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Destabilizing a stable crisis: Employment persistence and government intervention in macroeconomics

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ABSTRACT

The basic Keen model is a three-dimensional dynamical system describing the time evolution of the wage share, employment rate, and private debt in a closed economy. In the absence of government intervention this system admits, among others, two locally stable equilibria: one with a finite level of debt and nonzero wages and employment rate, and another characterized by infinite debt and vanishing wages and employment. We show how the addition of a government sector, modelled through appropriately selected functions describing spending and taxation, prevents the equilibrium with infinite debt. Specifically, we show that, by countering the fall in private profits with sufficiently high government spending at low employment, the extended system can be made uniformly weakly persistent with respect to the employment rate. In other words, the economy is guaranteed not to stay in a permanently depressed state with arbitrarily low employment rates.

1. Introduction

Among the many unintended consequences of the financial crisis of 2007–08, a pleasantly surprising one was the emergence of a Minsky revival. From Wall Street analysts to major newspapers to repentant mainstream economists, the ideas of Hyman Minsky attracted widespread interest because of the prescient and precise way in which they helped explain unfolding events. The term "Minsky crisis" was quickly coined to describe the processes leading up to the observed financial fragility and its consequences for the real economy. As highlighted by Wray (2011) in the New Palgrave Dictionary entry explaining the term, at the core of Minsky's analysis is the role of institutional ceilings and floors in stabilizing the inherently explosive dynamics of capitalist economies. The purpose of this paper is to investigate these stabilizing effects using the modern tools of persistence theory for dynamical systems.

Mathematical formalizations of Minsky's ideas are not exactly abundant, but are nonetheless identifiable as a growing strand in the economics literature. A useful survey up to 2005 is presented in Dos Santos (2005) and more recent contributions include Ryoo (2010) and Chiarella and Guilmi (2011). The vast majority of papers in this area, however, focus on the dynamic relationships that can lead to instability and explosive behaviour for the underlying variables, with the role of government somewhat restricted to playing second fiddle, say through regulation or by issuing bonds that can enter the portfolio decisions of more

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active players, such as firms and households. For example, as explained in Dos Santos (2005), because government policy is not specified in a sufficiently complete way in the influential early paper by Taylor and O'Connell (1985), the consequences of several "hidden" hypotheses that are necessary for stock-flow consistency issues are not fully analyzed. By contrast, we model government intervention explicitly and thoroughly analyze its relationships with the other dynamic variables in the economy.

After setting up a simplified yet sufficiently general closed system of accounts for households, firms, banks and the government sector in Section 2, we start by reviewing the special case of a model proposed in Keen (1995). In the absence of a government sector, the Keen model consists of the three-dimensional system (14) describing the dynamics of wages, employment rate and private debt. Its key insight is that, in boom times when profits are high, capitalists can choose to invest more than their profits by borrowing from the banking sector. If profits are low, on the other hand, capitalists might also want to invest less than their profits to pay down debt, thereby engaging in the familiar debt-deflation dynamics described in Fisher (1933). As shown in Grasselli and Costa Lima (2012), this behaviour by capitalists leads to the possibility of two very distinct equilibria recalled in Section 3.1: a "good equilibrium" characterized by finite private debt and nonzero wage share and employment rate, and a "bad equilibrium" characterized by infinite private debt and vanishing wage share and employment rate. Moreover, for typical parameter values, both equilibria are locally stable.

As emphasized throughout Minsky (1982), the debt-deflation mechanism can be halted by government intervention, since it follows from Kalecki's profit equation that government spending increases firm profits. We formalize this insight by introducing government expenditures, subsidies, and taxation into the Keen model in Section 2.2. Government intervention had already been proposed in Keen (1995), albeit in a different functional form. The key variable for firm behaviour is the profit share of output π given in (30), which depends on government policy only through subsidies and taxations, but not through expenditures, since the latter is part of total output. After isolating the core variables in the model from those whose evolution can be obtained separately, we are left with the five-dimensional system described by (32) for wage share, employment rate, stimulative subsidies and taxation, and profit share.

