MATH 4Q03, Sample Test for Test 1

1.Consider the function

$$f(x) = x^3 - 2x^2$$

on the three-point grid at $x_1 = 0$, $x_2 = 2$ and $x_3 = 3$.
a) Compute the Vandermonde interpolating polynomial $p_2(x)$.
b) Compute the Lagrange interpolating polynomial $p_2(x)$.
c) Compute the Newton interpolating polynomial $p_2(x)$.
d) Add a missing point to the grid at $x_4 = 1$ and upgrade the Newton interpolating polynomial $p_2(x)$ into $p_3(x)$. Confirm that $p_3(x) = f(x)$. Why this result could have been found without any calculations?

2. Consider the data points

$$(x_1, y_1) = (0, 2), \quad (x_2, y_2) = (-1, 1), \quad (x_3, y_3) = (1, 1), \quad (x_4, y_4) = (2, 0).$$

a) Compute the interpolating polynomial $p(x)$.
b) Find the linear regression of this set of data points.
c) Find the polynomial $p_2 \in \mathcal{P}_2$ of better approximation of these data points, using the basis $\{1, x, x^2\}$.
d) What would be the result of question c) if we were looking for $p_3 \in \mathcal{P}_3$?
e) Construct a basis $\{\pi_0, \pi_1, \pi_2\}$ of $\mathcal{P}_2$ using the algorithm seen in class.
f) Find again the result of question c) using this new basis.

3. If f is a continous function on $[a, b]$, if p_n is its interpolation polynomial constructed using an equally spaced grid with $n + 1$ points, the interpolation error

$$E_n = \max_{a \leq x \leq b} |f(x) - p_n(x)|$$

can be estimated by

$$E_n \leq \frac{C_n M_{n+1} h^n}{(n+1)!}, \qquad (1)$$

where h is the grid step, $M_n = \max_{a \leq x \leq b} |f^{(n)}(x)|$, $C_n = \max_{0 \leq z \leq n} g_n(z)$ and $g_n(z) = \prod_{j=0}^{n} |z - j|$.
a) For $a = 0$, $b = 5$ and $f(x) = \cos(x)$, can the Runge phenomenon occur? Justify. Same question for $f(x) = e^x$, for $f(x) = e^{2x}$. (you can use that $C_n/(n+1)! \leq 1$)
b) For $a = 0$, $b = 5$ and $f(x) = \cos(2x)$, according to (1), for which values of n are we sure that $E_n \leq 10^{-3}$?
c) We assume $0 < a < b$. Find M_n if $f(x) = 1/x^2$.
d) Evaluate $g_n(z)$ at $z = 1/2$ and deduce that $C_n \geq (2n)!/(2^{2n+1} \cdot n!)$. Hint:

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{(2 \cdot 1)(2 \cdot 2)(2 \cdot 3) \cdots (2n)}$$

e) (if you are not familiar with equivalents, you can skip this question) Using the Stirling formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as } n \to \infty,$$

show that (for $f(x) = 1/x^2$ and $0 < a < b$), for n sufficiently large,

$$\frac{C_n M_{n+1} h^n}{(n+1)!} \geq \frac{n(b-a)^n}{a^{n+3}}.$$

f) If $a = 1$, $b = 3$ and $f(x) = 1/x^2$, does (1) prevent the Runge phenomenon to happen?