y \to u_y$ uniformly on $\overline{\Omega}$, and hence it follows that

$$T^\delta L \subset \left\{(s,t) : u_y(s, \alpha(s)) + \frac{\varepsilon}{2} < t < u_y(s, \beta(s)) - \frac{\varepsilon}{2}, s \in P\Omega\right\}, \tag{2.8}$$

a precompact subset of $T^0\Omega^*$, for sufficiently small $\delta > 0$.

At this point we have that u_y^δ and all its derivatives are uniformly bounded, and hence equicontinuous, on compact subsets of Ω^*. Using the equation $u_{xx}^\delta u_{yy}^\delta - \left(u_{xy}^\delta\right)^2 = k + \delta$ together with

$$u_{yy}^\delta = \frac{1}{w_t^\delta} \geq c > 0 \tag{2.9}$$

on compact subsets of Ω^*, it now follows that the functions u_{xx}^δ, together with all their derivatives, are uniformly bounded, and hence equicontinuous, on compact subsets of Ω^*. Recall that $u^\delta \to u$ in $C^{0,1}\left(\overline{\Omega}\right)$. We can now extract a sequence $\left\{u^{\delta_n}\right\}_{n=1}^\infty$ with $\delta_n \to 0$ such that all derivatives of u^{δ_n}, of order at least two, converge uniformly on compact subsets of Ω^*. It follows that u is smooth in Ω^*. Finally, note that (2.9) shows that $u_{yy} > 0$ in Ω^*.

2.2. **Prescribed Gaussian curvature.** Here we prove Theorem 1.3 on prescribed Gaussian curvature.

Proof. (of Theorem 1.3) For $\delta \geq 0$ such that $\mathcal{K}(x,y) + \delta < \lambda$ for $(x,y) \in \overline{\Omega}$ (where λ is as in (1.11)), let u^δ be the solution in [15] of the prescribed Gaussian curvature boundary problem with homogeneous boundary data,

$$\left\{ \begin{array}{rcll} u_{xx}^\delta u_{yy}^\delta - \left(u_{xy}^\delta\right)^2 & = & \left(\mathcal{K}(x,y) + \delta\right)\left(1 + \left|\nabla u^\delta(x,y)\right|^2\right)^2, & (x,y) \in \Omega \\ u^\delta & = & 0, & (x,y) \in \partial\Omega \end{array} \right. .$$

Thus $u^0 = u$, and just as in the case of the Monge-Ampère equation, with $\mathbf{L}$ as in (2.1), $U = u^{\delta_1} - u^{\delta_2} \in C^{1,1}$ satisfies an elliptic equation with $C^{1,1}$ coefficients,

$$\begin{aligned} \mathbf{L}U &= (\delta_1 - \delta_2)\left(1 + \left|\nabla u^{\delta_1}(x,y)\right|^2\right)^2 \\ &\quad + \left(\mathcal{K}(x,y) + \delta_2\right)\left\{\left(1 + \left|\nabla u^{\delta_1}(x,y)\right|^2\right)^2 - \left(1 + \left|\nabla u^{\delta_2}(x,y)\right|^2\right)^2\right\} \\ &= (\delta_1 - \delta_2)\left(1 + \left|\nabla u^{\delta_1}(x,y)\right|^2\right)^2 \\ &\quad + \left(\mathcal{K}(x,y) + \delta_2\right)\int_0^1 \frac{d}{d\theta} F\left(\nabla u^{\delta_2} + \theta\nabla U\right) d\theta, \end{aligned}$$

where $F(p,q) = \left(1 + p^2 + q^2\right)^2$. When $\delta_1 > \delta_2 \geq 0$, we can rewrite this equation as

$$\mathbf{L}U - \left(\mathcal{K}(x,y) + \delta_2\right)\left\{\int_0^1 (\nabla F)\left(\nabla u^{\delta_2} + \theta\nabla U\right) d\theta\right\} \cdot \nabla U \geq 0$$

where the second term on the left is linear in ∇U with coefficients satisfying the hypotheses of the maximum principle, Theorem 9.6 in [9]. We conclude that $U \leq 0$ or $u^{\delta_1} \leq u^{\delta_2}$ in $\overline{\Omega}$. It follows that with $y = \alpha(x)$ and $y = \beta(x)$ parameterizing the bottom and top boundary curves of $\partial\Omega$,

$$\begin{aligned} u_y^{\delta_1}(x,\beta(x)) &= \lim_{y \to \beta(x)^-} \frac{0 - u_y^{\delta_1}(x,y)}{\beta(x) - y} \\ &\geq \lim_{y \to \beta(x)^-} \frac{0 - u_y^{\delta_2}(x,y)}{\beta(x) - y} = u_y^{\delta_2}(x,\beta(x)), \end{aligned}$$

and similarly,

$$u_y^{\delta_1}\left(x,\alpha\left(x\right)\right)\leq u_y^{\delta_2}\left(x,\alpha\left(x\right)\right),$$

for $\delta_1>\delta_2$. Note also that

$$u_y\left(x,\alpha\left(x\right)\right)\leq c_1<0<c_2\leq u_y\left(x,\beta\left(x\right)\right)$$

for x near 0, since u is a nonconstant convex function with zero boundary values.

Now apply the partial Legendre transform T^δ as in (1.15) with u^δ in place of u,

$$\begin{cases} s &= x \\ t &= u_y^\delta\left(x,y\right) \end{cases},$$

and use the above inequalities to obtain that the transformed regions

$$T^\delta\Omega=\left\{\left(x,u_y^\delta\left(x,y\right)\right):\left(x,y\right)\in\Omega\right\}$$

satisfy $T^{\delta_2}\Omega\subset T^{\delta_1}\Omega$ for $\delta_1>\delta_2$, as well as that the interior of $T^0\Omega$ is the domain $T^0\Omega^*$. Now for $\delta>0$, let $w^\delta\left(s,t\right)$ denote the inverse function $u_y^\delta\left(s,\cdot\right)^{-1}\left(t\right)$. Since

$$k^\delta\left(x,y,v,p,q\right)=\left(\mathcal{K}\left(x,y\right)+\delta\right)\left(1+p^2+q^2\right)^2$$

satisfies

$$\left\|\left(\frac{\left|k_1^\delta\right|+\left|k_5^\delta\right|+\left|k_{55}^\delta\right|}{\sqrt{k^\delta}}+\sum_{i=2}^{4}\frac{\left|k_i^\delta\right|}{\left(k^\delta\right)^{d(i)}}\right)\right\|_{L^\infty(L)}\leq C\left(L\right),\qquad L\ compact\subset\Omega,$$

uniformly in $1\geq\delta>0$, we can apply Theorem 1.4 to the transformed functions w^δ to show that they, together with all their derivatives, are uniformly bounded on compact subsets of $T^0\Omega^*$. Indeed, just as in the proof of Theorem 1.1 for the Monge-Ampère equation (but using instead the C^2 *a priori* estimates in [11] for the equation of prescribed Gaussian curvature), we can show that $u^\delta\to u$ in $C^{0,1}\left(\overline{\Omega}\right)$ as $\delta\to 0$, and that for every compact subset K of $T^0\Omega^*$, there is a compact subset L of $\Omega^*\times\mathbb{R}^3$ and $c>0$ such that

$$\left(x,w^\delta\left(x,y\right),r^\delta\left(x,y\right),z^\delta\left(x,y\right),y\right)\in L,\qquad for\ all\ \left(x,y\right)\in K,0<\delta<c.$$

Moreover, the inequalities in (1.25) also follow from the *a priori* estimates in [11]. Finally, as mentioned earlier, in the case of the prescribed Gaussian curvature equation, all of the inequalities in (1.26) hold automatically, save for $|k_2|\leq Ck^{\frac{3}{2}}$, which is part of the hypotheses of Theorem 1.3.