We perform local analysis for this system in Section 3. As before, we find a finite-value "good equilibrium" associated with non-zero wage share and employment rate and finite private debt. All other finite-value equilibria turn out to be related to vanishing wage shares, but none is locally stable for typical parameter values. We next move to the characterization of "bad equilibria", that is, those associated with collapsing profit shares even in the presence of government intervention. We find in Proposition 1 that provided the size of government subsidies in the vicinity of zero employment rates is large enough, all of these bad equilibria are either unstable or unachievable, even when the local stability condition for the corresponding bad equilibrium in the model without government is satisfied. In other words, government intervention successfully destabilizes an otherwise stable equilibrium point associated with an economic crisis.

Our main results are contained in Section 4. Persistence theory (see Smith and Thieme, 2011) studies the long term behaviour of dynamical systems, in particular the possibility that one or more variables remain bounded away from zero. Typical questions are, for example, which species in a model of interacting species will survive over the long term, or whether it is the case that in an endemic model an infection cannot persist in a population due to the depletion of the susceptible population. In our context, we are interested in establishing conditions in economic models that prevent one or more key economic variables, such as the employment rate, from vanishing. After preliminary technical results for profit levels in Propositions 2 and 3, we prove in Proposition 4 that under a variety of alternative mild conditions on government subsidies, the model describing the economy is uniformly weakly persistent with respect to the employment rate λ. The relevant precise definitions of persistence are reviewed in Appendix C, but the meaning of this result is easy enough to convey: we can guarantee that the employment rate does not remain indefinitely trapped at arbitrarily small values. This is in sharp contrast with what happens in the model without government intervention, where the employment rate is guaranteed to converge to zero and remain there forever if the initial conditions are in the basin of attraction of the bad equilibrium corresponding to infinite debt levels. Furthermore, as with any persistence result, Proposition 4 is a global one: no matter how disastrous the initial conditions are, a sufficiently responsive government can bring the economy back from a state of crises associated with zero employment rates. We end the paper with numerical examples illustrating these results in Section 5.

2. Derivation of the model

We consider the closed system of accounts shown in Table 1, where each entry represents a time-dependent quantity and a dot corresponds to differentiation with respect to time. As usual, balance sheet items are stocks measured in units of account, whereas both transactions and flow of funds items as flows measured in units of account per unit of time. For example, going down the first column, $M_h \equiv M_h(t)$, $r_{M_h} M_h \equiv r_{M_h}(t) M_h(t)$, and $\dot{M}_h \equiv \dot{M}_h(t)$ denote, respectively, the amount, the flow of interest payments, and the rate of change associate with deposits held by households at time t.

We see from Table 1 that the entire economy is subdivided in the Households, Firms, Banks, and Government sectors. Their balance sheet structure is fairly simple: the assets of households are bank deposits M_h and government debt B; the assets of firms are bank deposits M_f and capital goods K and they have liabilities in the form of bank loans L; banks have total deposits $M = M_h + M_f$ as their only liabilities and loans L as their only assets; government debt B is the only liability of the government sector. The empty cells in Table 1 represent the following simplifying assumptions: households do not take out bank loans; the government sector does not keep bank deposits or make bank loans;

Table 1
Balance sheet, transactions and flow of funds for the Keen model with government intervention.

