Malliavin Calculus

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1 Introduction

2 Malliavin Calculus

2.1 The Derivative Operator

Consider a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)$ generated by a one-dimensional Brownian motion W_t and let $L^2(\Omega)$ be the space of square integrable random variables.

Let $L^2[0,T]$ denote the Hilbert space of deterministic square integrable functions $h : [0,T] \to \mathbb{R}$. For each $h \in L^2[0,T]$, define the random variable

$$W(h) = \int_0^T h(t) dW_t$$

using the usual Ito integral with respect to a Brownian motion. Observe that this is a Gaussian random variable with $E[W(h)] = 0$ and that, due to the Ito isometry,

$$E[W(h)W(g)] = \int_0^T h(t)g(t)dt =< h, g >_{L^2[0,T]},$$

for all $h, g \in L^2[0,T]$.

The closed subspace $\mathcal{H}_1 \subset L^2(\Omega)$ of such random variables is then isometric to $L^2[0,T]$ and is called the space of zero-mean Gaussian random variables.

Now let $C_p^\infty(\mathbb{R}^n)$ be the set of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ with partial derivatives of polynomial growth. Denote by $\mathcal{S}$ the class of random variables of the form

$$F = f(W(h_1), ..., W(h_n))$$

where $f \in C_p^\infty$ and $h_1, ..., h_n \in L^2([0,T])$. Note that $\mathcal{S}$ is then a dense subspace of $L^2(\Omega)$.

Definition 2.1 *: For $F \in \mathcal{S}$, we define the stochastic process*

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n))h_i(t)$$

One can then show that $DF \in L^2(\Omega \times T)$.

So for we have obtained a linear operator $D : S \subset L^2(\Omega) \to L^2(\Omega \times T)$. We can now extend the domain of D by considering the norm

$$||F||_{1,2} = ||F||_{L^2(\Omega)} + ||D_t F||_{L^2(\Omega \times T)}$$

We then define $D^{1,2}$ as the closure of $\mathcal{S}$ in the norm $|| \cdot ||_{1,2}$. Then

$$D : D^{1,2} \subset L^2(\Omega) \to L^2(\Omega \times T)$$

is a closed unbounded operator with a dense domain $D^{1,2}$.

Properties:

1. Linearity: $D_t(aF + G) = aD_t F + D_t G, \quad \forall F, G \in D^{1,2}$
2. Chain Rule: $D_t(f(F)) = \sum f_i(F) D_t F, \quad \forall f \in C_p^\infty(\mathbb{R}), F_i \in D^{1,2}$
3. Product Rule:$D_t(FG) = F(D_t G) + G(D_t F), \quad \forall F, G \in D^{1,2}$ s.t. F and $||DF||_{L^2(\Omega \times T)}$ are bounded.

Simple Examples:

1. $D_t(\int_0^T h(t) dW_t) = h(t)$
2. $D_t W_s = 1_{\{t \leq s\}}$

3. $D_t f(W_s) = f^{'}(W_s) 1_{\{t \leq s\}}$

Exercises: Try Oksendal (4.1) (a), (b), (c), (d).

Fréchet Derivative Interpretation

Recall that the sample space Ω can be identified with the space of continuous function $C([0,T])$. Then we can look at the subspace

$$\mathbb{H}^1 = \{\gamma \in \Omega : \gamma = \int_0^t h(s)ds, h \in L^2[0,T]\},$$

that is, the space of continuous functions with square integrable derivatives. This space is isomorphic to $L^2[0,T]$ and is called the Cameron-Martin space. Then we can prove that $\int_0^T (D_t) F h(t) dt$ is the Fréchet derivative of F in the direction of the curve $\gamma = \int_0^t h(s)ds$, that is

$$\int_0^T D_t F h(t) dt = \frac{d}{d\varepsilon} F(w + \varepsilon \gamma)|_{\varepsilon=0}$$

2.2 The Skorohod Integral

Riemann Sums

Recall that for elementary adapted process

$$u_t = \sum_{i=0}^n F_i 1_{(t_i, t_{i+1}]}(t), \qquad F_i \in \mathcal{F}_{t_i}$$

the Ito integral is initially defined as

$$\int_0^T u_t dW_t = \sum_{i=0}^n F_i (W_{t_{i+1}} - W_{t_i})$$

Since ε is dense in $L_0^2(\Omega \times T)$, we obtain the usual Ito integral in $L_0^2(\Omega \times T)$ by taking limits.

Alternatively, instead of approximating a general process $u_t \in L^2(\Omega \times T)$ by a step process of the form above, we could approximate it by the step process

$$\hat{u}(t) = \sum_{i=0}^n \frac{1}{t_{i+1} - t_i} \left((\int_{t_i}^{t_{i+1}} E[u_s | \mathcal{F}_{[t_i, t_{i+1}]^c}] ds \right) 1_{(t_i, t_{i+1}]}(t)$$

and consider the Riemann sum

$$\sum_{i=0}^{n} \frac{1}{t_{i+1}-t_i} \Big(\int_{t_i}^{t_{i+1}} E[u_s|\mathcal{F}_{[t_i,t_{i+1}]^c}]ds \Big))(W_{t_{i+1}} - W_{t_i}).$$

If these sums converge to a limit in $L^2(\Omega)$ as the size of the partition goes to zero, then the limit is called the Skorohod integral of u and is denoted by $\delta(u)$. This is clearly not a friendly definition!

Skorohod Integral via Malliavin Calculus

An alternative definition for the Skorohod integral of a process $u \in L^2[T \times \Omega]$ is as the adjoint to the Malliavin derivate operator.

Define the operator $\delta : Dom(\delta) \subset L^2[T \times \Omega] \to L^2(\Omega)$ with domain

$$Dom(\delta) = \{u \in L^2[T \times \Omega] : |E(\int_0^T D_t F u_t dt)| \leq c(u)||F||_{L^2(\Omega)}, \forall F \in D^{1,2}\}$$

characterized by

$$E[F\delta(u)] = E[int_0^T D_t F u_t dt]$$

that is,

$$< F, \delta(u) >_{L^2} = < D_t F, u >_{L^2[T\times\Omega]}$$

That is, δ is a closed, unbounded operator taking square integrable process to square integrable random variables. Observe that $E[\delta(u)] = 0$ for all $u \in Dom\delta$.

