

Duality, monotonicity and the Wigner–Yanase–Dyson metrics

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June 9, 2003

Abstract

We show that, for each value of $\alpha \in (-1, 1)$, the only Riemannian metrics on the space of positive definite matrices for which the $\nabla^{(\alpha)}$ and $\nabla^{(-\alpha)}$ connections are mutually dual are matrix multiples of the Wigner-Yanase-Dyson metric. If we further impose that the metric be monotone, then this set is reduced to scalar multiples of the Wigner-Yanase-Dyson metric.

1 Introduction

Classical information geometry addresses the differential geometric properties of families of classical probability densities. Quantum information geometry is its noncommutative counterpart, dealing with the geometric structure of families of quantum probabilities. The classical theory has been already explored and extended substantially, to the point of treating the geometric structures of the infinite dimensional Banach manifold of all probability measures equivalent to a given one [28, 9]. All the ingredients of the original Amari's theory [1, 2], such as the Fisher metric, the exponential, mixture and α -connections, have been defined for this general manifold, from which the finite dimensional results follow by restricting them to its finite dimensional submanifolds [10]. In comparison, the quantum version

*Research supported by the Natural Sciences and Engineering Research Council of Canada

still has “miles to go before sleep” [6], being so far mostly restricted to the geometry of density matrices on finite dimensional Hilbert spaces. It stands as a proof of the richness of the quantum domain that even this limited setup already offers many challenging problems, completely absent in the classical case.

A central theme in the passage from classical to quantum information geometry is the breakdown of Chentsov’s result [4] that the Fisher metric is the unique Riemannian metric (up to scalar multiples) on finite dimensional classical information manifolds which is reduced by all Markov morphisms. As proved by Petz [25], there are infinitely many Riemannian metrics on a matrix space with the property of being reduced by stochastic maps (the quantum analogue of Markov morphisms). Having characterized all these possible monotone metrics in terms of operator monotone functions, Petz’s result opened the way to two different trends: to deal with the whole set of monotone metrics at once and try to find yet other characterizations [22, 8, 26] or to find out which among them are more natural than the others according to properties beyond monotonicity [17, 35]. This paper is dedicated to the second of these trends. Its general attitude could be rephrase as: if monotonicity is not enough to single out one particular metric, what are the other conditions that should be further imposed in order to obtain a unique metric on the information manifolds of density matrices ? The answer we offer is based on the concept of duality for affine connections with respect to a given metric.

There are two flat connections that can be introduced on information manifolds in a fundamental way: the mixture connection, coming from the linear structure of the manifold itself (either as a subset of L^1 in the classical case or as a subset of the trace class operators in the quantum case), and the exponential connection, coming from the linear structure of their logarithms. The former, denoted by $\nabla^{(-1)}$ or $\nabla^{(m)}$, arises naturally when we consider mixed states (classical or quantum), whereas the latter, denoted by $\nabla^{(1)}$ or $\nabla^{(e)}$, is intimately related to the concepts of moment generating functionals and partition functions. For infinite dimensional classical manifolds, the exponential and mixture connections were defined in [9], making use of Orlicz spaces of exponential and $L \log L$ types (recently this approach has been improved in [11]). Of course the nonparametric definitions are designed in such a way that when restricted to finite dimensional submanifolds they reduce to the long standing definitions of the parametric theory [2]. For infinite dimensional quantum information manifolds, the exponential connection was obtained in [33, 32, 12], using the technique of small perturbations of forms and operators in Hilbert spaces, but the mixture connection poses a much

harder problem, which is to some extent still open [34]. Fortunately, the situation is straightforward as far as finite dimensional quantum systems are concerned. Many authors have proposed essentially equivalent definitions for the exponential and mixture connections on manifolds of density matrices [15, 24, 21]. We summarize our views on these definitions for $\nabla^{(1)}$ and $\nabla^{(-1)}$ in [13], where we observed that they are flat connections by explicitly constructing affine coordinate systems for each of them.

Two connections are said to be dual with respect to a metric if the combined action of their parallel transport is compatible with the metric (see section 3 below for the technical definition). The same pair of connections can be dual with respect to a multitude of metrics. It is then meaningful to ask, for a given pair of connections, what are the all the possible metrics that make them dual. When we looked at the mixture and the exponential connection on finite dimensional quantum systems, we found in [13] that the only metrics with this duality property are *matrix* multiples of the Bogoliubov-Kubo-Mori inner product. Using Petz's characterization, we then obtained the improved result that the only monotone metrics which make the ± 1 -connections dual are *scalar* multiples of the *BKM* metric. The purpose of the present paper is to investigate the same kind of question for the more general pairs of $\pm\alpha$ -connections.

In the classical version of Information Geometry, there are two equivalent ways of defining the α -connections $\nabla^{(\alpha)}$ on an information manifold \mathcal{M} , for $\alpha \in (0, 1)$. The first approach consists of using the α -embeddings of the form $p \mapsto \frac{2}{1-\alpha} p^{\frac{1-\alpha}{2}}$ to map \mathcal{M} into the sphere of radius r in the Banach space L^r , for $r = \frac{2}{1-\alpha}$. One then looks at the natural connection on L^r , that is, the one for which the parallel transport is just the identity map, and its canonical projection onto the sphere of radius r . The pullback of the latter (again using the α -embedding) is then defined to be the α -connection on \mathcal{M} . For finite dimensional manifolds, this can be traced back to the early works of Amari [1] and [4], where they are introduced without explicit mention of what the target spaces for the α -embeddings should be. For infinite dimensional information manifolds, one has to explicitly make use of the functional analytic properties of the spaces L^r (namely that they are uniformly convex spaces), in order to unequivocally define what is meant by the *canonical* projection onto a sphere. This was done in detail for the first time in [9] and in a slightly different fashion in [11]. In any event, one can prove that

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2} \nabla^{(1)} + \frac{1-\alpha}{2} \nabla^{(-1)}, \quad (1)$$

which can then be taken as an equivalent definition for $\nabla^{(\alpha)}$. Proposals for the quantum analogues of α -connections, for both finite and infinite dimensional manifolds, have appeared in number of papers [16, 21, 7]. They all use the α -embeddings in one way or another. We present them in section 2, where we review some of their most relevant properties. As it turns out, the α -embedding definitions are no longer equivalent to (1), that is, to the definition based on the convex mixture of the ± 1 -connections. We shall have more to say about this point later on in the paper.

As it is well known, the *BKM* metric is a limiting case of the more general family of Wigner-Yanase-Dyson metrics, denoted by g^α (more about this notation later). The *WYD* metrics made their first appearance in the context of quantum information geometry in the work of Hasegawa [14]. It was later proved that they are monotone for all values of $\alpha \in [-3, 3]$ [27]. In the spirit of the α -embeddings discussed above, for which the target spaces are L^r , with $r = \frac{2}{1-\alpha}$, we restrict our discussion to the range $\alpha \in (-1, 1)$, thus corresponding to $r \in (1, \infty)$. It is straightforward to prove that, for each fixed value of α in this range, the $\pm\alpha$ -connections are dual with respect to the metric g^α [15]. The formal limits $\alpha \rightarrow \pm 1$ lead to the *BKM* metric and the exponential and mixture connections, for which the duality is established separately [24].