Balance sheet	Households	Firms		Banks	Government	Sum
Deposits	$+M_h$	$+M_f$		$-M$		0
Loans		$-L$		$+L$		0
Bills	$+B$				$-B$	0
Capital goods		$+K$				K
Sum (net worth)	V_h	V_f		0	$-B$	K
Transactions		**Current**	**Capital**			
Consumption	$-C$	$+C$				0
Investment		$+I$	$-I$			0
Government spending		$+G$			$-G$	0
Accounting memo [GDP]		$[Y]$				
Wages	$+W$	$-W$				0
Interest on deposits	$+r_{M_h}M_h$	$+r_{M_f}M_f$		$-r_{M_h}M_h - r_{M_f}M_f$		0
Interest on loans		$-r_L L$		$+r_L L$		0
Interest on bills	$+r_g B$				$-r_g B$	0
Subsidies		$+GS$			$-GS$	0
Taxes		$-T$			$+T$	0
Financial balances	S_h	Π_u	$-I$	0	S_g	0
Flow of funds						
Deposits	$+\dot{M}_h$	$+\dot{M}_f$		$-\dot{M}$		0
Loans		$-\dot{L}$		$+\dot{L}$		0
Bills	$+\dot{B}$				$-\dot{B}$	0
Capital goods		$+I$				I
Sum	S_h	Π_u		0	S_g	I
Change in net worth	S_h	$\Pi_u - \delta K$		0	S_g	$I - \delta K$

firms and banks do not hold government debt. The absence of equities as a balance sheet item corresponds to the following further simplifications in the ownership structure of banks and firms: neither sector issues or hold equities; the net worth of firms is the difference between capital and net debt to the banking sector; the net worth of banks is kept identically zero at all times. In particular, we have that

$$D := L - M_f = M_h. \tag{1}$$

Most of the transactions items in Table 1 are self-explanatory, except for our treatment of taxes and subsidies, which we assume to be restricted to the firms sector. This is because the main goal of this paper is to show how the government sector can effectively prevent a crisis caused by the collapse of firm profits. Because transfer payments to households and taxes from households do not play a significant role in this dynamics, we chose to leave them out of the transaction flows, since including them would not affect the results.

The only other nontrivial assumption about transactions refer to the rate of interest on loans and deposits. Consistently with our hypothesis of zero net worth for banks, the interest rate r_{M_h} paid to household deposits need to be related to the rates r_{M_f} and r_L for firm deposits and loans as follows:

$$r_{M_h}M_h = r_{M_f}M_f - r_L L. \tag{2}$$

Accordingly, since $M_h = D = L - M_f$, the net flow of interest payments from firms to banks equal $r_{M_h}D$.

The flow of funds presented in Table 1 reflect the stock-flow consistency condition: financial balances for each sector are used to change their holdings of balance-sheet items. For example, central to the model is the fact that firms finance investment using both their financial balance and net borrowing from the banking sector according to the accounting identity

$$I - \Pi_u = \dot{L} - \dot{M}_f = \dot{D}. \tag{3}$$

All the quantities in Table 1 are given in real rather than nominal terms, that is to say, already divided by an agreed price deflator.

2.1. Keen model without government

As with any stock-flow consistent model (see Godley and Lavoie, 2007), several alternative production specification and behavioural assumptions of the different sectors can be compatible with the basic accounting framework represented in Table 1. We start with the model proposed in Keen (1995), where it is assumed that labour productivity and total labour force in the economy are given by

$$a(t) = a_0 e^{\alpha t}, \quad N(t) = N_0 e^{\beta t}, \tag{4}$$

for constants α and β. Denoting the number of employed workers by ℓ and following Goodwin (1967), Keen assumes full capital utilization in a Leontief production function for two homogeneous factors given by

$$Y(t) = \min\left\{\frac{K(t)}{\nu}, a(t)L(t)\right\} = \frac{K(t)}{\nu} = a(t)L(t), \tag{5}$$

where ν is a constant capital-to-output ratio. For the first behavioural assumption in the model, denote real wages per worker by w and let

$$\dot{w} = \Phi(\lambda) w, \tag{6}$$

where $\Phi(\lambda)$, known as the Phillips curve, is an increasing function of the employment rate

$$\lambda(t) := \frac{\ell(t)}{N(t)}. \tag{7}$$

Finally, suppose that investment is given by

$$I(t) = \kappa(\pi(t))Y(t), \tag{8}$$

where κ is an increasing function of the net profit share $\pi(t) := \Pi_u(t)/Y(t)$, so that changes in capital are described by

$$\dot{K} = \kappa(\pi)Y - \delta K \tag{9}$$

for a constant depreciation rate δ.