It now follows from the inverse function theorem, just as in the proof for the Monge-Ampere equation, that the functions u_y^δ, together with all their derivatives, are uniformly bounded, and hence equicontinuous, on compact subsets of Ω^*. Indeed, for K a compact subset of Ω^*, the sets $T^\delta K$ lie in a fixed compact subset L of $T^0\Omega^*$ for δ small by (2.8), whose proof follows using the C^2 *a priori* estimates in [11] for the equation of prescribed Gaussian curvature. Using the equation

$$u_{xx}^\delta u_{yy}^\delta-\left(u_{xy}^\delta\right)^2=\left(\mathcal{K}\left(x,y\right)+\delta\right)\left(1+\left|\nabla u^\delta\left(x,y\right)\right|^2\right),$$

it follows that the functions u_{xx}^δ, together with all their derivatives, are uniformly bounded, and hence equicontinuous, on compact subsets of Ω^*. Just as in the proof for the Monge-Ampere equation, it now follows that u is smooth and $u_{yy}>0$ in Ω^*.

Remark 2.2. *We can obtain analogues of Theorems 1.1 and 1.3 for the more general boundary value problem (1.1) under the hypothesis (1.26) provided* ***(i)*** *there is an approximating family of smooth positive functions k_δ such that the corresponding elliptic problems have smooth solutions, and* ***(ii)*** *there are C^2 a priori estimates for the boundary value problems in (i). See* [12] *in connection with (ii) and* [9] *and the references given there in connection with (i).*

3. Appendix A: Pogorelov segments

Let Ω be an open set and u be a continuous function defined in either Ω or $\overline{\Omega}$. A segment L in $\overline{\Omega}$ is said to be a Pogorelov segment for u if both of the following hold:

(a) L is the projection onto $\overline{\Omega}$ of a *maximal* line segment in the graph of u,
(b) one endpoint of L lies in $\partial\Omega$ while the other endpoint lies in Ω.

Note that the endpoint of L in $\partial\Omega$ will or will not belong to L according as u is defined in $\overline{\Omega}$ or Ω. We prove here that if u is a nontrivial convex solution to the homogeneous Monge-Ampère boundary value problem

$$\left\{ \begin{array}{rcll} \det \nabla^2 u & = & k & in\ \Omega \\ u & = & 0 & on\ \partial\Omega \end{array} \right. , \tag{3.1}$$

with $k = 0$ on the y-axis and $\partial\Omega$ positively curved, then $u \notin C^2\left(\overline{\Omega}\right) \cap C^3\left(\Omega\right)$ if there is a Pogorelov segment in the y-axis. Moreover, if in addition Ω and k are symmetric about the y-axis, we obtain the stronger conclusion that $u \notin C^2\left(\overline{\Omega}\right)$ whether or not it has a Pogorelov segment, rendering u as irregular as it can possibly be when $k \in C^{1,1}\left(\overline{\Omega}\right)$ and $\partial\Omega$ is $C^{3,1}$ - indeed, in this case $u \in C^{1,1}\left(\overline{\Omega}\right)$ by the results in Guan [11]. A result for the nonhomogeneous problem is given in the second section below.

Before stating and proving our results in detail, we sketch our arguments in the simple case when $u \in C^4\left(\overline{\Omega}\right)$; the extra differentiablility permits the simplifying use of differential inequalities.

Theorem 3.1. *Suppose $u \in C^4\left(\overline{\Omega}\right)$ is a nontrivial convex solution to (3.1) where k vanishes when $x = 0$, and $\partial\Omega$ is positively curved. Then there are no Pogorelov segments in the y-axis.*

Proof. Suppose, in order to derive a contradiction, that L is a Pogorelov segment for u lying in the y-axis. We compute

$$k_{xx} = u_{xxxx}u_{yy} + 2u_{xxx}u_{xyy} + u_{xx}u_{xxyy} - 2\left(u_{xxy}\right)^2 - 2u_{xy}u_{xxxy},$$

Now k achieves its minimum value of 0 on the y-axis, so the second derivative test yields $k_{xx} \geq 0$ on L. Also, $u_{yy} = 0$ on L since u restricted to L is affine, and it follows that $u_{xy}^2 = u_{xx}u_{yy} - k = 0$ on L as well. Finally then, $u_{xyy} = \partial_y\left(u_{xy}\right) = 0$ on L and we have

$$0 \leq k_{xx} = u_{xx}\partial_y^2 u_{xx} - 2\left(\partial_y u_{xx}\right)^2, \quad on\ L.$$

Now by the homogeneous boundary condition and the positive curvature of $\partial\Omega$, the chain rule yields that u_{xx} is positive at the endpoint of L in $\partial\Omega$ (see below). The above inequality then shows that $\partial_y^2\left(\frac{1}{u_{xx}}\right) \leq 0$ where $u_{xx} > 0$ on L, and it follows easily that $\frac{1}{u_{xx}}$ is concave, and that u_{xx} is bounded below by a positive constant, on L (see below).

Next we compute

$$\partial_y k = u_{xx}\partial_y u_{yy} + u_{xxy}u_{yy} - 2u_{xy}\partial_x u_{yy},$$

which when restricted to the y-axis yields

$$u_{xx}\left|\partial_y u_{yy}\right| \le C u_{yy},$$

since both $|u_{xy}|$ and $|\partial_x u_{yy}|$ are dominated by $C\sqrt{u_{yy}}$ on the y-axis. Indeed, $u_{xy}^2 = u_{xx}u_{yy} - k$ yields the former, while the latter holds because u_{yy} is C^2 and nonnegative on Ω, and thus satisfies

$$\left|\partial_x u_{yy}\right| \le C\left\|\partial_x^2 u_{yy}\right\|_\infty^{\frac{1}{2}}\sqrt{u_{yy}},$$

by (1.24) with u_{yy} in place of k. In the previous paragraph we proved that u_{xx} is bounded below by a positive constant on L, and hence on an extension of L in Ω. Then unique continuation, or Gronwall's inequality, shows that u_{yy} vanishes on a segment of the y-axis strictly larger than L, contradicting the maximality of the Pogorelov segment L, and completing the proof of the theorem.

Our main technical results on Pogorelov segments do not involve boundary behaviour.