Following the same technique of [13], we obtain the converse of this result. We find in section 3 that, for each fixed value of $\alpha \in (-1, 1)$, the only metrics for which $\nabla^{(\alpha)}$ and $\nabla^{(-\alpha)}$ are dual are *matrix* multiples of the *WYD* metric g^α . Using Petz's characterization, we obtain in section 4 that the only monotone metrics on positive definite matrices which make the $\pm\alpha$ -connections dual are *scalar* multiples of g^α .

2 The quantum α -connections

2.1 The α -representation

Following the notation in [13], let \mathcal{H}^N be a finite dimensional complex Hilbert space, $\mathcal{B}(\mathcal{H}^N)$ the algebra of operators on \mathcal{H}^N , \mathcal{A} its N^2 -dimensional real vector subspace of self-adjoint operators and \mathcal{M} the n -dimensional submanifold of all invertible density operators on \mathcal{H}^N , with $n = N^2 - 1$. For $\alpha \in (-1, 1)$, define the α -embedding of \mathcal{M} into \mathcal{A} as

$$\begin{aligned} \ell_\alpha &: \mathcal{M} \rightarrow \mathcal{A} \\ \rho &\mapsto \frac{2}{1-\alpha} \rho^{\frac{1-\alpha}{2}}. \end{aligned}$$

Since \mathcal{A} is itself a vector space, its tangent vectors consist of the partial derivatives of curves in \mathcal{A} . Therefore we can use the α -embedding to obtain an explicit representation of the tangent bundle of \mathcal{M} in terms of operators in \mathcal{A} , provided we can efficiently take partial derivatives of functions of operators in \mathcal{A} . The noncommutative nature of quantum manifolds makes a full appearance at this point, since the derivative of a matrix with respect to its parameters does not necessarily commute with the original matrix. As a result, tools such as the chain rule do not hold in matrix calculus. To overcome this difficulty, at least for functions of density matrices, we make use of the following decomposition. In the sequel, for $A \in \mathcal{B}(\mathcal{H}^N)$, let $\mathcal{C}(A) = \{B \in \mathcal{B}(\mathcal{H}^N) : [A, B] = 0\}$ denote its commutant.

Lemma 2.1 (Hasegawa, 1997) *Let $\mathcal{S} = \rho(\theta)$ be a smooth manifold of invertible density matrices. Then there exist a anti-selfadjoint operator Δ_i such that*

$$\frac{\partial \rho}{\partial \theta^i} = \frac{\partial^c \rho}{\partial \theta^i} + [\rho, \Delta_i], \quad \frac{\partial^c \rho}{\partial \theta^i} \in \mathcal{C}(\rho), \quad [\rho, \Delta_i] \in \mathcal{C}(\rho)^\perp, \quad (2)$$

the orthogonality being with respect to the Hilbert-Schmidt inner product in $\mathcal{B}(\mathcal{H}^N)$. Moreover, for any function F which is differentiable on a neighbourhood of the spectrum of ρ we have

$$\frac{\partial F(\rho)}{\partial \theta^i} = \frac{\partial^c F(\rho)}{\partial \theta^i} + [F(\rho), \Delta_i], \quad \frac{\partial^c F(\rho)}{\partial \theta^i} \in \mathcal{C}(\rho), \quad [F(\rho), \Delta_i] \in \mathcal{C}(\rho)^\perp. \quad (3)$$

At each point $\rho \in \mathcal{M}$, consider the subspace of \mathcal{A} defined by

$$\mathcal{A}_\rho^{(\alpha)} = \left\{ A \in \mathcal{A} : \text{Tr} \left(\rho^{\frac{1+\alpha}{2}} A \right) = 0 \right\}.$$

Using (3) with $F(\rho) = \ell_\alpha(\rho)$, we obtain

$$\frac{\partial \ell_\alpha(\rho)}{\partial \theta^i} = \rho^{\frac{1-\alpha}{2}} \frac{\partial^c \log \rho}{\partial \theta^i} + \frac{2}{1-\alpha} [\rho^{\frac{1-\alpha}{2}}, \Delta_i]. \quad (4)$$

Therefore, it follows from the normalization condition $\text{Tr} \rho = 1$ and the cyclicity of the trace that

$$\text{Tr} \left(\rho^{\frac{1+\alpha}{2}} \frac{\partial \ell_\alpha(\rho)}{\partial \theta^i} \right) = \text{Tr} \left(\frac{\partial^c \rho}{\partial \theta^i} + \frac{2}{1-\alpha} [\rho, \Delta_i] \right) = 0,$$

so that $\frac{\partial \ell_\alpha(\rho)}{\partial \theta^i} \in \mathcal{A}_\rho^{(\alpha)}$.

We can then define the isomorphism

$$\begin{aligned} (\ell_\alpha)_{*(\rho)} &: T_\rho \mathcal{M} \rightarrow \mathcal{A}_\rho^{(\alpha)} \\ v &\mapsto (\ell_\alpha \circ \gamma)'(0), \end{aligned} \tag{5}$$

where $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}$ is a curve in the equivalence class of the tangent vector v . We call this isomorphism the α -representation of the tangent space $T_\rho \mathcal{M}$. If $(\theta^1, \dots, \theta^n)$ is a coordinate system for \mathcal{M} , then the α -representation of the basis $\{\frac{\partial}{\partial \theta^1}, \dots, \frac{\partial}{\partial \theta^n}\}$ of $T_\rho \mathcal{M}$ is $\{\frac{\partial \ell_\alpha(\rho)}{\partial \theta^1}, \dots, \frac{\partial \ell_\alpha(\rho)}{\partial \theta^n}\}$. The α -representation of a vector field X on \mathcal{M} is therefore the \mathcal{A} -valued function $(X)^{(\alpha)}$ given by $(X)^{(\alpha)}(\rho) = (\ell_\alpha)_{*(\rho)} X_\rho$.