Lemma 2.0.1 *:Let $u \in Dom(\delta), F \in D^{1,2}$ s.t. $Fu \in L^2[T \times \Omega]$. Then*

$$\delta(Fu) = F\delta(u) - \int_0^T D_t F u_t dt \tag{1}$$

in the sense that (Fu) is Skorohod integrals iff the right hand side belongs to $L^2(\Omega)$.

Proof:Let $G = g(W(G_1, ..., G_n))$ be smooth with g of compact support. Then from the product rule

$$
\begin{aligned}
E[GF\delta(u)] &= E[\int_0^T D_t(GF)u_t dt] \\
&= E[G\int_0^T D_tFu_t dt] + E[F\int_0^T D_tGu_t dt] \\
&= E[G\int_0^T D_tFu_t dt] + E[G\delta(Fu)]
\end{aligned}
$$

For the next result, let us denote by $L_a^2[T\times\Omega]$ the subset of $L^2[T\times\Omega]$ consisting of adapted processes.

Proposition 2.0.1 *: $L_a^2[T\times\Omega] \subset Dom(\delta)$ and δ restricted to L_a^2 coincides with the Ito integral.*

Exercises: Now do Oksendal 2.6, 2.1 (a),(b),(c),(d), 4.1 (f), as well as examples (2.3) and (2.4).

3 Wiener Chaos

Consider the Hermite polynomials

$$
H_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}, \qquad H_0(x) = 1
$$

which are the coefficients for the power expansion of

$$
\begin{aligned}
F(x,t) &= e^{xt-\frac{t^2}{2}} \\
&= e^{\frac{x^2}{2}}e^{-\frac{1}{2}(x-t)^2} \\
&= e^{\frac{x^2}{2}}\sum_{n=0}^{\infty}\frac{t^n}{n!}(\frac{d^n}{dt^n}e^{-\frac{1}{2}(x-t)^2})|_{t=0} \\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x)
\end{aligned}
$$

It then follows that we have the recurrence relation

$$
(x-\frac{d}{dx})H_n(x) = H_{n+1}(x).
$$

The first polynomials are

$$\begin{cases} H_0 = 1 \\ H_1 = x \\ H_2(x) = x^2 - 1 \\ H_3(x) = x^3 - 3x \\ \vdots \end{cases}$$

Lemma 3.0.2 *: Let x, y be two random variables with joint Gaussian distribution s.t. $E(x) = E(y) = 0$ and $E(x^2) = E(y^2) = 1$. Then for all $n, m \geq 0$,*

$$E[H_n(x)H_m(y)] = \begin{cases} 0, & \text{if } n \neq m \\ n!E(xy)^n & \text{if } n = m \end{cases}$$

Proof: From the characteristic function of a joint Gaussian random variable we obtain

$$E[e^{sX+vY}] = e^{\frac{s^2}{2}+svE[XY]+\frac{v^2}{2}} \Rightarrow E[e^{sX-\frac{s^2}{2}}e^{vY-\frac{v^2}{2}}] = e^{svE[XY]}$$

Taking the partial derivative $\frac{\partial^{n+m}}{\partial s^n \partial v^m}$ on both sides gives

$$E[H_n(x)H_m(y)] = \begin{cases} 0, & \text{if } n \neq m \\ n!E(xy)^n & \text{if } n = m \end{cases}$$

Let us now define the spaces

$$\mathcal{H}_n = \text{span}\{H_n(W(h)), \quad h \in L^2[0,T]\}$$

These are called the Wiener chaos of order n. We see that $\mathcal{H}_0$ corresponds to constants while $\mathcal{H}_1$ are the random variables in the closed linear space generated by $\{W(h) : h \in L^2[0,T]\}$ (as before).

Theorem 3.1 *: $L^2(\Omega, \mathcal{F}_T, P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$*

Proof: The previous lemma shows that the spaces $\mathcal{H}_n$, $\mathcal{H}_m$ are orthogonal for $n \neq m$.

Now suppose $X \in L^2(\Omega, \mathcal{F}_T, P)$ is such that $E[XH_n(W(h))] = 0, \forall n, \forall h \in L^2[0,T]$. Then

$$E[XW(h)^n] = 0, \quad \forall n, \forall h$$

$$\Rightarrow E[Xe^{W(h)}] = 0, \quad \forall h$$

$$\Rightarrow E[Xe^{\sum_{i=1}^m t_i W(h_i)}] = 0, \quad \forall t_1, \dots, t_m \in \mathbb{R} \quad \forall h_1, \dots, h_m \in \mathbb{R}$$

$$X = 0$$

.

Now let $\mathcal{O}_n = \{(t_1, \dots, t_n) \in [0,T]^n, 0 \le t_1 \le t_2 \le \dots \le t_n \le T\}$. We define the iterated integral for a deterministic function $f \in L^2[\mathcal{O}_n]$ as

$$J_n(f) = \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \dots, t_n) dW_{t_1} \cdots dW_{t_{n-1}} dW_{t_n}$$

Observe that, due to Ito isometry,

$$||J_n(f)||_{L^2(\Omega)} = ||f||_{L^2[\mathcal{O}_n]}$$

So the image of $L^2[S_n]$ under J_n is closed in $L^2(\Omega, \mathcal{F}_T, P)$. Moreover, in the special case where $f(t_1, \dots, t_n) = h(t_1) \dots h(t_n)$, for some $h \in L^2[0,T]$, we have

$$n! J_n(h(t_1)h(t_2)\dots h(t_n)) = ||h||^n H(\frac{W(h)}{||h||}),$$

as can be seen by induction. Therefore, $\mathcal{H}_n \subset J_n(L^2[\mathcal{O}_n])$. Finally, a further application of Ito isometry shows that

$$E[J_m(g)J_n(h)] = \begin{cases} 0, & \text{if } n \neq m \\ < g, h >_{L^2[\mathcal{O}_n]} & \text{if } n = m \end{cases}.$$

Therefore $J_m(L^2[\mathcal{O}_n])$ is orthogonal to $\mathcal{H}_n$ for all $n \neq m$. But this implies that $\mathcal{H}_n = J_n(L^2[\mathcal{O}_n])$. We have just given an abstract proof for the following theorem.