In the absence of a government sector, and choosing constant interest rates $r = r_{M_h} = r_{M_f} = r_L$, we can use the second column of transactions in Table 1 to find net profits for firms as

$$\Pi_u(t) = Y(t) - W(t) - rD(t), \tag{10}$$

so that the net profit share is

$$\pi(t) = 1 - \omega(t) - rd_k(t), \tag{11}$$

where

$$\omega(t) := \frac{w(t)L(t)}{Y(t)} = \frac{w(t)}{a(t)} \tag{12}$$

is the wage share in the economy and

$$d_k(t) := \frac{D(t)}{Y(t)} \tag{13}$$

is the firm net debt ratio. It follows from these assumptions that the wage share, employment rate, and firm debt ratio satisfy the following three-dimensional system of differential equations:

$$\begin{aligned} \dot{\omega} &= \omega\,[\Phi(\lambda) - \alpha] \\ \dot{\lambda} &= \lambda\left[\mu(\pi) - (\alpha + \beta)\right] \\ \dot{d}_k &= d_k\,[r - \mu(\pi)] + \nu[\mu(\pi) + \delta] - (1 - \omega) \end{aligned} \tag{14}$$

where

$$\mu(\pi) := \frac{\kappa(\pi)}{\nu} - \delta = \frac{\dot{Y}}{Y} \tag{15}$$

is the growth rate of output.

The first two equations are almost identical to the classical Goodwin (1967) model, with the exception of $\kappa(\pi)$ in place of $(1 - \omega)$. This represents the added feature of firms being able to invest more (or less) than their profits, with the difference corresponding to the change in net debt of firms to banks according to (3), whereas in the Goodwin model investment exactly equals the profits. In other words, the Keen model described in (14) reduces to the Goodwin model when $d = 0$ and $\kappa(\pi) = \pi = 1 - \omega$.

Observe that in both the Goodwin and Keen models the behaviour of households is fully accommodating in the sense that, given the investment function $\kappa(\pi)$, consumption is determined by the identity

$$C = Y - I = (1 - \kappa(\pi))Y, \tag{16}$$

precluding more general specification of households saving propensity. This obvious shortcoming of both models can be partially justified if one considers firm behaviour through investment to be the main determinant of cycles and instability, at least for long time scales. Independent specifications of both savings and investment require adjusting mechanisms either through quantity or prices. For example, an alternative to the Goodwin model where adjustment occurs through prices and changes in the capital-to-output ratio was proposed in Skott (1989). Because the purpose of this paper is to investigate the counterbalancing effect of government to the instability generated by the investment behaviour of firms, we keep household consumption as the accommodating variable as in (16). As a result of the closed system of accounts presented in Table 1 we have that, in the absence of government, household savings satisfy

$$\dot{M}_h = S_h = W + rM_h - C = W + rD - Y + I = I - \Pi_u = \dot{D}, \tag{17}$$

so that investment equals total savings, and the change in net debt of firms to the bank sector equals the change in household deposits.

We conclude this section by listing the technical assumptions we make on the Philips curve Φ and the investment function κ. To guarantee that the employment rate satisfies $0 \le \lambda(t) \le 1$ at all times, as well as existence of the relevant equilibria described next, we make the following additional assumptions on the increasing functions Φ and κ:

$$\Phi'(\lambda) > 0 \text{ on } (0, 1) \tag{18}$$

$$\Phi(0) < \alpha \tag{19}$$

$$\lim_{\lambda \to 1^-} \Phi(\lambda) = \infty \tag{20}$$

$$\kappa'(\pi) > 0 \text{ on } (-\infty, \infty) \tag{21}$$

$$\begin{aligned} \lim_{\pi \to -\infty} \kappa(\pi) = \kappa(-\infty) &< \nu\left(\max\{r, \alpha + \beta\} + \delta\right) \\ &< \lim_{\pi \to +\infty} \kappa(\pi) \le 1 \end{aligned} \tag{22}$$