2.2 The covariant derivative $\nabla^{(\alpha)}$

The ± 1 -connections have a simple definition in terms of their parallel transports, essentially because the ± 1 -embeddings map \mathcal{M} into sets with an affine structure (the density operators themselves in the -1 -embedding and their logarithms in the 1 -embedding). Once their (flat) parallel transports are defined, it is then a simple matter to find the coefficients of their covariant derivatives, as well as to exhibit affine coordinate systems for them, as explained for instance in the second section of [13]. However, as noted in the introduction, the α -embeddings can be viewed as a map from \mathcal{M} into the positive orthant of the sphere of radius $r = \frac{2}{1-\alpha}$ in \mathcal{A} when we equip \mathcal{A} with the r -norm

$$\|A\|_r := (\text{Tr}|A|^r)^{1/r}.$$

Indeed, we can readily verify that, for any $\rho \in \mathcal{M}$,

$$\|\ell_\alpha(\rho)\|_r = \left(\text{Tr} \left| r \rho^{1/r} \right|^r \right)^{1/r} = r,$$

so that $\ell_\alpha(\rho) \in S^r$, the sphere of radius r in \mathcal{A} . More interestingly, it can be shown that the tangent space at a point $0 \leq \sigma \in S^r$ is

$$T_\sigma S^r = \{A \in \mathcal{A} : \text{Tr}(A\sigma^{r-1}) = 0\}$$

(see the second section of [7] for a quick review of the geometry of spheres in the more general context of uniformly convex Banach spaces). If we put $\sigma = \ell_\alpha(\rho) = r\rho^{1/r}$, we find that

$$T_{r\rho^{1/r}} S^r = \left\{ A \in \mathcal{A} : \text{Tr}(A\rho^{1-1/r}) = 0 \right\} = \mathcal{A}_\rho^{(\alpha)},$$

so that the α -representation (5) is indeed an isomorphism between tangent spaces, as the push-forward notation suggests.

The sphere S^r inherits a natural connection obtained by projecting the trivial connection on \mathcal{A} (the one where parallel transport is just the identity map) onto its tangent space at each point. For each $0 \leq \sigma \in S^r$, the canonical projection from the tangent space $T_\sigma \mathcal{A}$ onto the tangent space $T_\sigma S^r$ is uniquely given by [7]

$$\begin{aligned} \Pi_\sigma &: T_\sigma \mathcal{A} \rightarrow T_\sigma S^r \\ A &\mapsto A - (r^{-r} \text{Tr} [A\sigma^{r-1}]) \sigma. \end{aligned}$$

For $\sigma = \ell_\alpha(\rho) = r\rho^{1/r}$, this gives

$$\begin{aligned} \Pi_{r\rho^{1/r}} &: T_{r\rho^{1/r}} \mathcal{A} \rightarrow T_{r\rho^{1/r}} S^r \\ A &\mapsto A - \left(\text{Tr} \left[\rho^{\frac{1+\alpha}{2}} A \right] \right) \rho^{\frac{1-\alpha}{2}}. \end{aligned}$$

We can now define the covariant derivative of the α -connection. Starting with a differentiable vector field $s \in S(TM)$, we first push it forward under the α -embedding along a curve γ to obtain $(\ell_\alpha)_{*(\gamma(t))} s \in T\mathcal{A}$. We then take its covariant derivative with respect to the trivial connection on \mathcal{A} , denoted by $\tilde{\nabla}$, in the direction of $(\ell_\alpha)_{*(\rho)} v$, that is, the push-forward of a tangent vector $v \in T_\rho \mathcal{M}$. The result is a vector in $T_{r\rho^{1/r}} \mathcal{A}$, which we then project down to $T_{r\rho^{1/r}} S^r$ using the operator $\Pi_{r\rho^{1/r}}$ above. Finally, we pull it back to $T_\rho \mathcal{M}$ using $(\ell_\alpha)_{*(\rho)}^{-1}$ and call it the α -covariant derivative of the vector field s in the direction of the tangent vector v at the point $\rho \in \mathcal{M}$. The formula for all these operations reads like the following.

Definition 1 For $\alpha \in (-1, 1)$, let $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}$ be a smooth curve such that $\rho = \gamma(0)$ and $v = \dot{\gamma}(0)$ and let $s \in S(TM)$ be a differentiable vector field. The α -connection on $T\mathcal{M}$ is given by

$$\left(\nabla_v^{(\alpha)} s \right) (\rho) = (\ell_\alpha)_{*(\rho)}^{-1} \left[\Pi_{r\rho^{1/r}} \tilde{\nabla}_{(\ell_\alpha)_{*(\rho)} v} ((\ell_\alpha)_{*(\gamma(t))} s) \right]. \quad (6)$$

Using the definition (6), we find that the α -representation of the α -covariant derivative of the vector field $\partial/\partial\theta^j$ in the direction of the tangent vector $\partial_i := \partial/\partial\theta^i$ is

$$\left(\nabla_{\partial_i}^{(\alpha)} \frac{\partial}{\partial\theta^j} \right)^{(\alpha)} = \frac{\partial^2 \ell_\alpha(\rho)}{\partial\theta^i \partial\theta^j} - \text{Tr} \left(\rho^{\frac{1+\alpha}{2}} \frac{\partial^2 \ell_\alpha(\rho)}{\partial\theta^i \partial\theta^j} \right) \rho^{\frac{1-\alpha}{2}}. \quad (7)$$

2.3 The α -parallel transport and the extended manifold $\widehat{\mathcal{M}}$

The α -parallel transport of a tangent vector from tangent spaces at different points in \mathcal{M} is the pull-back of the parallel transport of its α -representation in \mathcal{A} . The latter, by its turn, consists of identity map followed by the canonical projection onto the TS^r at all points along a curve on S^r . It is obviously path dependent, and therefore no longer flat, like the ± 1 -parallel transports were. This is a consequence of the fact that among all L^p -spaces, for $1 \leq p \leq \infty$, only the spaces L^1 and L^∞ have spheres which are flat with respect to their trivial connections (recall the shape of the unit circles in \mathbb{R}^2 for all the different L^p -norms).

Now let us consider the extended manifold of faithful weights $\widehat{\mathcal{M}}$ (the positive definite matrices). Observe first that the α -embedding in this case maps $\widehat{\mathcal{M}}$ to itself. Moreover, for any $\sigma \in \widehat{\mathcal{M}}$, $T_\sigma \widehat{\mathcal{M}} = T_\sigma \mathcal{A} \simeq \mathcal{A}$, so that there is no need to do any projection in order to obtain the parallel transport on $\widehat{\mathcal{M}}$ induced by the α -embedding. We can therefore define the α -parallel transport on $\widehat{\mathcal{M}}$ simply by

$$\begin{aligned} \widehat{\tau}_{\sigma_0, \sigma_1}^{(\alpha)} &: T_{\sigma_0} \widehat{\mathcal{M}} \rightarrow T_{\sigma_1} \widehat{\mathcal{M}} \\ v &\mapsto (\ell_\alpha)_{*(\sigma_1)}^{-1} \left((\ell_\alpha)_{*(\sigma_0)} v \right), \end{aligned}$$

and we find (using (6) without the projection step) that the α -representation of its covariant derivative is

$$\left(\widehat{\nabla}_{\partial_i}^{(\alpha)} \frac{\partial}{\partial \theta^j} \right)^{(\alpha)} = \frac{\partial^2 \ell_\alpha(\rho)}{\partial \theta^i \partial \theta^j}, \quad (8)$$

where $\theta = \{\theta^1, \dots, \theta^{n+1}\}$ is any coordinate system for the extended manifold $\widehat{\mathcal{M}}$. Now let $\{X_1, \dots, X_{n+1}\}$ be a basis for \mathcal{A} . For each $\sigma \in \widehat{\mathcal{M}}$, we have that $\sigma^{\frac{1-\alpha}{2}} \in \mathcal{A}$, so that there exist real numbers $\xi = \{\xi^1, \dots, \xi^{n+1}\}$ such that

$$\frac{2}{1-\alpha} \sigma^{\frac{1-\alpha}{2}} = \xi^1 X_1 + \dots + \xi^{n+1} X_{n+1}.$$