Theorem 3.2 *(time-ordered Wiener Chaos): Let $F \in L^2(\Omega, \mathcal{F}_T, P)$. Then*

$$F = \sum_{m=0}^{\infty} J_m(f_m)$$

for (unique) deterministic function $f_m \in L^2[\mathcal{O}_m]$. Moreover,

$$||F||_{L^2[\Omega]} = \sum_{m=0}^{\infty} ||f_m||_{L^2[\mathcal{O}_m]}$$

Example: Find the Wiener chaos expansion of $F = W_T^2$. Solution: We use the fact that $H_2(x) = x^2 - 1$. So writing $W_T = \int_0^T 1_{\{t \le T\}} dW_t = \int_0^T h(t) dW_T$ we obtain $||h|| = (\int_0^T 1_{\{t \le T\}}^2 dt)^{1/2} = T^{1/2}$, so

$$H_2(\frac{W_T}{||h||}) = \frac{W_T^2}{T} - 1$$

From

$$2 \int_0^T \int_0^{t_2} 1_{\{t \le T\}} 1_{\{t \le T\}} dW_{t_1} dW_{t_2} = T(\frac{W_T^2}{T} - 1)$$

We find

$$W_T^2 = T + 2J_2(1).$$

The connection between Wiener chaos and Malliavian calculus is best explained through the *symmetric* expansion, as opposed to the *time- ordered* expansion just presented.

We say that a function $g : [0,T]^n \to \mathbb{R}$ is symmetric if

$$g(X_{\sigma_1}, \ldots, X_{\sigma_n}) = g(X_1, \ldots, X_n)$$

for all permutations σ of the set $\{1, \ldots, n\}$. The closed subspace of square integrable symmetric functions is denoted $L_s^2([0,T]^n)$. Now observe that $\mathcal{O}_n$ occupies only the fraction $\frac{1}{n!}$ of the box $[0,T]^n$. Therefore, for a symmetric function we have

$$\begin{aligned} ||g||^2_{L^2[0,T]^n} &= \int_0^T \int_0^T \cdots \int_0^T g^2(t_1, \ldots, t_n) dt_1 dt_n \\ &= n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g^2(t_1, \ldots, t_n) dt_1 dt_n \\ &= n! ||g||^2_{L^2[\mathcal{O}_n]} \end{aligned}$$

Try to do a two dimensional example to convince yourself of this.

We can now extend the definition of multiple stochastic integrals to function $g \in L_s^2[0,T]^n$ by setting

$$\begin{aligned} I_n(g) &\equiv \int_0^T \int_0^T \cdots \int_0^T g(t_1, \ldots, t_n) dW_{t_1} \cdots dW_{t_n} \\ &:= n! J_n(g) \\ &= n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g(t_1, \ldots, t_n) dW_{t_1} \cdots dW_{t_n} \end{aligned}$$

It then follows that

$$E[I_n^2(g)] = E[n!^2 J_n^2(g)] = n!^2 ||g||_{L^2[\mathcal{O}_n]} = n!||g||_{L^2[0,T]^n}$$

so the image of $L_s^2[0,T]^n$ under I_n is closed. Also, for the particular case $g(t_1,\dots,t_n) = h(t_1)\dots h(t_n)$, the previous result for the time-ordered integral gives

$$I_n(h(t_1)\dots h(t_n)) = ||h||^n H_n(\frac{W(h)}{||h||})$$

Therefore we again have that $\mathcal{H}_n \subset I_n(L_s^2)$. Moreover, the orthogonality relation between integrals of different orders still holds, that is

$$E[J_n(g)J_m(h)] = \begin{cases} 0, & \text{if } n \neq m \\ n! < g, h > & \text{if } n = m \end{cases}$$

Therefore $I_n(L_s^2)$ is orthogonal to all $\mathcal{H}_m, \quad n \neq m$, which implies $I_n(L_s^2) = \mathcal{H}_n$.

Theorem 3.3 *(symmetric Wiener chaos) Let $L^2(\Omega, \mathcal{F}_T, P)$. Then*

$$F = \sum_{m=0}^{\infty} I_m(g_m)$$

for (unique) deterministic functions $g_m \in L_s^2[0,T]^n$. Moreover, $||F||_{L^2(s)} = \sum_{m=0}^{\infty} m!||g_m||_{L^2[0,T]^n}$

To go from the two ordered expansion to the symmetric one, we have to proceed as following. Suppose we found

$$F = \sum_{m=0}^{\infty} J_m(f_m), \quad f_m \in L^2[\mathcal{O}_n]$$

First extend $f_m(t_1,\dots,t_m)$ from $\mathcal{O}_m$ to $[0,T]^m$ by setting

$$f_m(t_1,\dots,t_m) = 0 \quad if \quad (t_1,\dots,t_m) \in [0,T]^m \backslash \mathcal{O}_m$$

Then define a symmetric function

$$g_m(t_1,\dots,t_m) = \frac{1}{m!} \sum_{\sigma} f_m(t_{\sigma_1},\dots,t_{\sigma_m})$$

Then

$$I_m(g_m) = m! J_m(g_m) = J_m(f_m)$$

Example: What is the symmetric Wiener chaos expansion of $F = W_t(W_T - W_t)$? Solution:

Observe that

$$\begin{aligned} W_t(W(T) - W_t) &= W_t \int_t^T dW_{t_2} \\ &= \int_t^T \int_0^t dW_{t_1} dW_{t_2} \\ &= \int_0^T \int_0^{t_2} 1_{\{t_1 < t < t_2\}} dW_{t_1} dW_{t_2} \end{aligned}$$

Therefore $F = J_2(1_{\{t_1<t<t_2\}})$. To find the symmetric expansion, we have to find the symmetrization of $f(t_1, t_2) = 1_{\{t_1<t<t_2\}}$. This is

$$\begin{aligned} g(t_1, t_2) &= \frac{1}{2}[f(t_1, t_2) + f(t_2, t_1)] \\ &= \frac{1}{2}[1_{\{t_1<t<t_2\}} + 1_{\{t_2<t<t_1\}}] \end{aligned}$$

Then $F = I_2[\frac{1}{2}(1_{\{t_1<t<t_2\}} + 1_{\{t_2<t<t_1\}})]$

Exercises: Oksendal 1.2(a),(b),(c),(d).