$$\lim_{\pi \to -\infty} \pi^2 \kappa'(\pi) < \infty. \tag{23}$$

2.2. Introducing government

We see from Table 1 that government intervention is modelled through expenditures G, subsidies GS, and taxes T. We postpone the specification of government expenditures until the end of this section. For now, let us write subsidies and taxes in the form

$$GS(t) = G_b(t) + G_s(t), \tag{24}$$

$$T(t) = T_b(t) + T_s(t), \tag{25}$$

where

$$\dot{G}_b = \Gamma_b(\lambda)Y, \quad \dot{G}_s = \Gamma_s(\lambda)G_s, \tag{26}$$

$$\dot{T}_b = \Theta_b(\pi)Y, \quad \dot{T}_s = \Theta_s(\pi)T_s. \tag{27}$$

We interpret G_b and T_b as base-level subsidies and taxation, whose dynamics depends primarily on the overall state of the economy as measured by the level of output Y and are weakly dependent on the employment rate and firm profits through slow-varying functions Γ_b and Θ_b. On the other hand, we interpret G_s and T_s as stimulative subsidies and taxation, deemed to react quickly to changes in employment and firm profits through fast-varying functions Γ_s and Θ_s. For now, we only make the following general assumptions on the subsidies and taxation structural functions:

$$\Gamma'_b(\lambda) < 0 \quad \text{and} \quad \Gamma'_s(\lambda) < 0 \quad \text{on} \quad (0, 1) \tag{28}$$

$$\Theta'_b(\pi) > 0 \quad \text{and} \quad \Theta'_s(\pi) > 0 \quad \text{on} \quad (-\infty, \infty). \tag{29}$$

Defining $g_b = G_b/Y$, $g_s = G_s/Y$, $\tau_b = T_b/Y$, $\tau_s = T_s/Y$, it follows that the profit share of firms is now

$$\pi(t) = 1 - \omega(t) - rd_k(t) + g_b(t) + g_s(t) - \tau_b(t) - \tau_s(t). \tag{30}$$

A bit of algebra leads to the following seven-dimensional system:

$$\begin{aligned}
\dot{\omega} &= \omega\,[\Phi(\lambda) - \alpha] \\
\dot{\lambda} &= \lambda\left[\mu(\pi) - \alpha - \beta\right] \\
\dot{d}_k &= \nu\mu(\pi) + \nu\delta - \pi - d_k\mu(\pi) \\
\dot{g}_b &= \Gamma_b(\lambda) - g_b\mu(\pi) \\
\dot{\tau}_b &= \Theta_b(\pi) - \tau_b\mu(\pi) \\
\dot{g}_s &= g_s\,[\Gamma_s(\lambda) - \mu(\pi)] \\
\dot{\tau}_s &= \tau_s\left[\Theta_s(\pi) - \mu(\pi)\right].
\end{aligned} \tag{31}$$

Observe now that we can write

$$\begin{aligned}
\dot{\pi} &= -\dot{\omega} - r\dot{d}_k + \dot{g}_b + \dot{g}_s - \dot{\tau}_b - \dot{\tau}_s \\
&= -\omega(\Phi(\lambda) - \alpha) - r(\nu\mu(\pi) + \nu\delta - \pi) + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_b(\pi) - \tau_s\Theta_s(\pi) + (rd_k - g + \tau)\mu(\pi) \\
&= -\omega(\Phi(\lambda) - \alpha) - r(\nu\mu(\pi) + \nu\delta - \pi) + (1 - \omega - \pi)\mu(\pi) + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_b(\pi) - \tau_s\Theta_s(\pi),
\end{aligned}$$