Then $\xi = \{\xi^1, \dots, \xi^{n+1}\}$ is a $\widehat{\nabla}^{(\alpha)}$ -affine coordinate system for $\widehat{\mathcal{M}}$, since (8) gives

$$\left(\widehat{\nabla}_{\partial_i}^{(\alpha)} \frac{\partial}{\partial \xi^j} \right)^{(\alpha)} = \frac{\partial^2 \ell_\alpha(\rho)}{\partial \xi^i \partial \xi^j} = \frac{\partial X_j}{\partial \xi^i} = 0.$$

Therefore, $\widehat{\mathcal{M}}$ is $\widehat{\nabla}^{(\alpha)}$ -flat, even though its submanifold \mathcal{M} is not $\nabla^{(\alpha)}$ -flat. We note in passing that the connection $\nabla^{(\alpha)}$ on the submanifold \mathcal{M} is a restriction of the connection $\widehat{\nabla}^{(\alpha)}$, which acts on the larger manifold $\widehat{\mathcal{M}}$,

obtained without the use of any metric on \mathcal{M} , but rather using the canonical projection existing in \mathcal{A} , the target space for the α -embedding.

We finish this section with a couple of comparative remarks. Definition 1 is the verbatim analogue for finite dimensional quantum systems of the general definition for α -connections for infinite dimensional classical information manifolds [9, 11] and are, consequently, the quantum analogue of the original definition by Amari [1] and Chentsov [4] as well. Formulae (7) and (8) are special cases of those obtained by Jenčová using an embedding by a more general monotone function g , which include the α -embeddings (see respectively line 3, page 150 and line 10, page 149 of [20]). Finally, quantum α -connection in the spirit we present here had been hinted before by Hasegawa in [15, equation 35] and [16, equation 16], although in the less general form of Christoffel's symbols, which depend on a metric to be defined, as opposed to covariant derivatives and parallel transports, which are therefore more intrinsic. Infinite dimensional quantum α -connections were proposed in [7], making heavy use of the geometry of uniformly convex Banach spaces, of which the definitions given here are concrete finite dimensional realizations.

3 Duality and the *WYD* metrics

We recall some purely geometrical definitions of duality, which apply to any statistical manifold, classical or quantum: dual affine connections and dual coordinate systems.

Two connections ∇ and ∇^* on a Riemannian manifold (\mathcal{M}, g) are dual with respect to g if and only if

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z), \quad (9)$$

for any vector fields X, Y, Z on \mathcal{M} [1, 23]. Equivalently, if $\tau_{\gamma(t)}$ and $\tau_{\gamma(t)}^*$ are the respective parallel transports along a curve $\{\gamma(t)\}_{0 \leq t \leq 1}$ on \mathcal{M} , with $\gamma(0) = \rho$, then ∇ and ∇^* are dual with respect to g if and only if for all $t \in [0, 1]$,

$$g_\rho(Y, Z) = g_{\gamma(t)}(\tau_{\gamma(t)} Y, \tau_{\gamma(t)}^* Z). \quad (10)$$

Two coordinate systems $\theta = (\theta^i)$ and $\eta = (\eta_i)$ on a Riemannian manifold (\mathcal{M}, g) are dual with respect to g if and only if their natural bases for $T_\rho \mathcal{M}$ are *biorthogonal* at every point $\rho \in \mathcal{M}$, that is,

$$g\left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \eta_j}\right) = \delta_j^i.$$

Equivalently, $\theta = (\theta^i)$ and $\eta = (\eta_i)$ are dual with respect to g if and only if

$$g_{ij} = \frac{\partial \eta_i}{\partial \theta^j} \quad \text{and} \quad g^{ij} = \frac{\partial \theta^i}{\partial \eta_j}$$

at every point $\rho \in \mathcal{M}$, where, as usual, $g^{ij} = (g_{ij})^{-1}$.

The next two theorems establishes the role of potential functions as well as the relation between dual connections and dual coordinate systems for the case of flat manifolds. In the sense used in this paper, a connection ∇ on manifold \mathcal{M} is said to be flat if \mathcal{M} admits a global ∇ -affine coordinate system. This is equivalent to its curvature and torsion both being zero.

Theorem 3.1 (Amari, 1985) *When a Riemannian manifold (\mathcal{M}, g) has a pair of dual coordinate systems (θ, η) , there exist potential functions $\Psi(\theta)$ and $\Phi(\eta)$ such that*

$$g_{ij}(\theta) = \frac{\partial^2 \Psi(\theta)}{\partial \theta^i \partial \theta^j} \quad \text{and} \quad g^{ij} = \frac{\partial^2 \Phi(\eta)}{\partial \eta_i \partial \eta_j}.$$

Conversely, when either potential function Ψ or Φ exists from which the metric is derived by differentiating it twice, there exist a pair of dual coordinate systems. The dual coordinate systems and the potential functions are related by the following Legendre transforms

$$\theta^i = \frac{\partial \Phi(\eta)}{\partial \eta_i}, \quad \eta_i = \frac{\partial \Psi(\theta)}{\partial \theta^i}$$

and

$$\Psi(\theta) + \Phi(\eta) - \theta^i \eta_i = 0$$

Theorem 3.2 (Amari, 1985) *Suppose that ∇ and ∇^* are two flat connections on a manifold \mathcal{M} . If they are dual with respect to a Riemannian metric g on \mathcal{M} , then there exists a pair (θ, η) of dual coordinate systems such that θ is ∇ -affine and η is a ∇^* -affine.*

Let us now consider the definition of a Riemannian metric for our manifold \mathcal{M} of density matrices. Using the α -representation to obtain a concrete realization of tangent vectors on \mathcal{M} in terms of operators in \mathcal{A} , a Riemannian metric on \mathcal{M} is deemed to be provided by the smooth assignment of an inner product $\langle \cdot, \cdot \rangle_\rho$ in $\mathcal{A} \subset B(\mathcal{H}^N)$ for each point $\rho \in \mathcal{M}$.