3.1 Wiener Chaos and Malliavin Derivative

Suppose note that $F \in L^2(\Omega, \mathcal{F}, P)$ with expansion

$$F = \sum_{m=0}^{\infty} I_m(g_m), \quad g_m \in L_s^2[0, T]^m$$

Proposition 3.3.1 *$F \in D^{1,2}$ if and only if*

$$\sum_{m=1}^{\infty} mm! ||g_m||^2_{L^2(T^m)} < \infty$$

and in this case

$$D_t F = \sum_{m=1}^{\infty} m I_{m-1}(g_m(\cdot, t)) \tag{2}$$

Moreover $||D_t F||^2_{L^2(\Omega \times T)} = \sum_{m=1}^{\infty} mm! ||g_m||^2_{L^2}$.

Remarks:

1. If $F \in D^{1,2}$ with $D_t F = 0 \quad \forall t$ then the series expansion above shows that F must be constant, that is $F = E(F)$.

2. Let $A \in F$. Then

$$D_t(1_A) = D_t(1_A)^2 = 21_A D_t(1_A).$$

Therefore

$$D_t(1_A) = 0.$$

But since $1_A = E(1_A) = P(A)$, the previous item implies that

$$P(A) = 0 \text{ or } 1$$

Examples:(1) (Oksendal Exercises 4.1(d)) Let

$$\begin{aligned} F &= \int_0^T \int_0^{t_2} cos(t_1 + t_2) dW_{t_1} dW_{t_2} \\ &= \frac{1}{2} \int_0^T \int_0^T cos(t_1 + t_2) dW_{t_1} dW_{t_2} \end{aligned}$$

Using the (2) we find that

$$D_t F = 2\frac{1}{2} \int_0^T cos(t_1 + t) = \int_0^T cos(t_1 + t).$$

(2) Try doing $D_t W_T^2$ using Wiener chaos.

Exercise: 4.2(a),(b).

3.2 Wiener Chaos and the Skorohod Integral

Now let $u \in L^2(\Omega \times T)$. Then it follows from the Wiener-Ito expansion that

$$u_t = \sum_{m=0}^{\infty} I_m(g_m(\cdot, t)), \quad g_m(\cdot, t) \in L_s^2[0, T]^m$$

Moreover

$$||u_t||^2_{L^2(\Omega \times T)} = \sum_{m=0}^{\infty} m! ||g_m||_{L^2[0,T]^{m+1}}$$

Proposition 3.3.2 : *$u \in Dom\delta$ if and only if*

$$\sum_{m=0}^{\infty}(m+1)!||\tilde{g}_m||^2_{L^2(T^{m+1})} < \infty$$

in which case

$$\delta(u) = \sum_{m=0}^{\infty} I_{m+1}(\tilde{g}_m)$$

<u>Examples</u>: (1) Find $\delta(W_T)$ using Wiener Chaos. <u>Solution</u>:

$$W_T = \int_0^T 1dW_{t_1} = I_1(1), \quad f_0 = 0, \quad f_1 = 1, \quad f_n = 0 \quad \forall n \geq 2$$

Therefore

$$\delta(W_T) = I_2(1) = 2J_2(1) = W_T^2 - T$$

(2) Find $\delta(W_T^2)$ using Wiener Chaos. <u>Solution</u>:

$$W_T^2 = T + I_2(1), \quad f_0 = T, \quad f_1 = 0, \quad f_2 = 1, \quad f_n = 0 \quad \forall n \geq 3$$

$$\begin{aligned}
\delta(W_T^2) &= I_1(T) + I_2(0) + I_3(1) \\
&= TW_T + T^{3/2}H_3(\frac{W(T)}{T^{1/2}}) \\
&= TW_T + T^{3/2}(\frac{W(T)^3}{T^{3/2}} - \frac{3W(T)}{T^{1/2}}) \\
&= TW_T + W(T)^3 - 3W(T)T \\
&= W(T)^3 - 2TW(T)
\end{aligned}$$

3.3 Further Properties of the Skorohod Integral

The Wiener chaos decomposition allows us to prove two interesting properties of Skorohod integrals.

Proposition 3.3.3 : *Suppose that $u \in L^2(\Omega \times T)$. Then $u = DF$ for some $F \in D^{1/2}$ if and only if the Kernels for*

$$u_t = \sum_{m=0}^{\infty} I_m(f_m(\cdot, t))$$

is symmetric functions of all variables.

Proof: Put $F = \sum_{m=0}^{\infty} \frac{1}{m+1} I_{m+1}(f_m(\cdot, t))$.

Proposition 3.3.4 : *Every process $u \in L^2(\Omega \times T)$ can be uniquely decomposed as $u \subset DF + u^0$, where $F \in D^{1/2}$ and $E(\int D_t G u_t^0 dt) = 0$ for all $G \in D^{1/2}$. Furthermore, $u^0 \in Dom(\delta)$ and $\delta(u^0) = 0$.*

Proof: It follows from the previous proposition that an element of the form $DF, \quad F \in D^{1/2}$ form a closed subspace of $L^2(\Omega \times T)$.

4 The Clark-Ocone formula (Optional)

Recall from Ito calculus that any $F \in L^2(\omega)$ can be written as

$$F = E(F) + \int_0^T \phi_t dW_t$$

for a unique process $\phi \in L^2(\Omega \times T)$. If $F \in D^{1/2}$, this can be made more explicit, since in this case

$$F = E(F) + \int_0^T E[D_t F | \mathcal{F}_t] dW_t$$

We can also obtain a generalized Clark-Ocone formula by considering

$$d\bar{W}_t^Q = dW + \lambda_t dt$$

and the measure $\frac{dQ}{dP} = Z_T = e^{-\int_0^T \lambda_t dW_t - \frac{1}{2}\int_0^T \lambda_t^2 dt}$. Then for $F \in D^{1/2}$ with additional technical conditions we obtain

$$F = E_Q[F] + \int_0^T E_Q[(D_t F - \int_t^T D_t \lambda_s dW_s^Q) | \mathcal{F}_t] dW^Q$$

Exercises: 5.2 (a),(b),(c),(d),(e),(f) 5.3 (a),(b),(c)

5 The Malliavin Derivative of a Diffusion

Let us begin with a general result concerning the commutator of the Malliavin derivative and the Skorohod integral.

