Question 1 (2 points)

Here is the table for an operation on a four element set G whose elements are: a, b, c, d.

$\cdot$	a	b	c	d
a	b	a	d	c
b	a	b	c	d
c	d	c	a	b
d	c	d	b	a

Is this the Cayley table for a group on the set {a,b,c,d}? Answer "Yes" or "No". Justify your answer to receive full credit.

Format

Yes. This is the Cayley table for a cyclic group of order 4 that has the element b as the identity element and the element a as a generator.

Question 2 (2 points)

Suppose that G is a group and N is a normal subgroup of G such that N is abelian and the factor group G/N is abelian. Must G be abelian? Answer "Yes" or "No".

To receive full credit, justify your answer by providing a proof if you think that the answer is "Yes", or by providing an example that shows that G need not be abelian if you think that the answer is "No".

Format

No. Let $G = S_3$, the group of symmetries on $\{1,2,3\}$. Let $N = \{id, (1,2,3), (1,3,2)\}$. Then N is an abelian group that is a normal subgroup of G and G/N has order 2, and so is abelian. But G is not abelian.

Question 3 (2 points)

Let Q be the group of rational numbers, with addition, and let Z be the group of integers, with addition. Is the factor group Q/Z isomorphic to Q? Answer "Yes" or "No". Justify your answer to receive full credit.

Format

No. The factor group Q/Z has elements of finite order, besides the identity element.
For example, the element $(1/2) + Z$ is not the identity element, and it has order 2,
since $(1/2) + Z + (1/2) + Z = 1 + Z = Z$.
The group Z does not have any element of order 2.

Question 4 (2 points)

Suppose that G is a group that only has a finite number of distinct subgroups. Must G be a finite group? Answer "Yes" or "No". Justify your answer to receive full credit.

Format

Yes. G cannot have an element of infinite order, or else it has an infinite number of subgroups. Each element a of G can be used to generate a finite subgroup <a>. But if there are only a finite number of subgroups, then there can only be a finite number of subgroups of the form <a>, which implies that G is finite.

Question 5 (5 points)

(a) Express the following permutation as a product of disjoint cycles. Place your answer in the first textbox below.

$$f = (1\ 2\ 3)(4\ 5)(1\ 6\ 7)(1\ 5).$$

(b) Express the following permutation as a product of transpositions. Place your answer in the second textbox below.

$$g = (1\ 4\ 3)(2\ 5\ 1)(6\ 7).$$

(c) Which of f and g are even permutations? Place your answer in the third textbox below.

Enter either "f", "g", "f, g", or "none" as your answer, depending on which of these transpositions are even. For example, if you think that just g is even, then enter "g".

Blank # 1 (1,4,5,6,7,2,3)

Blank # 2 (1,3)(1,4)(2,1)(2,5)(6,7)

Blank # 3 f

Question 6 (4 points)

(a) Find an element of the group S_5 that has order 6.

(b) Does this group have elements that have order greater than 6? Justify your answer.

(a) (1,2)(3,4,5)

(b) No. Such an element can be written as a product of disjoint cycles. But then it is either a cycle of length at most 5, or the product of a 2-cycle and a 3-cycle and so has order 6.

Question 7 (3 points)

Let R and S be rings and let f: R -> S be an **onto** ring homomorphism from R to S. Let I be the kernel of f.

(a) What theorem can be used to establish that the factor ring R/I is isomorphic to the ring S?

(b) Give the definition of a map g: R/I -> S that is an isomorphism from the factor ring R/I to the ring S. You do not need to justify your answer, just give the definition of a function g with this property.

Format

(a) The First Isomorphism Theorem for Rings.

(b) Define g:R/I -> S by g(r+I) = f(r).

Question 8 (2 points)

Consider the following polynomials from the polynomial ring

$$\mathbb{Z}_7[x]:$$

$$f(x) = -3x^4 + 2x^3 - x^2 + 4x + 5$$

$$\text{and}$$

$$g(x) = 3x^2 + 2.$$

There are polynomials q(x) and r(x) from this ring such that

$$f(x) = q(x)g(x) + r(x)$$

with the degree of r(x) equal to 0 or to 1, or with r(x) the zero polynomial. From the following list, select the unique polynomial r(x) with this property.