Theorem 3.2. *Suppose $u \in C^2(\Omega)$ is convex and $L = \{(0,y) : y \in I\}$ is a Pogorelov segment for u in the y-axis. Let*

$$f(y) = u_{xx}(0,y), \qquad y \in I.$$

Then ***either*** *f is identically zero on I,* ***or*** *f is positive on I and $\frac{1}{f}$ is concave on I.*

Theorem 3.3. *Suppose u is convex in Ω and that* ***either*** *$u \in C^2(\Omega)$ and is symmetric about the y-axis,* ***or*** *$u \in C^3(\Omega)$. Moreover, we assume that $\det D^2 u = 0$ on the y-axis. Then if L is a Pogorelov segment for u in the y-axis, the Hessian of u has rank zero on L, i.e. $D^2 u = 0$ on L.*

If we assume that u is defined on $\overline{\Omega}$ and vanishes on the boundary of Ω, then we are able to obtain some negative regularity results.

Corollary 3.4. *Suppose k vanishes on the y-axis, and $\partial\Omega$ is positively curved (at the intersection of the y-axis and Ω). Then a nontrivial convex solution to (3.1) has no Pogorelov segments on the y-axis if $u \in C^2\left(\overline{\Omega}\right) \cap C^3(\Omega)$. If in addition both Ω and k are symmetric about the y-axis, then there is no nontrivial convex $C^2\left(\overline{\Omega}\right)$ solution u to (3.1).*

Proof. (of Theorem 3.2) We have that $u(0,y)$ is affine for y belonging to some nontrivial *maximal* interval $(\rho(0), a]$, where the lower half of $\partial\Omega$ has graph $(x, \rho(x))$. Then $u(0,y) = \beta(y - \rho(0))$ for $\rho(0) < y \le a$ for some $\beta < 0$ (since u is nontrivial). Let $k = \det D^2 u$. Now $u_{yy} = 0$ on L together with $k = u_{xx}u_{yy} - u_{xy}^2 \ge 0$ on y-axis, yields $u_{xy} = 0$ on L, and so u_x is a constant on L, say γ. By Taylor's formula in the x-variable with y fixed, and with $h(y) = \beta(y - \rho(0))$ and $f(y) = \frac{\partial^2}{\partial x^2}u(0,y)$, we then have

$$u(x,y) = h(y) + \gamma x + \frac{1}{2}f(y)x^2 + \eta(x,y)x^2,$$

for $\rho(0) \le y \le a$, where $\eta(x,y)$ is continuous in x for each fixed y and satisfies

$$\lim_{x\to 0}\eta(x,y) = 0, \qquad \rho(0) < y \le a. \tag{3.2}$$

We will now establish that either f vanishes identically on $(\rho(0), a]$, or f is positive on $(\rho(0), a]$ and $\frac{1}{f}$ is concave on $(\rho(0), a]$. Fix $\theta \in (0,1)$, $\rho(0) < y_1 < y_2 \leq a$, and for x_1, x_2 nonnegative, small and not both zero, we set

$$\begin{aligned} x_\theta &= (1-\theta) x_1 + \theta x_2, \\ y_\theta &= (1-\theta) y_1 + \theta y_2. \end{aligned}$$

The convexity of u implies

$$\begin{aligned} &h(y_\theta) + \gamma x_\theta + \left\{\frac{1}{2} f(y_\theta) + \eta(x_\theta, y_\theta)\right\}(x_\theta)^2 \\ &= u(x_\theta, y_\theta) \leq (1-\theta) u(x_1, y_1) + \theta u(x_2, y_2) \\ &= (1-\theta) h(y_1) + \theta h(y_2) + (1-\theta)\gamma x_1 + \theta\gamma x_2 \\ &\quad + (1-\theta)\left\{\frac{1}{2} f(y_1) + \eta\right\} x_1^2 + \theta\left\{\frac{1}{2} f(y_2) + \eta\right\} x_2^2. \end{aligned} \tag{3.3}$$

Since h is affine on $(\rho(0), a]$, then $h(y_\theta) = (1-\theta) h(y_1) + \theta h(y_2)$, and we obtain

$$f(y_\theta)(x_\theta)^2 = f(y_1)(1-\theta) x_1^2 + f(y_2)\theta x_2^2 + \circ\left(x_1^2 + x_2^2\right), \tag{3.4}$$

since

$$(1-\theta)\eta(x_1, y_1) x_1^2 + \theta\eta(x_2, y_2) x_2^2 - \eta(x_\theta, y_\theta)(x_\theta)^2 = \circ\left(x_1^2 + x_2^2\right),$$

by (3.2) with $y = y_1$, y_2 and y_θ. Now using that

$$(x_\theta)^2 = ((1-\theta) x_1 + \theta x_2)^2 \approx x_1^2 + x_2^2,$$

with constants depending on θ (but θ is fixed in $(0,1)$), we obtain upon dividing (3.4) throughout by $(x_\theta)^2$,

$$f(y_\theta) \leq \frac{f(y_1)}{(1-\theta)}\left(\frac{(1-\theta) x_1}{x_\theta}\right)^2 + \frac{f(y_2)}{\theta}\left(\frac{\theta x_2}{x_\theta}\right)^2 + \circ(1) \tag{3.5}$$

as $x_1^2 + x_2^2 \to 0$.

Now if $f(y_1)$ (respectively $f(y_2)$) is zero, we may choose x_2 (respectively x_1) to be zero, and obtain in the limit from (3.5) that $f(y_\theta) = 0$. Thus if f vanishes at any point of $(\rho(0), a]$, then f vanishes identically on $(\rho(0), a)$, and by continuity on $(\rho(0), a]$ as well. So we may now assume that f is positive on $(\rho(0), a]$.

For $A, B > 0$ the function $A(1-\alpha)^2 + B\alpha^2$ has minimum value equal to the harmonic mean $\frac{1}{\frac{1}{A}+\frac{1}{B}}$, and is minimized by choosing $\alpha = \frac{A}{A+B}$. Thus if we let the pair $(x_1, x_2) \to (0,0)$ through values satisfying the linear relation

$$\frac{\theta x_2}{(1-\theta) x_1 + \theta x_2} = \alpha = \frac{A}{A+B} = \frac{\frac{f(y_1)}{1-\theta}}{\frac{f(y_1)}{(1-\theta)} + \frac{f(y_2)}{\theta}}, \tag{3.6}$$

then we conclude from (3.5) that

$$f(y_\theta) \leq \frac{1}{\frac{1}{A} + \frac{1}{B}} = \frac{1}{\frac{1-\theta}{f(y_1)} + \frac{\theta}{f(y_2)}},$$

or $\frac{1}{f(y_\theta)} \geq \left(\frac{1-\theta}{f(y_1)} + \frac{\theta}{f(y_2)}\right)$, and this establishes the concavity of $\frac{1}{f}$ on $(\rho(0), a]$.

Remark 3.1. *If we replace the regularity assumption $u \in C^2(\Omega)$ by $u \in C^1(\Omega)$, then the above argument demonstrates that the function $f^+(y) = \limsup_{x\to 0^+} \frac{u(x,y)}{x^2}$ either vanishes identically on $(\rho(0), a)$, or f^+ is positive on $(\rho(0), a)$ and $\frac{1}{f^+}$ is concave on $(\rho(0), a)$ (with a similar result for f^- defined with $x \to 0^-$). Indeed, simply use*

$$u(x,y) \leq h(y) + \{f(y) + \circ(1)\} x^2$$

to estimate both $u(x_1, y_1)$ and $u(x_2, y_2)$ in (3.3), and then choose a sequence of pairs $(x_1^n, x_2^n) \to (0,0)$ satisfying (3.6) such that the corresponding sequence of points $x_\theta^n = (1-\theta)x_1^n + \theta x_2^n$ satisfy $f^+(y_\theta) = \lim_{n\to\infty} \frac{u(x_\theta^n, y_\theta)}{(x_\theta^n)^2}$.