Theorem 5.1 : *Let $u \in L^2(\Omega \times T)$ be a process such that $u_s \in D^{1/2}$ for each $s \in [0,T]$. Assume further that, for each fixed t, the process $D_t u_s$ is Skorohod Integrable ($D_t u_s \in Dom(\delta)$). Furthermore, suppose that $\delta(D_t u) \in L^2(\Omega \times T)$. Then $\delta(u) \in D^{1/2}$ and*

$$D_t(\delta(u)) = u_t + \delta(D_t u) \tag{3}$$

Proof: Let $u_s = \sum_{m=0}^{\infty} I_m(f_m(\cdot, s))$. Then

$$\delta(u) = \sum_{m=0}^{\infty} I_{m+1}(\tilde{f}_m)$$

where $\tilde{f}_m$ is the symmetrization of $f_m(\cdot, s)$. Then

$$D_t(\delta(u)) = \sum_{m=0}^{\infty} (m+1) I_m(\tilde{f}_m(\cdot, t))$$

Now note that

$$\begin{aligned}\tilde{f}_m(t_1, \ldots, t_m, t) \quad &= \tfrac{1}{m+1}[f_m(t_1, \ldots, t_m, t) + f_m(t, t_2, \ldots, t_m, t_1) + \ldots + f_m(t_1, \ldots, t_{m-1}, t, t_m)] \\ &= \tfrac{1}{m+1}[f_m(t_1, \ldots, t_m, t) + f_m(t_1, \ldots, t_{m-1}, t, t_m) + f_m(t_m, t_2, \ldots, t_{m-1}, t, t_1) + \\ &\qquad \ldots + f_m(t_1, \ldots, t_m, t, t_{n-1})]\end{aligned}$$

Therefore

$$D_t(\delta(u)) = \sum_{m=0}^{\infty} I_m(f_m(\cdot, t)) + \sum_{m=0}^{\infty} m I_m(symm f_m(\cdot, t, \cdot).$$

On the other hand

$$\begin{aligned}\delta(D_t u) \quad &= \quad \delta[D_t(\sum_{m=0}^{\infty} I_m(f_m(\cdot, s)))] \\ &= \quad \delta[\sum_{m=0}^{\infty} m I_{m-1}(f_m(\cdot, t, s))] \\ &= \quad \sum_{m=0}^{\infty} m I_m(symm f_m(\cdot, t, \cdot))\end{aligned}$$

Comparing the two expressions now gives the result.

Corollary 5.1.1 : *If, in addition to the conditions of the previous theorem, u_s is $\mathcal{F}_s$ adapted, we obtain*

$$D_t(\int_0^T u_s dW_s) = u_t + \int_t^T D_t u_s dW_s$$

Now suppose that

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

for functions b and σ satisfying the usual Lipschitz and growth conditions to enssure existence and uniqueness of the solution in the form

$$X_t = X_0 + \int_0^t b(u, X_u)du + \int_0^t \sigma(u, X_u)dW_u$$

It is then possible to prove that $X_t \in D^{1,2}$ for each $t \in [0, T]$. Moreover, its Malliavian derivative satisfies the linear equation

$$\begin{aligned} D_t(X_t) &= D_s(\int_0^t b(u, X_u)du) + D_s(\int_0^t \sigma(u, X_u)dW_u) \\ &= \int_s^t b^{'}(u, X_u)D_sX_u du + \sigma(s, X_s) + \int_s^t \sigma^{'}(u, X_u)D_sX_u dW_u \end{aligned}$$

That is,

$$D_sX_t = \sigma(s, X_s)exp[\int_s^t (b^{'} - \frac{1}{2}(\sigma^{'})^2)du + \int_s^t \sigma^{'} dW_u]$$

In other words,

$$D_sX_t = \frac{Y_t}{Y_s}\sigma(s, X_s)1_{\{s\leq t\}}$$

where Y_t is the solution to

$$dY_t = b^{'}(t, X_t)Y_t dt + \sigma^{'}(t, X_t)Y_t dW_t, \quad Y_0 = 1.$$

This is called the *first variation* process and plays a central role in what follows.

Examples: (1) $dX_t = r(t)X_t dt + \sigma(t)X_t dW_t, \quad X_0 = x$

$$\Longrightarrow dY_t = r(t)Y_t dt + \sigma(t)Y_t dW_t$$

$$\Longrightarrow Y_t = \frac{X_t}{x}$$

$$\Longrightarrow D_sX_t = \frac{X_t}{X_s}\sigma(s)X_s = \sigma(s)X_t$$

(2) $dX_t = (\theta(t) - kX_t)dt + \sigma dW_t, \quad X_0 = x$

$$\Longrightarrow dY_t = -kY_t dt \Longrightarrow Y_t = e^{-kt}$$

$$D_s X_t = e^{-k(t-s)}\sigma$$

(3) $dX_t = k(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t, \quad X_0 = x$

$$dY_t = -kY_t dt + \frac{1}{2}\frac{\sigma}{\sqrt{X_t}}Y_t dW_t$$

$$D_s X_t = \sigma\sqrt{X_s}exp[-\int_s^t (k + \frac{1}{4}\frac{\sigma^2}{X_u})du + \frac{1}{2}\int_s^t \frac{\sigma}{\sqrt{X_u}}dW_u]$$

6 Malliavin Weighted Scheme

Suppose now that we want to calculate $E[\phi'(X)G]$ where X is an underlying asset, ϕ is a pay-off function and $\frac{X_\lambda - X}{\lambda} \to G$ as $\lambda \to 0$, corresponding to a perturbation of the process X. Then using the chain rule for the Malliavin derivative we have that, for an arbitrary process h_s,

$$Gh_s D_s(\phi(X)) = Gh_s\phi'(X)D_s.X$$

Integrating both sides of the equation gives

$$\Rightarrow \int_0^T Gh_s D_s(\phi(X))ds = G\phi'(X)\int_0^T h_s D_s X ds.$$

That is

$$G\phi'(X) = \frac{\int_0^T Gh_s D_s(\phi(X))ds}{\int_0^T h_s D_s X ds}$$

Define $u_s = \frac{Gh_s}{\int_0^T h_s D_s X ds}$. Then from duality

$$E[\phi' G] = E[\int_0^T D_s(\phi(X))u_s ds] = E[\phi(X)\delta(u)]$$

Therefore $E[\phi'(X)G] = E[\phi(X)\pi]$, where

$$\pi = (\frac{Gh_s}{\int_0^T h_s D_s X ds}). \qquad (4)$$

Example: Let $S_T = S_0 e^{(r - \frac{\sigma^2}{2})T} + \sigma W_T$. Then its *delta* is given by

$$\begin{aligned} \Delta &= E[e^{-rT}\phi(S_T)] \\ &= E[e^{-rT}\phi'(S_T)\frac{\partial S_T}{\partial S_0}] \\ &= \frac{e^{-rT}}{S_0}E[\phi'(S_T)S_T] \end{aligned}$$