For a fixed $\alpha \in (-1, 1)$, the *WYD* (Wigner-Yanase-Dyson) metric on \mathcal{M} is given by

$$g_\rho^{(\alpha)}(A, B) := \text{Tr} \left(A^{(\alpha)} B^{(-\alpha)} \right), \quad A, B \in T_\rho \mathcal{M}. \quad (11)$$

The symmetry properties of this definition are more apparent if one express it in a coordinate system $(\theta^1, \dots, \theta^n)$ for \mathcal{M} . By virtue of the decomposition lemma 2.1, we have that

$$\begin{aligned} g_{ij}^{(\alpha)}(\theta) &:= g_\rho^{(\alpha)} \left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j} \right) = \text{Tr} \left(\frac{\partial \ell_\alpha(\rho)}{\partial \theta^i} \frac{\partial \ell_{-\alpha}(\rho)}{\partial \theta^j} \right) \\ &= \text{Tr} \left(\rho \frac{\partial^c \log \rho}{\partial \theta^i} \frac{\partial^c \log \rho}{\partial \theta^j} \right) + \frac{4}{1-\alpha^2} \text{Tr} \left[\rho^{\frac{1-\alpha}{2}}, \Delta_i \right] \left[\rho^{\frac{1+\alpha}{2}}, \Delta_j \right]. \end{aligned} \quad (12)$$

It is then clear that $g_{ij}^{(\alpha)} = g_{ji}^{(\alpha)} = g_{ij}^{(-\alpha)}$. Observe also that for the extreme cases $\alpha \rightarrow \pm 1$, formula (11) leads to the familiar *BKM* (Bogoliubov-Kubo-Mori) metric

$$g_\rho^{(\pm 1)}(A, B) = g_\rho^B(A, B) = \text{Tr} \left(A^{(-1)} B^{(1)} \right) \quad (13)$$

where $A^{(\pm 1)}, B^{(\pm 1)}$ are the ± 1 -representations of the tangent vectors $A, B \in T_\rho \mathcal{M}$, as explained, for instance, in [13]. In coordinates, the *BKM* metric assumes the form

$$\begin{aligned} g_{ij}^B(\theta) &:= g_\rho^B \left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j} \right) = \text{Tr} \left(\frac{\partial \log \rho}{\partial \theta^i} \frac{\partial \rho}{\partial \theta^j} \right) \\ &= \text{Tr} \left(\rho \frac{\partial^c \log \rho}{\partial \theta^i} \frac{\partial^c \log \rho}{\partial \theta^j} \right) + \text{Tr}[\log \rho, \Delta_i][\rho, \Delta_j]. \end{aligned} \quad (14)$$

It follows directly from the definition (11), as has been observed in a number of papers [16, 21], that the $\pm\alpha$ -connections are dual with respect to the metric $g^{(\alpha)}$ for each fixed value of $\alpha \in (-1, 1)$ (just as the ± 1 -connections are dual with respect to the *BKM* metric). Our purpose is to discover what other metrics have the same property.

As suggested by the statement in theorem 3.2, most of the ingredients of Amari's theory, such as statistical divergences and the projection theorems [1, pp. 84-93], can only be *a priori* defined for flat manifolds. Only in a later stage, one consider what happens when they are applied to curved submanifolds of flat manifolds. Following this trend, we from now on confine our attention to those metrics on \mathcal{M} which are obtained as restrictions of

metrics on the extended manifold $\widehat{\mathcal{M}}$, which is $\widehat{\nabla}^{(\pm\alpha)}$ -flat, and treat the latter as our primary objects

Observe first that the WYD metric extends quite naturally to $\widehat{\mathcal{M}}$, simply using the $\pm\alpha$ -representations of tangent vectors \widehat{A}, \widehat{B} (that is, the representation induced by the $\pm\alpha$ -embedding of $\widehat{\mathcal{M}}$ into \mathcal{A}):

$$\widehat{g}_\sigma^{(\alpha)}(\widehat{A}, \widehat{B}) := \text{Tr}\left(\widehat{A}^{(\alpha)}\widehat{B}^{(-\alpha)}\right), \quad \widehat{A}, \widehat{B} \in T_\sigma\widehat{\mathcal{M}}. \quad (15)$$

It is also obvious that $\widehat{g}^{(\alpha)}$ has the same symmetry and duality properties of $g^{(\alpha)}$. We now show how $\widehat{g}^{(\alpha)}$ can be obtained from a potential function on $\widehat{\mathcal{M}}$.

Lemma 3.3 *If $(\theta^1, \dots, \theta^{n+1})$ is a $\widehat{\nabla}^{(\alpha)}$ -affine coordinate system for the extended manifold $\widehat{\mathcal{M}}$, then the function*

$$\widetilde{\Psi}_\alpha(\theta) = \frac{2}{1+\alpha} \text{Tr}\sigma(\theta), \quad \sigma(\theta) \in \widehat{\mathcal{M}} \quad (16)$$

satisfies

$$\widehat{g}_{ij}^{(\alpha)}(\theta) = \frac{\partial^2 \widetilde{\Psi}_\alpha(\theta)}{\partial \theta^i \partial \theta^j}. \quad (17)$$

Moreover,

$$\widetilde{\eta}_i = \frac{\partial \widetilde{\Psi}_\alpha(\theta)}{\partial \theta^i} \quad (18)$$

is a $\widehat{\nabla}^{(-\alpha)}$ -affine coordinate system for $\widehat{\mathcal{M}}$.

Proof: Since θ is $\widehat{\nabla}^{(\alpha)}$ -affine, there exist linearly independent operators $\{X_1, \dots, X_{n+1}\}$ such that

$$\ell_\alpha(\sigma) = \frac{2}{1-\alpha} \sigma^{\frac{1-\alpha}{2}} = \theta^1 X_1 + \dots + \theta^{n+1} X_{n+1}. \quad (19)$$

Since the point $\sigma \in \widehat{\mathcal{M}}$ is fixed in the course of this proof, we omit it from the notation and just write ℓ_α and $\ell_{-\alpha}$ for $\ell_\alpha(\sigma)$ and $\ell_{-\alpha}(\sigma)$, respectively. From lemma 2.1 we obtain that

$$X_i = \frac{\partial \ell_\alpha}{\partial \theta^i} = \frac{\partial^c \ell_\alpha}{\partial \theta^i} + [\ell_\alpha, \Delta_i], \quad (20)$$

that is

$$\frac{\partial^c \ell_\alpha}{\partial \theta^i} = X_i + [\Delta_i, \ell_\alpha]. \quad (21)$$

Also, since

$$\ell_{-\alpha} = \frac{2}{1+\alpha} \sigma^{\frac{1+\alpha}{2}} = \left(\frac{2}{1+\alpha} \right) \left(\frac{1-\alpha}{2} \right)^{\frac{1+\alpha}{1-\alpha}} \ell_{\alpha}^{\frac{1+\alpha}{1-\alpha}}$$

we have that

$$\frac{\partial^c \ell_{-\alpha}}{\partial \theta^j} = \left(\frac{1-\alpha}{2} \right)^{\frac{2\alpha}{1-\alpha}} \ell_{\alpha}^{\frac{2\alpha}{1-\alpha}} \frac{\partial^c \ell_{\alpha}}{\partial \theta^j}. \quad (22)$$