Proof. (of Theorem 3.3) Suppose L is a Pogorelov segment as in Theorem 3.2 above. With the notation of the previous proof, we now claim that

$$f(a) = u_{xx}(0,a) = 0. \tag{3.7}$$

By Theorem 3.2, this shows that u_{xx} vanishes on L, and since u_{yy} vanishes on L by definition, we conclude that $D^2 u$ vanishes on L as well, completing the proof of the theorem. There are two cases to consider in proving (3.7).

If u is symmetric about the y-axis, then $u_x = 0$ on the y-axis. Thus $u_{xy} = 0$ on the y-axis, and along with $k = 0$ there, we conclude that $u_{xx}u_{yy} = 0$ on the y-axis. Thus u_{xx} vanishes on the set $E = \{(0,y) : u_{yy}(0,y) > 0\}$, and since $(0,a)$ is a limit point of E (by maximality of the Pogorelov segment L), we have (3.7) as $u \in C^2(\Omega)$.

In the other case there is no symmetry, but there is additional regularity instead, namely $u \in C^3(\Omega)$. We compute as before

$$\partial_y k = u_{xx}\partial_y u_{yy} + u_{xxy}u_{yy} - 2u_{xy}\partial_x u_{yy}. \tag{3.8}$$

We assume, in order to derive a contradiction, that $u_{xx}(0,a) > 0$, and show that $u_{yy}(0,y)$ satisfies a differential inequality of Gronwall type for y in a neighbourhood of a. Again we use $|u_{xy}| \leq \sqrt{u_{xx}u_{yy}} \leq C\sqrt{u_{yy}}$ on the y-axis, but since we are no longer assuming $u \in C^4(\Omega)$, we must calculate $\partial_x u_{yy}$ directly. Now for some $\varepsilon > 0$, $u_{xx}(0,a) > 0$ for $\rho(0) < y < a + \varepsilon$, and in this range we have $u_{yy} = \frac{k + u_{xy}^2}{u_{xx}}$, and so

$$\partial_x u_{yy} = \partial_x \left(\frac{k + u_{xy}^2}{u_{xx}} \right) = \frac{k_x + 2u_{xy}u_{xxy}}{u_{xx}} - \frac{k + u_{xy}^2}{u_{xx}^2} u_{xxx}. \tag{3.9}$$

Now $|k_x| \leq C\sqrt{k} \leq C\sqrt{u_{yy}}$ near $(0,a)$ by (1.24) since k is smooth and nonnegative, and $|u_{xy}| \leq C\sqrt{u_{yy}}$ on the y-axis was established above. Since $u \in C^3(\Omega)$, we thus obtain from (3.9) that

$$|\partial_x u_{yy}(0,y)| \leq C\sqrt{u_{yy}(0,y)}, \qquad \rho(0) < y < a + \varepsilon.$$

Altogether, we have from (3.8) and the fact that $k = 0$ on the y-axis ,

$$|\partial_y u_{yy}(0,y)| \leq \frac{|u_{xxy}|\, u_{yy}}{u_{xx}} + 2\frac{|u_{xy}|\,|\partial_x u_{yy}|}{u_{xx}} \leq C u_{yy}(0,y),$$

at least for y near a. With $g(y) = u_{yy}(0,y)$, we thus have

$$\begin{cases} |g'(y)| & \leq & Cg(y) \\ g(a) & = & 0 \end{cases},$$

which yields $g \equiv 0$ on $(\rho(0), a+\varepsilon)$ by Gronwall's inequality, which contradicts the maximality of the Pogorelov segment L, and provides the desired contradiction. Thus we have established (3.7) and as indicated above, this completes the proof of the theorem.

Proof. (of Corollary 3.4) We first claim that if $u \in C^2\left(\overline{\Omega}\right)$ is a nontrivial convex solution to (3.1), then

$$u_{xx}(0, \rho(0)) > 0, \tag{3.10}$$

provided that either the data are symmetric about the y-axis, or that u has a Pogorelov segment with an endpoint at $(0, \rho(0))$. To see this, we first differentiate $0 = u(x, \rho(x))$ twice with respect to x to get

$$0 = u_{xx} + 2u_{xy}\rho' + u_{yy}(\rho')^2 + u_y\rho''.$$

Thus we obtain

$$u_{xx} + 2u_{xy}\rho' + u_{yy}(\rho')^2 = -u_y\rho'' > 0 \tag{3.11}$$

at the boundary point $(0, \rho(0))$ since $u_y(0, \rho(0)) < 0$ for u a nontrivial convex function with zero boundary data, and since $\rho''(0) > 0$ as $\partial\Omega$ is positively curved. Now if the data are symmetric about the y-axis, then $\rho'(0) = 0$ and (3.10) follows from (3.11). If u has a Pogorelov segment ending at $(0, \rho(0))$, then $u_{yy}(0, \rho(0)) = 0$, and so also $u_{xy}(0, \rho(0)) = 0$. Thus (3.10) again follows from (3.11).

The first part of the corollary now follows from (3.10) since $u \in C^2\left(\overline{\Omega}\right)$ then implies $u_{xx}(0, y) > 0$ for y near $\rho(0)$ and this contradicts Theorem 3.3 (this is where we use $u \in C^3(\Omega)$). To prove the second part of the corollary, we note that the symmetry of the data, together with uniqueness, shows that u is symmetric about the y-axis. Thus $u_x = 0$ on the y-axis, and so then does u_{xy}. Since $u \in C^2\left(\overline{\Omega}\right)$, we have $u_{xx} > 0$ in a neighbourhood of $(0, \rho(0))$, and so $0 = k = u_{xx}u_{yy} - u_{xy}^2$ on the y-axis implies that $u_{yy}(0, y) = 0$ for y near $\rho(0)$, thus establishing the existence of a Pogorelov segment in the y-axis. This contradicts Theorem 3.3 (again using $u \in C^3(\Omega)$) and completes the proof of Corollary 3.4.

3.1. **More general boundary values.** The condition of homogeneous boundary data in Corollary 3.4 can be relaxed somewhat. For convenience we consider only the case where all the data are symmetric about the y-axis. In order to state the result, we begin with a quantitative version of the Hopf boundary point lemma. We denote by $\mathbf{n}u(P)$ the inward normal derivative at $P \in \partial\Omega$ of a differentiable function u defined on a domain Ω with smooth boundary $\partial\Omega$. We let

$$\rho_{\max} = \rho_{\max}(\Omega) = \max_{P\in\partial\Omega} \frac{1}{\Gamma(P)}, \quad \rho_{\min} = \rho_{\min}(\Omega) = \min_{P\in\partial\Omega} \frac{1}{\Gamma(P)},$$

where $\Gamma(P)$ is the curvature of $\partial\Omega$ at P (i.e. $\rho_{\max}$ and $\rho_{\min}$ are the maximum and minimum radii of curvature). For $\varepsilon > 0$, we define $\Omega_\varepsilon = \{(x, y) \in \Omega : dist((x, y), \partial\Omega) \geq \varepsilon\}$. The following estimate of Hopf type is essentially sharp - see Remark 3.4 below.