Using (4) with $h_s = 1$ we get

$$\begin{aligned}\pi &= \delta(\frac{S_T}{\int_0^T D_s S_T ds}) \\ &= \delta(\frac{S_T}{\sigma\S_T int_0^T 1^{\{s\leq T\}} ds}) \\ &= \delta(\frac{1}{\sigma T}) \\ &= \frac{W_T}{\sigma T}\end{aligned}$$

Therefore

$$\Delta = \frac{e^{-rT}}{S_0} E[\phi(S_T)\frac{W_T}{\sigma T}].$$

For the *vega* we have

$$\begin{aligned}\mathcal{V} &= \frac{\partial}{\partial\sigma} E[e^{-rT}\phi(S_T)] \\ &= e^{-rT} E[\phi^{'}(S_T)\frac{\partial S_T}{\partial\sigma}] \\ &= e^{-rT} E[\phi^{'}(S_T)(W_T - \sigma T)S_T]\end{aligned}$$

Again, applying (4) with $h_s = 1$ we obtain

$$\begin{aligned}\pi &= \delta(\frac{(W_T - \sigma T)S_T}{\int_0^T D_s S_T ds}) \\ &= \delta(\frac{(W_T - \sigma T)}{\sigma T}) \\ &= \delta(\frac{W_T}{\sigma T} - 1) \\ &= \frac{1}{\sigma T}\delta(W_T) - W_T \\ &= \frac{W_T^2 - T}{\sigma T} - W_T \\ &= \frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma},\end{aligned}$$

that is

$$\mathcal{V} = e^{-rT} E[\phi^{'}(S_T)(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma})]$$

Finally, for the *gamma* we have to evaluate a second derivative. That is

$$\begin{aligned}\Gamma &= \frac{\partial^2}{\partial S_0^2}E[e^{-rT}\phi(S_T)] \\ &= \frac{e^{-rT}}{S_0^2}E[\phi^{''}(S_T)S_T^2].\end{aligned}$$

We begin by applying (4) to this last exapression. This gives

$$E[\phi^{''}(S_T)S_T^2] = E[\phi^{'}(S_T)\pi_1]$$

$$\pi_1 = \delta(\frac{S_T^2}{\sigma T S_T}) = \delta(\frac{S_T}{\sigma T}) = \frac{S_T W_T}{\sigma T} - S_T$$

That is

$$E[\phi^{''}(S_T)S_T^2] = E[\phi^{'}(S_T)(\frac{S_T W_T}{\sigma T} - S_T)].$$

We now use the formula again for

$$\begin{aligned}\pi_2 &= \delta(\frac{\frac{W_T}{\sigma T} - 1}{\sigma T}) \\ &= \delta(\frac{W_T}{\sigma^2 T^2}) - \delta(\frac{1}{\sigma T}) \\ &= \frac{W_T^2 - T}{\sigma^2 T^2} - \frac{W_T}{\sigma T}\end{aligned}$$

$$\Gamma = \frac{e^{-rT}}{S_0^2 \sigma T}E[\phi(S_T)(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma})]$$

such that $\gamma = \frac{\mathfrak{v}}{S_0^2 \sigma T}$

To formalize our discussion, suppose $f \in D^{1/2}$ and denote by W the set of random variables π such that

$$E[\phi^{'}(F)G] = E[\phi(F)\phi], \quad \forall \quad \phi \in C_p^{\infty}$$

Proposition 6.0.1 *: A necessary and sufficient condition for a weight to be of the form* $\pi = \delta(u)$ *where* $u \in Dom(\delta)$, *is that*

$$E[\int_0^T D_t F u_t dt|\mathcal{F}(F)] = E[G|\mathcal{F}(F)] \qquad (*)$$

Moreover, $\pi_0 = E[\pi|\mathcal{F}(F)]$ *is the minimum over all weights of the correct functional*

$$Var = E[(\phi(F)\pi - E[\phi^{'}(F)G])^2]$$

Proof: Suppose that $u \in Dom(\delta)$ satisfies

$$E[\int_0^T D_t F u_t dt | \sigma(F)] = E[G|\sigma(F)]$$

Then

$$\begin{aligned} E[\phi^{'}(F)G] &= E[E[\phi^{'}(F)G|\sigma(F)]] \\ &= E[\phi^{'}(F)E[G|\sigma(F)]] \\ &= E[\phi^{'}(F)E[\int_0^T D_t F u_t dt|\sigma(F)]] \\ &= E[\int_0^T D_t \phi(F) u_t dt] \\ &= E[\phi(F)\delta(u)dt] \end{aligned}$$

so $\pi = \delta(u)$ is a weight.

Conversely, if $\pi = \delta(u)$ for some $u \in Dom(\delta)$ is a weight, then

$$\begin{aligned} E[\phi^{'}(F)G] &= E[\phi\delta(u)] \\ &= E[\int_0^T D_t \phi(F) u_t dt] \\ &= E[\phi^{'}(F) \int_0^T D_t F u_t dt] \end{aligned}$$

Therefore,

$$E[\int_0^T D_t F u_t dt | \sigma(F)] = E[G|\sigma(F)].$$

To prove the minimal variance claim, observe first that for any two weights $\pi_1, \quad \pi_2$ we must have $E[\pi_1|\sigma(F)] = E[\pi_2|\sigma(F)]$. Therefore, setting $\pi_0 = E[\pi|\sigma(F)]$ for a generic weight π we obtain

$$\begin{aligned} \text{var}^{\pi} &= E[(\phi(F)\pi - E[\phi^{'}(F)G])^2] \\ &= E[(\phi(F)(\pi - \pi_0) + \phi(F)\pi_0 - E[\phi^{'}(F)G])^2] \\ &= E[(\phi(F)(\pi - \pi_0))^2] + E[(\phi(F)\pi_0 - E[\phi^{'}(F)G])^2] \\ &+ 2E[\phi(F)(\pi - \pi_0)(\phi(F)\pi_0 - E[\phi^{'}(F)G])] \end{aligned}$$

But

$$E[\phi(F)(\pi-\pi_0)(\phi(F)\pi_0 - E[\phi^{'}(F)G])] = E[E[\phi(F)(\pi-\pi_0)(\phi(F)\pi_0 - E[\phi^{'}(F)G])]|\sigma(F)] = 0.$$

Therefore the minimum must be achieved for $\pi = \pi_0$.