Now observe that

$$\begin{aligned} \frac{\partial^2 \tilde{\Psi}_{\alpha}(\theta)}{\partial \theta^i \partial \theta^j} &= \frac{\partial^2}{\partial \theta^i \partial \theta^j} \left(\frac{2}{1+\alpha} \text{Tr} \sigma \right) = \frac{2}{1+\alpha} \text{Tr} \left(\frac{\partial^2 \sigma}{\partial \theta^i \partial \theta^j} \right) \\ &= \frac{2}{1+\alpha} \text{Tr} \left(\frac{\partial^2 \sigma^{\frac{1-\alpha}{2}} \sigma^{\frac{1+\alpha}{2}}}{\partial \theta^i \partial \theta^j} \right) = \frac{1-\alpha}{2} \text{Tr} \left(\frac{\partial^2 \ell_{\alpha} \ell_{-\alpha}}{\partial \theta^i \partial \theta^j} \right) \\ &= \frac{1-\alpha}{2} \frac{\partial}{\partial \theta^i} \text{Tr} \left(X_j \ell_{-\alpha} + \ell_{\alpha} \frac{\partial \ell_{-\alpha}}{\partial \theta^j} \right) \\ &= \frac{1-\alpha}{2} \frac{\partial}{\partial \theta^i} \text{Tr} \left[X_j \ell_{-\alpha} + \ell_{\alpha} \left(\frac{\partial^c \ell_{-\alpha}}{\partial \theta^j} + [\ell_{-\alpha}, \Delta_j] \right) \right]. \quad (23) \end{aligned}$$

Now using (22) and taking into account that the trace of the term involving the commutator vanishes, we obtain that

$$\begin{aligned} \frac{\partial^2 \tilde{\Psi}_{\alpha}(\theta)}{\partial \theta^i \partial \theta^j} &= \frac{1-\alpha}{2} \frac{\partial}{\partial \theta^i} \text{Tr} \left[X_j \ell_{-\alpha} + \ell_{\alpha} \left(\frac{1-\alpha}{2} \right)^{\frac{2\alpha}{1-\alpha}} \ell_{\alpha}^{\frac{2\alpha}{1-\alpha}} \frac{\partial^c \ell_{\alpha}}{\partial \theta^j} \right] \\ &= \frac{1-\alpha}{2} \frac{\partial}{\partial \theta^i} \text{Tr} \left[X_j \ell_{-\alpha} + \left(\frac{1-\alpha}{2} \right)^{\frac{2\alpha}{1-\alpha}} \ell_{\alpha}^{\frac{1+\alpha}{1-\alpha}} (X_j + [\Delta_j, \ell_{\alpha}]) \right] \\ &= \frac{1-\alpha}{2} \frac{\partial}{\partial \theta^i} \text{Tr} \left[X_j \ell_{-\alpha} + \left(\frac{1-\alpha}{2} \right)^{\frac{2\alpha}{1-\alpha}} \ell_{\alpha}^{\frac{1+\alpha}{1-\alpha}} X_j \right] \\ &= \frac{1-\alpha}{2} \frac{\partial}{\partial \theta^i} \text{Tr} \left[X_j \ell_{-\alpha} + \left(\frac{1+\alpha}{2} \right) \left(\frac{2}{1-\alpha} \right) \ell_{-\alpha} X_j \right] \\ &= \text{Tr} \left[X_j \frac{\partial \ell_{-\alpha}}{\partial \theta^i} \right]. \quad (24) \end{aligned}$$

But by definition, the *WYD* in this $\widehat{\nabla}^{\alpha}$ -affine coordinate system assumes the form

$$g_{ij}^{(\alpha)}(\theta) = \text{Tr} \left(X_j \frac{\partial \ell_{-\alpha}}{\partial \theta^i} \right) \quad (25)$$

which proves the first assertion of the lemma. For the second part of the lemma, we have seen in the previous section that there exists a $\widehat{\nabla}^{-\alpha}$ -affine coordinate system $\xi = \{\xi_1, \dots, \xi_{n+1}\}$ in terms of which we can write

$$\ell_{-\alpha} = \xi_1 Y^1 + \dots + \xi_{n+1} Y^{n+1},$$

for some other set of linearly independent operators $\{Y^1, \dots, Y^{n+1}\}$. Now following the same reasoning that led to (23) we obtain that

$$\begin{aligned} \frac{\partial \widetilde{\Psi}_\alpha(\theta)}{\partial \theta^i} &= \frac{1-\alpha}{2} \text{Tr} \left(\frac{\partial \ell_\alpha \ell_{-\alpha}}{\partial \theta^i} \right) \\ &= \frac{1-\alpha}{2} \text{Tr} \left(X_i \ell_\alpha + \ell_\alpha \frac{\partial \ell_{-\alpha}}{\partial \theta^i} \right) \\ &= \frac{1-\alpha}{2} \text{Tr} \left[\left(1 + \frac{1+\alpha}{1-\alpha} \right) X_i \ell_{-\alpha} \right] \\ &= \text{Tr} [X_i (\xi_1 Y^1 + \dots + \xi_{n+1} Y^{n+1})] \\ &= \xi_1 \text{Tr} (X_i Y^1) + \dots + \xi_{n+1} \text{Tr} (X_i Y^{n+1}) \quad (26) \\ &= \sum_{j=1}^{n+1} \text{Tr} (X_i Y^j) \xi_j. \quad (27) \end{aligned}$$

This means that the coordinate system $(\widetilde{\eta})$ is affinely related to (ξ) and therefore it is itself $\widehat{\nabla}^{-\alpha}$ -affine.

We end this section with the next theorem, which is the extension for a general α -connections of the result proved in [13] for the case $\alpha = \pm 1$.

Theorem 3.4 *For a fixed value of $\alpha \in (-1, 1)$, suppose that the connections $\nabla^{(\alpha)}$ and $\nabla^{(-\alpha)}$ are dual with respect to a Riemannian metric \widehat{g} on $\widehat{\mathcal{M}}$. Then there exist a constant (independent of σ) $(n+1) \times (n+1)$ matrix M , such that $(\widehat{g}_\sigma)_{ij} = \sum_{k=1}^{n+1} M_i^k (\widehat{g}_\sigma^{(\alpha)})_{kj}$, in some α -affine coordinate system.*