Remember that the coefficients are from the ring of integers, modulo 7, so your calculations and answers should take this into account. For example, in that ring, the elements -3 and 4 are equal, and, for instance, the product of the element 3 with the element 4 is the same as the element 5.

- ◯ 4x + 1
- ◯ 0
- ⓧ 5x + 2
- ◯ 3x
- ◯ 1
- ◯ 2x -3
- ◯ x-1

Question 9 (4 points)

Consider the following list of rings:

$$
\begin{array}{ll}
(a) & \mathbb{Z} \\
(b) & \mathbb{Q} \\
(c) & \mathbb{R}[x] \\
(d) & \mathbb{M}_2(\mathbb{R}) \text{ (the 2x2 matrices over } \mathbb{R}\text{)} \\
(e) & 3\mathbb{Z} \\
(f) & \mathbb{Z}_{11} \\
(g) & \mathbb{Z}_{12} \\
(h) & \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}
\end{array}
$$

In the first textbox, enter all of these rings that are **integral domains**. Use the letters (a) through (h) to refer to particular rings. For example, if you think that the ring of integers modulo 11 is an integral domain, include "(f)" as part of your answer.

(c) as well.

In the second textbox enter all of the rings from this list that are **fields**.

Blank # 1 (a), (b), (f), (h)

Blank # 2 (b), (f)

Question 10 (4 points)

For each of the following rings R and subsets X of R, determine if X is an **ideal** of R:

$$
\begin{aligned}
&(\#1) \quad R = \mathbb{Z}[x], \\
&\qquad\quad X = \{p(x) \in \mathbb{Z}[x] \mid p(x) = 0 \text{ or } \deg(p(x)) \leq 2\} \\
&(\#2) \quad R = \mathbb{Z}, X = 5\mathbb{Z} \\
&(\#3) \quad R = \mathbb{Q}, X = \mathbb{Z} \\
&(\#4) \quad R = \mathbb{R}[x], X = \{p(x) \in \mathbb{R}[x] \mid p(5) = 0\}
\end{aligned}
$$

Place your answer ("Yes" if X is an ideal, and "No" if it isn't) to part (#1) in the first textbox. Place your answers to parts (#2), (#3), and (#4) in the second, third, and fourth textboxes, respectively. You do not need to justify your answers.

Blank # 1 No

Blank # 2 Yes

Blank # 3 No

Blank # 4 Yes

Question 11 (3 points)

Let H and K be normal subgroups of the group G.

(a) Show that the map

$$f : G \to G/H \times G/K$$

defined by

$$f(g) = (gH, gK)$$

is a group homomorphism.

(b) Which elements of G lie in the kernel of f? Do not just give the definition of the kernel as your answer.

Format

(a) Let a, b belong to G. Then f(ab) = (abH, abK) = (aHbH, aKbK) = (aH, aK)(bH, bK) = f(a)f(b). So, f is a homomorphism.

(b) Let b lie in the kernel of f. Then f(b) = (bH,bK) = (H,K), the identity element of the product. So bH = H and bK = K. This implies that b is in H and in K.
Conversely every element in the intersection of H and K can be seen to be in the kernel. So the kernel is the intersection of H with K.

Question 12 (4 points)

Let p and q be prime numbers and let G be a group of order pq.

(a) What are the possible orders of elements in G (don't forget to consider the identity element)? Justify your answer.

(b) Is every proper non-trivial subgroup of G cyclic? Justify your answer.

Paragraph

(a) By Lagrange's theorem, the possibilities are 1, p, q, and pq.

(b) Yes. If H is a proper subgroup of G, then it has order p or q. Since they are prime numbers then H is cyclic. This was proved in class (or can be seen by observing that any non-identity element of H must generate all of H).

Question 13 (3 points)

Consider the group

$$G = \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5 .$$

(a) List all of the elements of the group G that have order 3? Place your answer in the first textbox.

(b) How many elements of order 5 does the group G have? Place your answer in the second textbox.

(c) Is the group G cyclic? Place your answer, "Yes" or "No", in the third textbox.

You do not need to provide justifications for your answers to this question.

Blank # 1 (1,0,0) (2,0,0)

Blank # 2 4 [(0,0,1), (0,0,2), (0,0,3), (0,0,4)]

Blank # 3 Yes (it is iso. to Z_60.

Submit Quiz

0 of 13 questions saved