7 Generalized Greeks

Consider now

$$dX_t = r(t)X_t dt + \sigma(t, X_t)dW_t, \quad X_0 = x$$

where $r(t)$ is a deterministic function and $\sigma(\cdot,\cdot)$ satisfies the Lipschitz, boundedness and uniform ellipticity condition. Let $\tilde{r}(t)$, $\tilde{\sigma}(\cdot,\cdot)$ be two directions such that $(r+\epsilon\tilde{r}t)$ and $(\sigma+\epsilon\tilde{\sigma})$ satisfy the same conditions for any $\epsilon \in [-1,1]$.

Define

$$dX_t^{\epsilon_1} = (r(t)+\epsilon_1\tilde{r})X_t^{\epsilon_1} + \sigma(t, X_t^{\epsilon_1})dW_t$$

$$dX_t^{\epsilon_2} = r(t)X_t^{\epsilon_2} + [\sigma(t, X_t^{\epsilon_2}) + \epsilon_2\tilde{\sigma}(t, X_t^{\epsilon_2})]dW_t$$

Consider also the price functionals, for a square integral pay-off function of the form $\phi : \mathbb{R}^m \to \mathbb{R}$.

$$P(x) = E_x^Q[e^{-\int_0^T r(t)dt}\phi(X_{t_1},\ldots,X_{t_m})]$$

$$P^{\epsilon_1}(x) = E_x^Q[e^{-\int_0^T (r(t)+\epsilon_1\tilde{r}(t))dt}\phi(X_{t_1}^{\epsilon_1},\ldots,X_{t_m}^{\epsilon_1})]$$

$$P^{\epsilon_2}(x) = E_x^Q[e^{-\int_0^T r(t)dt}\phi(X_{t_1}^{\epsilon_2},\ldots,X_{t_m}^{\epsilon_2})]$$

Then the generalized Greeks are defined as

$$\Delta = \frac{\partial P(x)}{\partial x}, \qquad \Gamma = \frac{\partial^2 P}{\partial x^2}$$

$$\rho = \frac{\partial P^{\epsilon_1}}{\partial \epsilon_1}|_{\epsilon_1=0,\tilde{r}}, \qquad \mathcal{V} = \frac{\partial P^{\epsilon_2}}{\partial \epsilon_2}|_{\epsilon_2=0,\tilde{\sigma}}$$

The next proposition shows that the variations with respect to both the drift and the diffusion coefficients for the process X_t can be expressed in terms of the first variation process Y_t defined previously.

Proposition 7.0.2 : *The following limits hold in L^2–convergence*

1. $\lim_{\epsilon_1 \to 0} \frac{X_t^{\epsilon_1} - X_t}{\epsilon_1} = \int_0^t \frac{Y_t \tilde{r}(s) X_s}{Y_s} ds$

2. $\lim_{\epsilon_2 \to 0} \frac{X_t^{\epsilon_2} - X_t}{\epsilon_2} = \int_0^t Y_t \frac{\tilde{\sigma}(s, X_s)}{Y_s} dW_s - \int_0^t Y_t \frac{\sigma'(s, X_s) \tilde{\sigma}(s, X_s)}{Y_s} ds$

8 General One–Dimensional Diffusions

Now we return to the general diffusion SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

where the deterministic functions b, σ satisfy the usual conditions for existence and uniqueness. We will find several ways to specify Malliavin weights for delta: similar formulas can be found for the other greeks. An important point not addressed in these notes is to understand the computational issues when using Monte Carlo simulation. This will involve the practical problem of knowing which processes in addition to X_t itself need to be sampled to compute the weight: the answer in general is to compute the variation processes $Y_t, Y_t^{(2)}, Y_t^{(3)}, \ldots$.

Recall the conditions for $\pi = \delta(w_t)$ to be a weight for delta:

$$E[\int_0^T \frac{\sigma_t}{Y_{t0}} w_t dt | \sigma(X_{t_i})] = E[Y_{T0} | \sigma(X_{t_i})]$$

Our solutions will solve the stronger, sufficient conditions:

$$\int_0^T \frac{\sigma_t}{Y_{t0}} w_t dt = Y_{T0}$$

We investigate in more detail two special forms for w. We will see the need from this to look at higher order Malliavin derivatives: the calculus for this is given in the final section.

1. We obtain a t independent weight analogous to the construction in Ben-hamou (2001) by letting

$$w_t = w_T = \left[\int_0^T \frac{\sigma_t}{Y_{t0}} dt \right]^{-1}$$

To compute $\delta(w)$ use (1) to give $\delta(w) = wW_T - \int_0^T D_t w dt$. From the quotient rule $D_t(A^{-1}) = -A^{-1} D_t A \ A^{-1}$ and the commutation relation (3) we obtain

$$\begin{aligned} D_t w &= -w^2 Y_{t0}^{-1} \int_t^T \left[\frac{D_t \sigma_s Y_{st} - \sigma_s D_t Y_{st}}{Y_{st}^2} \right] ds & (5) \\ &= -w^2 Y_{t0}^{-1} \int_t^T \left[\frac{D_t \sigma_s Y_{st} - \sigma_s D_t Y_{st}}{Y_{st}^2} \right] ds & (6) \\ & & (7) \end{aligned}$$

where we use the general formula for $D_t Y_{st}$ derived in the next section. This yields the final formula

$$\pi = -\int_0^T \int_t^T w^2 Y_{t0}^{-1} \left[\frac{\sigma_s' \sigma_t - Y_{st} - \sigma_s D_t Y_{st}}{Y_{st}^2} \right] ds\ dt$$

Computing this weight will be computationally intensive: in addition to sampling X_t, one needs $Y_t, Y_t^{(2)}$.