Proof: Since the two connections are flat on the extended manifold $\widehat{\mathcal{M}}$, theorem 3.2 tell us that there exist dual coordinate systems (θ, η) such that θ is $\nabla^{(\alpha)}$ -affine and η is $\nabla^{(-\alpha)}$ -affine. Using lemma 3.3, we know that the function $\widetilde{\Psi}_\alpha(\theta) = \frac{2}{1+\alpha} \text{Tr} \sigma(\theta)$ satisfies

$$\widehat{g}_{ij}^{(\alpha)}(\theta) = \frac{\partial^2 \widetilde{\Psi}_\alpha(\theta)}{\partial \theta^i \partial \theta^j} \quad (28)$$

and also that

$$\tilde{\eta}_i = \frac{\partial \tilde{\Psi}_\alpha(\theta)}{\partial \theta^i} \quad (29)$$

is a another $\widehat{\nabla}^{(-\alpha)}$ -affine coordinate system for $\widehat{\mathcal{M}}$. Therefore, the coordinate systems (η) and $(\tilde{\eta})$ are related by an affine transformation, so there must exist a matrix M and numbers (a_1, \dots, a_{n+1}) such that

$$\eta_i = \sum_{k=1}^{n+1} M_i^k \tilde{\eta}_k + a_i. \quad (30)$$

But from theorem 3.1, there exists a potential function $\Psi(\theta)$ such that

$$\widehat{g}_{ij}(\theta) = \frac{\partial^2 \Psi(\theta)}{\partial \theta^i \partial \theta^j}$$

and

$$\eta_i = \frac{\partial \Psi(\theta)}{\partial \theta^i}.$$

Equation (30) then gives

$$\frac{\partial \Psi(\theta)}{\partial \theta^i} = \sum_{k=1}^{n+1} M_i^k \frac{\partial \tilde{\Psi}_\alpha(\theta)}{\partial \theta^k} + a_i,$$

and differentiating this equation with respect to θ^j leads to

$$\widehat{g}_{ij}(\theta) = \frac{\partial^2 \Psi(\theta)}{\partial \theta^i \partial \theta^j} = \sum_{k=1}^{n+1} M_i^k \frac{\partial^2 \tilde{\Psi}_\alpha(\theta)}{\partial \theta^j \partial \theta^k} = \sum_{k=1}^{n+1} M_i^k \widehat{g}_{kj}^{(\alpha)}(\theta). \quad (31)$$

4 The condition of monotonicity

We have seen in the previous section that requiring duality between the $\nabla^{(\alpha)}$ and $\nabla^{(-\alpha)}$ connections reduces the set of possible Riemannian metrics on $\widehat{\mathcal{M}}$ to matrix multiples of the *WYD* metric. Following [13], we now investigate the effect of imposing a monotonicity property on this set.

Recall that the -1 -representation is the limiting case $\alpha = -1$ of the α -representations defined in section 2.1. If we use it to define to define a Riemannian metric \widehat{g} on $\widehat{\mathcal{M}}$ by means of the inner product $\langle \cdot, \cdot \rangle_\rho$ in $\mathcal{A} \subset B(\mathcal{H}^N)$, then we say that \widehat{g} is *monotone* if and only if

$$\left\langle S(A^{(-1)}), S(A^{(-1)}) \right\rangle_{S(\rho)} \leq \left\langle A^{(-1)}, A^{(-1)} \right\rangle_\rho, \quad (32)$$

for every $\rho \in \mathcal{M}$, $A \in T_\rho \mathcal{M}$, and every completely positive, trace preserving map $S : \mathcal{A} \rightarrow \mathcal{A}$.

For any metric \widehat{g} on $T\widehat{\mathcal{M}}$, define the positive (super) operator K_σ on \mathcal{A} by

$$\widehat{g}_\sigma(\widehat{A}, \widehat{B}) = \left\langle \widehat{A}^{(-1)}, K_\sigma \left(\widehat{B}^{(-1)} \right) \right\rangle_{HS} = \text{Tr} \left(\widehat{A}^{(-1)} K_\sigma \left(\widehat{B}^{(-1)} \right) \right). \quad (33)$$

Note that our K is denoted K^{-1} by Petz in [25]. Define also the (super) operators, $L_\sigma X := \sigma X$ and $R_\sigma X := X\sigma$, for $X \in \mathcal{A}$, which are also positive. The aforementioned characterization of monotone metrics obtained by Petz is the content of the following theorem.

Theorem 4.1 (Petz 96) *A Riemannian metric g on \mathcal{A} is monotone if and only if*

$$K_\sigma = \left(R_\sigma^{1/2} f(L_\sigma R_\sigma^{-1}) R_\sigma^{1/2} \right)^{-1},$$

where K_σ is defined in (33) and $f : R^+ \rightarrow R^+$ is an operator monotone function satisfying $f(t) = tf(t^{-1})$.

In particular, the WYD metric is monotone and its corresponding operator monotone function is

$$f_p(x) = \frac{p(1-p)(x-1)^2}{(x^p-1)(x^{1-p}-1)}, \quad (34)$$

for $p = \frac{1+\alpha}{2}$ [27].

Combining this characterization with our theorem (3.4), we obtain the following improved uniqueness result.

Theorem 4.2 *If the connections $\nabla^{(\alpha)}$ and $\nabla^{(-\alpha)}$ are dual with respect to a monotone Riemannian metric \widehat{g} on $\widehat{\mathcal{M}}$, then \widehat{g} is a scalar multiple of the WYD metric.*

Proof: Let $\theta = (\theta^1, \dots, \theta^n)$ be the $\nabla^{(\alpha)}$ -affine coordinate system of theorem 3.4. Given $\sigma \in \widehat{\mathcal{M}}$, we have that $T_\sigma \mathcal{M} \simeq \mathcal{A}$. In particular, $\left\{ \frac{\partial \sigma}{\partial \theta^1}, \dots, \frac{\partial \sigma}{\partial \theta^n} \right\}$ is the basis for \mathcal{A} obtained as the -1 -representation of $\left\{ \frac{\partial}{\partial \theta^1}, \dots, \frac{\partial}{\partial \theta^n} \right\}$. Now let K^g and $K^{(\alpha)}$ be the kernels of \widehat{g} and $\widehat{g}^{(\alpha)}$, respectively. Then it follows

from theorem 3.4 that

$$\begin{aligned}
\left\langle \frac{\partial \sigma}{\partial \theta^i}, K_\sigma^g \left(\frac{\partial \sigma}{\partial \theta^j} \right) \right\rangle_{HS} &= \widehat{g}_\sigma \left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j} \right) = (\widehat{g}_\sigma)_{ij} \\
&= \sum_{k=1}^{n+1} M_i^k (\widehat{g}_\sigma^{(\alpha)})_{kj} \\
&= \sum_{k=1}^{n+1} M_i^k \widehat{g}_\sigma^{(\alpha)} \left(\frac{\partial}{\partial \theta^k}, \frac{\partial}{\partial \theta^j} \right) \\
&= \sum_{k=1}^{n+1} M_i^k \left\langle \frac{\partial \sigma}{\partial \theta^k}, K_\sigma^{(\alpha)} \left(\frac{\partial \sigma}{\partial \theta^j} \right) \right\rangle_{HS}. \quad (35)
\end{aligned}$$

Thus, as operators on \mathcal{A} , the kernels K^g and $K^{(\alpha)}$ are related by

$$K_\sigma^g = M K_\sigma^{(\alpha)}. \quad (36)$$

Therefore, if f^g and $f^{(\alpha)}$ are the operator monotone functions corresponding respectively to g and $g^{(\alpha)}$, from theorem 4.1, we have

$$\begin{aligned}
\left(R_\sigma^{1/2} f^g (L_\sigma R_\sigma^{-1}) R_\sigma^{1/2} \right)^{-1} &= M \left(R_\sigma^{1/2} f^{(\alpha)} (L_\sigma R_\sigma^{-1}) R_\sigma^{1/2} \right)^{-1} \\
\left(R_\sigma^{1/2} f^g (L_\sigma R_\sigma^{-1}) R_\sigma^{1/2} \right) M &= \left(R_\sigma^{1/2} f^{(\alpha)} (L_\sigma R_\sigma^{-1}) R_\sigma^{1/2} \right) \\
M &= f^g (L_\sigma R_\sigma^{-1})^{-1} f^{(\alpha)} (L_\sigma R_\sigma^{-1}),
\end{aligned}$$

as everything commutes. Thus, the operator M is given as a function of the operator $L_\sigma R_\sigma^{-1}$, but it is itself independent of the point σ , so we conclude that it must be a scalar multiple of the identity operator.