Lemma 3.5. *Let Ω be convex with $\partial\Omega$ smooth and positively curved. The for all nonpositive convex functions $u \in C^{1,1}\left(\overline{\Omega}\right)$, and all $P \in \partial\Omega$with $u(P) = 0$,*

$$\mathbf{n}u(P) \leq -\frac{1}{2\sqrt{\pi}\rho_{\max}} \sup_{\varepsilon>0} \varepsilon \left(\int\int_{\Omega_\varepsilon} \det D^2u\right)^{\frac{1}{2}}, \tag{3.12}$$

and with $c_0 = \frac{1}{4\sqrt{\pi}\rho_{\max}|\partial\Omega|}$*, we also have*

$$\mathbf{n}u\left(P\right) \leq -c_0 \frac{\left(\int\int_\Omega \det \nabla^2 u\right)^{\frac{3}{2}}}{\left\|\det D^2 u\right\|_{L^\infty(\Omega)}}. \tag{3.13}$$

Proof. Given $P \in \partial\Omega$ with $u\left(P\right) = 0$, let Q be the point in Ω of distance $\rho_{\min}$ from P such that PQ is normal to $\partial\Omega$ at P. Let $A \in \Omega_\varepsilon$, $\varepsilon > 0$, and let m be the line through A and Q. If $m \cap \Omega$ has endpoints S and T, chosen so that $SQAT$ are consecutive in m, then $|SA| \leq 2\rho_{\max}$ and $|SQ| \geq 2\rho_{\min}$ since the balls $B_{\min}$ and $B_{\max}$ tangent to $\partial\Omega$ at P of radius $\rho_{\min}$ and $\rho_{\max}$ respectively, satisfy $B_{\min} \subset \Omega \subset B_{\max}$. Since u is convex and nonpositive, we conclude that

$$u\left(Q\right) \leq \frac{|QA|}{|SA|} u\left(S\right) + \frac{|SQ|}{|SA|} u\left(A\right) \leq \frac{|SQ|}{|SA|} u\left(A\right) \leq \frac{\rho_{\min}}{2\rho_{\max}} u\left(A\right).$$

Now the modulus of the slope (p, q) of the tangent plane at $(A, u\left(A\right))$ is at most $\frac{|u(A)|}{\varepsilon}$ since u is convex and nonpositive on the ball of radius ε centered at A. From the previous inequality we thus obtain

$$\sqrt{p^2 + q^2} \leq \frac{|u\left(A\right)|}{\varepsilon} \leq \frac{2\rho_{\max}}{\varepsilon\rho_{\min}} \left|u\left(Q\right)\right|.$$

If

$$(p, q) = \chi_u\left(x, y\right) = \left(u_x\left(x, y\right), u_y\left(x, y\right)\right)$$

is the gradient map of u, then the Jacobian of χ_u is $\det D^2 u$, and we have

$$\begin{aligned} \int_{\Omega_\varepsilon} \det D^2 u &= \int_{\Omega_\varepsilon} \frac{\partial\left(p, q\right)}{\partial\left(x, y\right)} dxdy = \int_{\chi_u(\Omega_\varepsilon)} dpdq = \left|\chi_u\left(\Omega_\varepsilon\right)\right| \\ &\leq \pi \left(\frac{2\rho_{\max}}{\varepsilon\rho_{\min}} u\left(Q\right)\right)^2 = \frac{4\pi\rho_{\max}^2}{\varepsilon^2} \left(\frac{u\left(Q\right)}{\rho_{\min}}\right)^2, \end{aligned}$$

where the second inequality follows from the change of variable formula applied to $\chi_u + \delta I$ and then letting $\delta \to 0$. Since $u\left(P\right) = 0$, we have

$$\mathbf{n}u\left(P\right) \leq \frac{u\left(Q\right) - u\left(P\right)}{|QP|} = \frac{u\left(Q\right)}{\rho_{\min}} \leq -\frac{\varepsilon}{2\sqrt{\pi}\rho_{\max}} \left(\int_{\Omega_\varepsilon} \det D^2 u\right)^{\frac{1}{2}},$$

which proves (3.12).

To obtain (3.13), we note that

$$\begin{aligned} \int_{\Omega_\varepsilon} \det D^2 u &= \int_\Omega \det D^2 u - \int_{\Omega \setminus \Omega_\varepsilon} \det D^2 u \\ &\geq \int_\Omega \det D^2 u - \varepsilon\left|\partial\Omega\right| \left\|\det D^2 u\right\|_{L^\infty(\Omega)} \\ &\geq \frac{1}{2} \int_\Omega \det D^2 u \end{aligned}$$

if we choose $\varepsilon = \frac{1}{2|\partial\Omega|\|\det D^2 u\|_{L^\infty(\Omega)}} \int_\Omega \det D^2 u$. Plugging this choice of ε into (3.12) yields (3.13) with $c_0 = \frac{1}{4\sqrt{\pi}\rho_{\max}|\partial\Omega|}$.

Remark 3.2. *This lower bound for* $|\mathbf{n}u(P)|$ *should be compared with the formula in Lemma 7.1 of* [17] *for the* L^2 *average of* $|\mathbf{n}u(P)|$ *with respect to the probability measure* $\frac{\Gamma ds}{2\pi}$, *valid for those* $u \in C^{1,1}\left(\overline{\Omega}\right)$ *that vanish on* $\partial\Omega$*;*

$$\left(\int_\Omega |\mathbf{n}u(P)|^2 \frac{\Gamma ds}{2\pi}\right)^{\frac{1}{2}} = \left(\frac{1}{\pi}\int_\Omega \det D^2 u\right)^{\frac{1}{2}}.$$

We now have the following generalization of Theorem 3.3 and Corollary 3.4 for the general boundary value problem

$$\left\{\begin{array}{rcll} \det D^2 u & = & k & \text{in } \Omega \\ u & = & \phi & \text{on } \partial\Omega \end{array}\right. . \tag{3.14}$$

Let the lower half of $\partial\Omega$ be given by the graph $(x, \rho(x))$.

Theorem 3.6. *Suppose* Ω, k *and* ϕ *are symmetric about the* y*-axis,* $k \in C^{1,1}\left(\overline{\Omega}\right)$ *vanishes on the* y*-axis,* $k \geq 0$ *and nontrivial on* Ω, $\phi \in C^{3,1}(\partial\Omega)$, *and* $\partial\Omega$ *is smooth and positively curved. Suppose moreover that* Ω, k *and* ϕ *are related at the boundary point* $(0, \rho(0))$ *by the condition*

(3.15)

$$\frac{d^2}{dx^2}\phi(x,\rho(x))\,|_{x=0} > \rho''(0)\left\{\sup_{(x,y)\in\partial\Omega}\frac{\phi(x,y)-\phi(0,\rho(0))}{y-\rho(0)} - c_0\frac{\left(\int\int_\Omega k\right)^{\frac{1}{2}}}{\|k\|_{C^0(\Omega)}}\right\},$$

where c_0 *is as in Lemma 3.5. If* $u \in C^{1,1}\left(\overline{\Omega}\right)$ *is the convex solution to the boundary value problem (3.14), then* $u \notin C^2\left(\overline{\Omega}\right)$.