2. We obtain a t dependent weight analogous to the construction in Ben-hamou (2001) by letting

$$w_t = \frac{Y_{t0}}{T\sigma_t}$$

. This yields the weight as an ordinary Ito integral:

$$\pi = \int_0^T \frac{Y_{t0}}{T\sigma_t} dW_t$$

Numerical simulation of X_t, Y_t will be sufficient to compute this weight.

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9 Higher Order Malliavin Derivatives

In this section, we sketch out the properties of the higher order variation processes

$$Y_{st}^{(k)} = \frac{\partial^k X_s}{\partial X_t^k}, \qquad t < s$$

and use them to compute multiple Malliavin derivatives $D_{t_1} \ldots D_{t_k} X_s$. These formulas are usually necessary for higher order greeks like γ. But as seen in the previous section it may also enter formulas for delta when certain weights are chosen.

Here are the SDEs satisfied by the first three variation processes:

$$\begin{aligned}
dY_t &= b_t' Y_t dt + \sigma_t' Y_t dW_t & (8)\\
dY_t^{(2)} &= \left[b_t' Y_t^{(2)} + b_t'' Y_t^2 \right] dt + \left[\sigma_t' Y_t^{(2)} + \sigma_t'' Y_t^2 \right] dW_t & (9)\\
dY_t^{(3)} &= \left[b_t' Y_t^{(3)} + 3b_t'' Y_t Y_t^{(2)} + b_t''' Y_t^3 \right] dt + \left[\sigma_t' Y_t^{(3)} + 3\sigma_t'' Y_t Y_t^{(2)} + \sigma_t''' Y_t^3 \right] dW_t & (10)\\
& & (11)
\end{aligned}$$

One can see that the pattern is

$$dY_t^{(k)} = b_t' Y_t^{(k)} dt + \sigma_t' Y_t^{(k)} dW_t + F_t^{(k)} dt + G_t^{(k)} dW_t$$

where $F^{(k)}, G_t^{(k)}$ are explicit functions of $t, X, (Y^{(j)})_{j<k}$. This particular form of SDE can be integrated by use of the semigroup defined by Y_{ts}.

Proposition 9.0.3

$$Y_t^{k)} = Y_{t0} \int_0^t Y_{u0}^{-1} \left[(F_u^{(k)} - G_u^{(k)} \sigma_u) du + G_u^{(k)} dW_u \right]$$

Proof: From the product rule $dXY = dX(Y + dY) + XdY$ for stochastic processes

$$\begin{aligned} d(RHS) &= [b_t' Y_{t0} dt + \sigma_t' Y_{t0} dW_t] \left[Y_{t0}^{-1} Y_t^{(k)} + Y_{t0}^{-1} (F_t^{(k)} - G_t^{(k)} \sigma_t') dt + G_t^{(k)} dW_t \right] \\ & +Y_{t0} \left[Y_{t0}^{-1} (F_t^{(k)} - G_t^{(k)} \sigma_t') dt + G_t^{(k)} dW_t \right] \\ &= b_t' Y_t^{(k)} + \sigma_t' Y_t^{(k)} dW_t + G_t^{(k)} \sigma_t' dt + (F_t^{(k)} - G_t^{(k)} \sigma') dt + G_t^{(k)} dW_t \\ &= d(LHS) \end{aligned}$$

□

Since the higher variation processes have the interpretation of higher derivatives of X with respect to the initial value x, we can extend the notion to derivatives with respect to X_t, any t. For that we define

$$\tilde{Y}_{t0}^{(k)} = \frac{Y_t^{(k)}}{Y_{t0}}$$

and then note

$$\tilde{Y}_{s0}^{(k)} - \tilde{Y}_{t0}^{(k)} = \int_t^s Y_{u0}^{-1} \left[(F_u^{(k)} - G_u^{(k)} \sigma_u) du + G_u^{(k)} dW_u \right]$$

If we define

$$\tilde{Y}_{st}^{(k)} := Y_{t0} [\tilde{Y}_{s0}^{(k)} - \tilde{Y}_{t0}^{(k)}]$$

then $\tilde{Y}_{st}^{(k)}$ solves (9.0.3) for $s > t$, subject to the initial condition $\tilde{Y}_{tt}^{(k)} = 0$. Therefore $\tilde{Y}_{st}^{(k)}$ has the interpretation of $\frac{\partial^k X_s}{\partial X_t^k}$.

The following rules extend the Malliavin calculus to higher order derivatives:

1. Chain rule:

$$D_t F(X_s) = F'(X_s) D_t X_s, \quad t < s;$$

2.

$$\frac{\partial X_s}{\partial X_t} := Y_{st} I(t < s) := \tilde{Y}_{st}^{(1)} I(t < s);$$

3.

$$\frac{\partial \tilde{Y}_{st}^{(k)}}{\partial X_t} = \tilde{Y}_{st}^{(k+1)} := \frac{\partial^{k+1} X_s}{\partial X_t^{k+1}}, \quad t < s;$$

4.

$$\tilde{Y}_{st}^{(k)} = Y_{st}^{-1} Y_{t0}^{(k)} - Y_{s0}^{(k)};$$

5.

$$D_t X_t = \sigma_t.$$

Examples:

1.

$$\begin{aligned} D_t X_T &= \frac{\partial X_T}{\partial X_t} D_t X_t \qquad \text{by chain rule} \\ &= Y_{Tt} \sigma_t \qquad \text{by (2), (5)} \end{aligned}$$

2.

$$D_t Y_{s0} = (D_t Y_{st}) Y_{t0} = \tilde{Y}_{st}^{(2)} (D_t X_t) Y_{t0}$$

3. For $t < s < T$:

$$\begin{aligned} D_t [D_s X_T] &= \frac{\partial}{\partial X_s} \left[Y_{Ts} \sigma_s I(s < T) \right] D_t X_s \qquad \text{by chain rule} \\ &= \left[\tilde{Y}_{Ts}^{(2)} \sigma_t + Y_{Ts} \sigma_s' \right] Y_{st} \sigma_t \end{aligned}$$

4. For $t < s < T$:

$$D_s [D_t X_T] = D_s [Y_{Ts} Y_{st} \sigma_t] = \tilde{Y}_{Ts}^{(2)} \sigma_s Y_{st} \sigma_t \tag{12}$$