5 Discussion

With the result of this paper, we have completed the programme initiated in [13] of characterizing the *BKM* and the *WYD* metrics in terms of the combining requirement of monotonicity and duality. The monotonicity condition has an appealing motivation coming from estimation theory. If we interpret the geodesic distance between two density matrices as a measure of their statistical distinguishability, then (32) tells us that they will become less distinguishable if we introduce randomness into the system under consideration. In other words, their distance decreases under coarse-graining.

As it is, estimation theory is more basic than physics itself, since it does not assume any particular underlying physical process, being just a

tool to help analyze statistical data. Nevertheless, the interpretation above carries over to statistical mechanical systems as well, where stochastic (i.e. completely positive, trace-preserving) maps appear as a mathematical implementation of the time evolution of a system whose states are described by density matrices [29]. In this case, monotonicity means that the distance between different states decreases under the same time evolution. If it decreases asymptotically to zero for any two points in a certain set of ‘initial’ states, then we are in the presence of a fixed point for the dynamics, or in other words, an equilibrium state. From all this, it seems that imposing a monotonicity condition on the possible Riemannian metrics on a statistical manifold is not at all an artificial technicality.

Our motivation behind Amari’s duality is less general and ultimately rests upon quantum statistical mechanics alone [31, 30]. Recall that the von Neumann entropy for a state $\rho \in \mathcal{M}$ is defined as [36]

$$S(\rho) := -\text{Tr}(\rho \log \rho) \quad (37)$$

and that the relative (Kullback-Leibler) entropy of the state ρ given the state σ is

$$S(\rho|\sigma) = \text{Tr}[\rho(\log \rho - \log \sigma)] \quad (38)$$

Now let us choose a set of $m \leq n$ observables Y_1, \dots, Y_m such that the set $\{\mathbf{1}, Y_1, \dots, Y_m\}$ is a basis for \mathcal{A} . Among all possible observables in \mathcal{A} , these ones represent the *slow variables* of the theory, that is, those whose means we can measure at any given time. Then it is an easy exercise, using the Lagrange multipliers technique, to show that the states which maximize the von Neumann entropy subject to keeping the means of all $\{Y_i\}$, $i = 1, \dots, m$, constant are the Gibbs states of the form

$$\rho = \exp(\theta^1 Y_1 + \dots + \theta^m Y_m - \Psi \mathbf{1}), \quad (39)$$

where $\Psi(\theta)$ is determined by the normalization condition $\text{Tr} \rho = 1$. For example, if $Y_1 = H$ is the energy operator, then we obtain the so called canonical ensemble, whereas if we have $Y_1 = H, Y_2 = N$ where N is the number of particles, we get the grand canonical ensemble. We immediately recognize these states as constituting a $\nabla^{(1)}$ -flat, m -dimensional, submanifold $\mathcal{S}_m \subset \mathcal{S}$, which is determined by our choice of Y_1, \dots, Y_m , that is, by our choice of the level of description adopted.

Inasmuch as entropy is negative information, the principle of *maximum entropy*, advocated in information theory and statistical physics by Jaynes [18, 19], tells us that, if the only information available about the system

under consideration are the means of the random variables Y_1, \dots, Y_m , then we should take as the state of the system the element in \mathcal{S}_m with these means. The replacement of the *true* state $\rho \in \mathcal{S}$ by the one in \mathcal{S}_m with the same means for Y_1, \dots, Y_m is a reflection of our ignorance of what really goes on with the system. It is the least biased choice of state given the information available.

The point of view in statistical dynamics [29] is somewhat different, in the sense that it regards the same *replacement* as part of the true dynamics of the system. For instance, the heat transfer in a local region of a fluid happens 10^8 times faster than most chemical reactions [5], so we can choose to regard the concentrations of the chemicals reacting as the slow variables while all other observables are thermalized (maximum entropy) along each time step in the dynamics. The skill of the scientist using statistical dynamics thus resides in correctly identifying which are the slow variables of the problem at hand and then following the time evolution of the system, which involves, apart from a stochastic dynamics particular to each problem, successive *projections* onto \mathcal{S}_m .

Information geometry provides a mathematical meaning for this projection [3, 31]. It is well known that the relative entropy (38) is the statistical divergence associated with the dualistic triple $(g^B, \nabla^{(1)}, \nabla^{(-1)})$ [24]. It then follows from the general theory [2] that, given an arbitrary point $\rho \in \mathcal{S}$, the point in \mathcal{S}_m (which is $\nabla^{(1)}$ -flat) that minimizes $S(\rho|\sigma)$ is obtained uniquely by following a -1 -geodesic from ρ that intercepts \mathcal{S} orthogonally with respect to the *BKM* metric g^B . This is equivalent to the projection described above (maximum entropy subject to constant means) precisely because a path preserving the mean parameters (or mixture coordinates) is a -1 -geodesic, that is, a straight line for the mixture connection.

However, if g is a general monotone metric, with respect to which $\nabla^{(1)}$ and $\nabla^{(-1)}$ are not necessarily dual, then the relative entropy might fail to be a divergence for $(g, \nabla^{(1)}, \nabla^{(-1)})$ and nothing guarantees that minimizing $S(\rho|\sigma)$ will produce a point in \mathcal{S}_m connected to ρ by a -1 -geodesic intersecting \mathcal{S}_m perpendicularly with respect to g . Information geometry no longer provides a mathematical implementation for statistical dynamics anymore.

As a final word for this paper, let us mention that a corollary to theorem 4.2 is the fact that the relation (1) *does not* hold for the quantum α -connections defined using the α -representations as in section 2. If it did, a simple calculation shows that $\nabla^{(\alpha)}$ and $\nabla^{(-\alpha)}$ would then be dual with respect to the *BKM* metric (since the ± 1 -connections are). But from theorem 4.2, this would imply that the *BKM* is a scalar multiple of the *WYD*, which is only true in the extreme cases $\alpha = \pm 1$.

Acknowledgements: The result of this paper was first announced during the conference Information Geometry and its Applications, held in Pescara, July 2002. I would like to thank the local organizers, especially P. Gibilisco, for their display of Italian hospitality. I also thank A. Jenčová, H. Hasegawa, R.F. Streater and the anonymous referee for their comments and suggestions.

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