Remark 3.3. *A useful special case of the theorem occurs when* $\phi(0,\rho(0)) = 0$ *and* $\phi \leq 0$ *on* $\partial\Omega$, *and we assume the following inequality stronger than (3.15):*

$$\frac{\frac{d^2}{dx^2}\phi(x,\rho(x))\,|_{x=0}}{\rho''(0)} > -c_0\frac{\left(\int\int_\Omega k\right)^{\frac{1}{2}}}{\|k\|_{C^0(\Omega)}}. \tag{3.16}$$

Proof. We begin by proving the special case in Remark 3.3 above. So assume, in order to derive a contradiction, that $u \in C^2\left(\overline{\Omega}\right)$. Differentiating $\phi(x,\rho(x)) = u(x,\rho(x))$ twice with respect to x, and setting $x = 0$ and using $\rho'(0) = 0$ yields

$$\frac{d^2}{dx^2}\phi(x,\rho(x))\,|_{x=0} = u_{xx}(0,\rho(0)) + u_y(0,\rho(0))\,\rho''(0).$$

Thus by (3.16) and Lemma 3.5 we have

$$u_{xx}(0,\rho(0)) > -c_0\frac{\left(\int\int_\Omega k\right)^{\frac{1}{2}}}{\|k\|_{C^0(\Omega)}}\rho''(0) - u_y(0,\rho(0))\,\rho''(0) \geq 0,$$

since $\mathbf{n}u(0,\rho(0)) = u_y(0,\rho(0))$. With this inequality, the proofs of Theorem 3.3 and Corollary 3.4 carry over without change.

Now let $\theta = \sup_{(x,y)\in\partial\Omega}\frac{\phi(x,y)-\phi(0,\rho(0))}{y-\rho(0)}$, and define

$$\widetilde{u}(x,y) = u(x,y) - \phi(0,\rho(0)) - \theta(y-\rho(0))$$

and

$$\widetilde{\phi}(x,y) = \phi(x,y) - \phi(0,\rho(0)) - \theta(y-\rho(0))$$

so that $\widetilde{\phi}(0,\rho(0)) = 0$ and $\widetilde{\phi} \leq 0$ on $\partial\Omega$, and (3.14) holds with $\widetilde{u}$ and $\widetilde{\phi}$ in place of u and ϕ. Then (3.15) implies that $\widetilde{\phi}$ satisfies (3.15), and applying Remark 3.3, proved in the previous paragraph, completes the proof of the theorem.

Remark 3.4. *The exponent $\frac{3}{2}$ in conclusion (3.13) of Lemma 3.5 is sharp. This is most easily seen by first noting that Lemma 3.5 carries over to arbitrary dimension n in the form*

$$\mathbf{n}u(P) \leq -c_0 \frac{\left(\int\int_\Omega \det \nabla^2 u\right)^{\frac{n+1}{n}}}{\|\det D^2 u\|_{C^0(\Omega)}} \tag{3.17}$$

with $c_0 = \frac{1}{4(\omega_n)^{\frac{1}{n}}\rho_{\max}|\partial\Omega|}$ (ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $\rho_{\max}$ is the maximum principle radius of curvature of $\partial\Omega$) for appropriate u and P. It is easy to construct examples to show that (3.17) is sharp in one dimension, and from there it is straightforward, although messy, to construct analogous examples in higher dimension. In the case $n = 1$, simply let $0 < \beta < 1$, $\Omega = [0,1]$, and define u by requiring $u(0) = u(1) = 0$ and

$$u''(t) = \begin{cases} 0, & 0 \leq t \leq 1-\beta \\ 1, & 1-\beta \leq t \leq 1 \end{cases}.$$

Then

$$\begin{aligned} 0 &= u(1) - u(0) = \int_0^1 \left\{u'(0) + \int_0^x u''(t)\,dt\right\} dx \\ &= u'(0) + \int_0^1 (1-t)\,u''(t)\,dt \\ &= u'(0) + \frac{\beta^2}{2}, \end{aligned}$$

and thus we have

$$\mathbf{n}u(0) = u'(0) = -\frac{\beta^2}{2} = -\frac{1}{2}\left(\int_0^1 u''(t)\,dt\right)^2 = -\frac{1}{2}\left(\int_\Omega \det D^2 u\right)^2,$$

where $\left\|\det D^2 u\right\|_{C^0(\Omega)} = 1$ and $\int_\Omega \det D^2 u$ can be arbitrarily small.

4. Appendix B

We collect some standard material here for the reader's convenience.

4.1. **Generalized solutions.** Let μ be a nonnegative Borel measure on Ω. A. D. Alexandrov introduced in [1] the concept of a generalized convex solution u to the Monge-Ampère equation in Ω,

$$u_{xx}u_{yy} - (u_{xy})^2 = \mu, \tag{4.1}$$

based on the change of variable formula

$$\int\int f(s,t)\,dsdt = \int\int f(u_x, u_y)\left\{u_{xx}u_{yy} - (u_{xy})^2\right\} dxdy. \tag{4.2}$$

In particular, if u is smooth and convex, and the change of variables is one-to-one, then

$$|\{(u_x(x,y), u_y(x,y)) : (x,y) \in E\}| = \int\int_E \left(u_{xx}u_{yy} - (u_{xy})^2\right) dxdy$$

for all Borel subsets E of Ω. Given any convex function u on Ω and a point $(x,y)\in\Omega$, define as in [5] the set

$$B(x,y)=\left\{(a,b)\in\mathbb{R}^2: u(s,t)\geq u(x,y)+a(s-x)+b(t-y),(s,t)\in\Omega\right\}$$

of slopes of all supporting planes of the graph of u at (x,y). To each Borel set E in Ω, let

$$B(E)=\cup_{(x,y)\in E}B(x,y)$$

and define $\mu_u(E)$ to be the Lebesgue measure of the set $B(E)$. In [1] (see also [5]) it is proved that $B(E)$ is Lebesgue measurable if E is Borel, and that μ_u is a nonnegative Borel measure on Ω, referred to as the representing measure of u (and to be thought of as the generalized determinant of the Hessian of u). One says that u is a generalized convex solution of (4.1) if its representing measure is μ. In particular, u is a generalized convex solution of $u_{xx}u_{yy}-(u_{xy})^2=k$ if u is convex and $d\mu_u=kdxdy$.

In the case that u is $C^{1,1}$, it turns out that (4.2) implies that μ_u is absolutely continuous with density $u_{xx}u_{yy}-(u_{xy})^2$. Thus the above notion of solution generalizes the classical notion of solution for the Monge-Ampère equation. One of the key properties of generalized solutions is the following weak convergence theorem.

Theorem 4.1. *If a sequence of convex functions $\{u_n\}_{n=1}^{\infty}$ converges uniformly on compact subsets of Ω to a convex function u, then the associated measures μ_{u_n} converge weakly to the measure μ_u in the sense that* $\lim_{n\to\infty}\int f d\mu_{u_n}=\int f d\mu_u$ *for all continuous functions f with compact support in Ω.*

See Proposition 1 and the subsequent corollary in [5] for a proof.

4.2. **Radial solutions.** Radial solutions to the Monge-Ampère equation,

$$\left\{\begin{array}{rcll} u_{xx}u_{yy}-(u_{xy})^2 &=& k(x,y), & (x,y)\in D\\ u &=& C, & (x,y)\in\partial D\end{array}\right. \tag{4.3}$$

in the unit disk D are easily characterized. Since the determinant of the Hessian is rotation invariant, u is radial in (1.1) if and only if k is radial and ϕ is a constant C on the unit circle ∂D. Let

$$k(x,y)=f\left(\frac{r^2}{2}\right)\text{ and }u(x,y)=g\left(\frac{r^2}{2}\right),$$

where $r=\sqrt{x^2+y^2}$. Then one easily computes

$$u_{xx}u_{yy}-u_{xy}^2=g''\left(\frac{r^2}{2}\right)g'\left(\frac{r^2}{2}\right)r^2+g'\left(\frac{r^2}{2}\right)^2$$

and so

$$f(t)=g''(t)g'(t)2t+g'(t)^2=\left[tg'(t)^2\right]'.$$

Thus

$$g(t)=C+\int_0^t\sqrt{\frac{1}{s}\int_0^s f}\,ds$$

and $u(x,y)=g\left(\frac{x^2+y^2}{2}\right)$ solves (4.3). In particular, if $f(t)=t^N$ so that $k(x,y)=cr^{2N}$, then $u-C\approx r^{N+2}$, which fails to be smooth for N a positive odd integer.

Note that if we add a positive constant of integration K to $\int_0^s f$, then $u(r) = C + 2\sqrt{K}r + O(r^2)$ satisfies $u_{xx}u_{yy} - u_{xy}^2 = k$ almost everywhere in D, but fails to be $C^{1,1}$ in D since it has a conical singularity at the origin. Moreover, u fails to be a solution of (4.3) in the generalized sense since the representing measure of u has a Dirac component at the origin.

We can just as easily compute radial solutions u to the prescribed Gaussian curvature equation in the unit disk,

$$\left\{ \begin{array}{rcl} u_{xx}u_{yy} - (u_{xy})^2 & = & \mathcal{K}(x,y)\left(1 + u_x^2 + u_y^2\right)^2, \quad (x,y) \in D \\ u & = & C, \quad (x,y) \in \partial D \end{array} \right. , \tag{4.4}$$

where again, since the Hessian and the gradient are rotation invariant (the curvature is actually an isometry invariant), the data must be rotation invariant if the solution u is. With

$$\mathcal{K}(x,y) = f\left(\frac{r^2}{2}\right) \text{ and } u(x,y) = g\left(\frac{r^2}{2}\right),$$

as above, we have

$$\left(1 + u_x^2 + u_y^2\right)^2 = \left(1 + g'\left(\frac{r^2}{2}\right)^2 r^2\right)^2$$

and

$$f(t) = \frac{g''(t)\,g'(t)\,2t + g'(t)^2}{\left(1 + 2t\left|g'(t)\right|^2\right)^2} = \frac{\left[tg'(t)^2\right]'}{\left(1 + 2tg'(t)^2\right)^2} = -\frac{1}{2}\left[\frac{1}{\left(1 + 2tg'(t)^2\right)}\right]'.$$

Integrating, we obtain

$$\frac{1}{\left(1 + 2tg'(t)^2\right)} - 1 = -2\int_0^t f \text{ and } g(t) = C_1 + \int_0^t \sqrt{\left(\frac{\frac{1}{s}\int_0^s f}{1 - 2\int_0^s f}\right)}ds.$$

Thus $u(x,y) = g\left(\frac{r^2}{2}\right)$ solves (4.4) provided $2\int_0^{\frac{1}{2}} f < 1$. This is of course the familiar condition

$$\int\int_D \mathcal{K}(x,y)\,dxdy = \int_0^{2\pi}\int_0^1 f\left(\frac{r^2}{2}\right) rdrd\theta = 2\pi\int_0^{\frac{1}{2}} f(t)\,dt < \pi, \tag{4.5}$$

which is necessary if $\mathcal{K}$ is the curvature of a convex $C^{1,1}$ function u on $\overline{D}$. The necessity of (4.5) follows from the change of variable $(s,t) = T(x,y)$ given by $s = u_x(x,y)$, $t = u_y(x,y)$ in the computation

$$\begin{aligned} \pi &= \int_0^{2\pi}\int_0^\infty \left(1 + r^2\right)^{-2} rdrd\theta = \int\int_{\mathbb{R}^2} \left(1 + s^2 + t^2\right)^{-2} dsdt \\ &> \int\int_{T(D)} \left(1 + s^2 + t^2\right)^{-2} dsdt = \int\int_D \mathcal{K}(x,y)\,dxdy, \end{aligned}$$

where we have used that $\mathcal{K}$ is the curvature of u, the Jacobian of T is $u_{xx}u_{yy} - (u_{xy})^2$, and that $T(D)$ is bounded if u has bounded derivatives on D. See for example [29] for a discussion of prescribed Gaussian curvature and condition (4.5).

4.3. **Almost one-variable.** Our condition (1.4),

$$|\partial_y k(x,y)| \leq Ck(x,y)^{\frac{3}{2}}, \tag{4.6}$$

says that k is close to being a function of x alone when k is small. In fact, the zero set of such a function k is a union of vertical lines by Gronwall's inequality. Thus for a given x, we may suppose that $k(x,y) \neq 0$ for all y and rewrite (4.6) for such (x,y) as $\left|\partial_y \frac{1}{\sqrt{k(x,y)}}\right| \leq C$, which implies

$$\left|\frac{1}{\sqrt{k(x,y)}} - \frac{1}{\sqrt{k(x,0)}}\right| \leq C|y|.$$

Since k is bounded, we conclude that $k(x,y) \approx k(x,0)$, and so

$$k(x,y) = f(x)\left[1 + \sqrt{f(x)}h(x,y)\right] \tag{4.7}$$

where $f(x) = k(x,0)$ is nonnegative and both $h(x,y) = \frac{k(x,y)-k(x,0)}{k(x,0)^{\frac{3}{2}}}$ and $\partial_y h$ are bounded:

$$\begin{aligned}
|h(x,y)| &= \left|\frac{\sqrt{k(x,y)} - \sqrt{k(x,0)}}{k(x,0)^{\frac{3}{2}}}\right| \left(\sqrt{k(x,y)} + \sqrt{k(x,0)}\right) \\
&\leq C|y| \left(\frac{\sqrt{k(x,y)} + \sqrt{k(x,0)}}{\sqrt{k(x,0)}}\right) \leq 2C|y|; \\
|\partial_y h(x,y)| &= \left|\frac{\partial_y k(x,y)}{k(x,0)^{\frac{3}{2}}}\right| \leq C.
\end{aligned}$$

Conversely, functions k of the form (4.7) with f, h and $\partial_y h$ bounded, are themselves bounded and satisfy (4.6). This observation led to the refinement of the Sibony example in section 1.4 above. We thank C. Rios for a suggestion to combine this with the example of Sibony.

4.4. **Interpolation inequalities.** We begin with the classical interpolation inequality for a smooth nonnegative function k defined on a domain Ω,

$$|\nabla k(x,y)| \leq B\sqrt{k(x,y)}, \qquad (x,y) \in \mathcal{R}, \tag{4.8}$$

valid for a compact subset $\mathcal{R}$ of Ω. We first note that (4.8) holds for k nonnegative and smooth on all of $\mathbb{R}^2$ with bounded second derivatives. Indeed, the inequality follows from the one-dimensional version, which in turn follows from Taylor's formula,

$$\begin{aligned}
0 &\leq k(y) = k(x) + k'(x)(y-x) + \frac{1}{2}k''(c)(y-x)^2 \\
&\leq k(x) + k'(x)(y-x) + \frac{1}{2}\|k''\|_\infty (y-x)^2,
\end{aligned}$$

upon choosing $y = x - \frac{k'(x)}{\|k''\|_\infty}$ (if $\|k''\|_\infty = 0$, the result is trivial). The inequality $|\nabla k(x,y)| \leq \sqrt{2}\left\|\nabla^2 k\right\|_\infty^{\frac{1}{2}} \sqrt{k(x,y)}$, valid for k nonnegative with bounded second derivatives on $\mathbb{R}^2$, remains true in the form

$$|\nabla k(x,y)| \leq C\left\{\left\|\nabla^2 k\right\|_\infty^{\frac{1}{2}} + (dist((x,y),\partial\Omega))^{-\frac{1}{2}}\right\}\sqrt{k(x,y)}, \qquad (x,y) \in \Omega,$$

if k is merely nonnegative with bounded first and second derivatives on a domain Ω. To see this recall that Taylor's formula in one dimension yields

$$\begin{aligned} 0 &\leq k(y) = k(x) + k'(x)(y-x) + \frac{1}{2}k''(c)(y-x)^2 \\ &\leq k(x) + k'(x)(y-x) + \frac{1}{2}\|k''\|_\infty (y-x)^2 . \end{aligned}$$

We obtain $|k'(x)| \leq 2\|k''\|_\infty^{\frac{1}{2}} \sqrt{k(x)}$ upon choosing $y = x - \frac{k'(x)}{\|k''\|_\infty}$, but if $\frac{|k'(x)|}{\|k''\|_\infty}$ exceeds the distance d to the boundary (or if k is linear), then with $y = x - d\, sign(k'(x))$ we can only achieve

$$0 \leq k(x) - |k'(x)|\, d + \frac{1}{2}\|k''\|_\infty d^2 \leq k(x) - \frac{1}{2}|k'(x)|\, d,$$

which yields $|k'(x)| \leq 2\frac{k(x)}{d}$. Since k' is bounded, we may write $|k'(x)| \leq C\sqrt{\frac{k(x)}{d}}$ as claimed.

References

[1] A. D. Alexandrov, Dirichlet's problem for the equation $Det\|z_{ij}\| = \Phi(z_1, ...z_n, z, x_1, ...x_n)$, I, *Vestnik Leningrad Univ. Ser. Mat. Mekh. Astr.* **13** (1958), 5-24.

[2] E. Bedford and J. E. Fornaess, Counterexamples to regularity for the complex Monge-Ampère equation, *Inventiones Math.* **50** (1979), 129-134.

[3] L. Caffarelli and L. Cabré, *Fully Nonlinear Elliptic Equations,* Colloquium Publications, Volume 43, Amer. Math. Soc., 1995.

[4] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations, I. Monge-Ampère equations, *Comm. Pure Appl. Math.* **37** (1984), 369-402.

[5] S.-Y. Cheng and S.-T. Yau, On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial x_j) = F(x, u)$, *Comm. Pure Appl. Math.* **30** (1977), 41-68.

[6] M. Christ, Hypoellipticity in the infinitely degenerate regime, preprint on website.

[7] V. S. Fediĭ, On a criterion for hypoellipticity, *Math. USSR Sbornik* **14** (1971), 15-45.

[8] B. Franchi, Weighted Sobolev-Poincaré inequalities and pointwise estimates for a class of degenerate elliptic equations, *Trans. Amer. Math. Soc.* **327** (1991), 125-158.

[9] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, revised 3rd printing, 1998.

[10] P. Guan, Regularity of a class of quasilinear degenerate elliptic equations, *Advances in Mathematics* **132** (1997), 24-45.

[11] P. Guan, C^2 *a priori* estimates for degenerate Monge-Ampère equations, *Duke Math. J.* **86** (1997), 323-346.

[12] P. Guan, N. S. Trudinger and X.-J. Wang, On the Dirichlet problem for degenerate Monge-Ampère equations, *Acta Math.* **182** (1999), 87-104.

[13] E. Heinz, On elliptic Monge-Ampère equations and Weyl's embedding problem, *J. Analyse Math.* **7** (1959), 1-52.

[14] E. Heinz, Interior estimates for solutions of elliptic Monge-Ampère equations, *Proc. Sympos. Pure Math. Vol.* **4** (1960), 149-155.

[15] N. Ivochkina, Solution of the Dirichlet problem for curvature equations of order m, *Math. USSR Sb.* **67** (1990), 317-339.

[16] J. J. Kohn, Hypoellipticity of some degenerate subelliptic operators, *preprint* (1997), 1-13.

[17] O. A. Ladyzhenskaya and N. N. Ural'tseva. Schulz, *Linear and Quasilinear Elliptic Equations*, Moscow: Izdat. "Nauka" 1965 [Russian] English translation: New York: Academic Press 1968. 2nd Russian ed. 1973.

[18] H. Lewy, A priori limitations for solutions of Monge-Ampère equations I, II, *Trans. Amer. Math. Soc.* **37** (1935), 417-434.

[19] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.* **42** (1936), 689-692.

[20] J. Moser, On Harnack's theorem for elliptic differential equations, *Comm. Pure Appl. Math.* **14** (1961), 577-591.

[21] L. Nirenberg, The Weyl and Minkowski problems in differential geometry in the large, *Comm. Pure Appl. Math.* **6** (1953), 337-394.

[22] A. V. Pogorelov, The regularity of a convex surface with a given Gauss curvature, *Russian, Mat. Sbornik N. S.* **31** (1952), 88-103.

[23] C. Rios, E. Sawyer and R. L. Wheeden, A higher dimensional partial Legendre transform, and regularity of degenerate Monge-Ampère equations, *to appear in Advances in Math.*

[24] C. Rios, E. Sawyer and R. L. Wheeden, Regularity of degenerate quasilinear equations, *preprint.*

[25] E. Sawyer, A symbolic calculus for rough pseudodifferential operators, *Indiana U. Math. J.* **45** (1996), 289-332.

[26] E. Sawyer and R. L. Wheeden, *A priori* estimates for quasilinear equations related to the Monge-Ampère equation II, *preprint.*

[27] E. Sawyer and R. L. Wheeden, Regularity of Degenerate Monge-Ampère and Prescribed Gaussian Curvature Equations in Two Dimensions, *preprint available at http://www.math.mcmaster.ca/~sawyer.*

[28] F. Schulz, Regularity Theory for Quasilinear Elliptic Systems and Monge-Ampère Equations in Two Dimensions, *Lecture Notes in Math.* **1445** (1990), Springer-Verlag.

[29] N. Trudinger and J. Urbas, The Dirichlet problem for the equation of prescribed Gauss curvature, *Bull. Austral. Math. Soc.* **28** (1983), 217-231.

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