so that the previous system reduces to the five-dimensional system

$$\begin{aligned}
\dot{\omega} &= \omega\,[\Phi(\lambda) - \alpha] \\
\dot{\lambda} &= \lambda\left[\mu(\pi) - \alpha - \beta\right] \\
\dot{g}_s &= g_s\,[\Gamma_s(\lambda) - \mu(\pi)] \\
\dot{\tau}_s &= \tau_s\left[\Theta_s(\pi) - \mu(\pi)\right] \\
\dot{\pi} &= -\omega(\Phi(\lambda) - \alpha) - r(\nu\mu(\pi) + \nu\delta - \pi) + (1 - \omega - \pi)\mu(\pi) + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_b(\pi) - \tau_s\Theta_s(\pi).
\end{aligned} \tag{32}$$

In other words, since the variables (d_k, g_b, τ_b) do not affect the dynamics of the variables $(\omega, \lambda, g_s, \tau_s, \pi)$, the reduced system (32) can be solved separately. Moreover, the trajectories $(\pi(t), \lambda(t))$ arising as solutions of (32) can be treated as time-dependent coefficients for the remaining uncoupled differential equations

$$\begin{aligned}
\dot{d}_k &= \nu\mu(\pi) + \nu\delta - \pi - d_k\mu(\pi) \\
\dot{g}_b &= \Gamma_b(\lambda) - g_b\mu(\pi) \\
\dot{\tau}_b &= \Theta_b(\pi) - \tau_b\mu(\pi)
\end{aligned} \tag{33}$$

In particular, if the system (32) is at an equilibrium state $(\overline{\omega}, \overline{\lambda}, \overline{g_s}, \overline{\tau_s}, \overline{\pi})$, then the remaining variables must converge exponentially fast, with rate $\mu(\overline{\pi})$, to their equilibrium values

$$\overline{d}_k = \frac{\nu\mu(\overline{\pi}) + \nu\delta - \overline{\pi}}{\mu(\overline{\pi})}, \quad \overline{g}_b = \frac{\Gamma_b(\overline{\lambda})}{\mu(\overline{\pi})}, \quad \overline{\tau}_b = \frac{\Theta_b(\overline{\pi})}{\mu(\overline{\pi})}. \tag{34}$$

We shall base our analytic results on the reduced system (32), since this will be enough to characterize the equilibria in which the economy either prospers or collapses. Observe that when working with the reduced system (32), we cannot recover d_k, g_b and τ_b separately, but rather the combination

$$rd_k - g_b + \tau_b = 1 - \omega - \pi + g_s - \tau_s. \tag{35}$$

For numerical simulations, however, we compute the trajectories for the full system (31), so that the evolution of each individual variable can be followed separately.

We now return to the specification of government expenditures G. Observe that, since G does not affect the profit share in (30), its dynamics can be freely chosen without altering the solution of either the reduced system (32) or the full system (31). In fact, the only other variable affected by G is government debt, which according to Table 1, satisfies

$$\dot{B} = r_gB + G + GS - T. \tag{36}$$

For example, if we define $d_g = B/Y$ and $g_e = G/Y$ and postulate the dynamics for expenditures in the form

$$\dot{G} = \Gamma(t, \omega, \lambda, \pi, g_s, \tau_s, G, Y), \tag{37}$$

we obtain

$$\dot{g}_e = \frac{\Gamma(\omega, \lambda, \pi, g_s, \tau_s, G, Y)}{Y} - g_e\mu(\pi) \tag{38}$$

$$\dot{d}_g = g_e + g_b + g_s - \tau_b - \tau_s - d_g\left(\mu(\pi) - r\right). \tag{39}$$

In other words, as long as the dynamics for government expenditures does not depend explicitly on the level of government debt, Eq. (38) can be solved separately first and then used to solve Eq. (39). Equivalently, we can model the government expenditure ratio directly as a function

$g_e = g_e(t, \omega, \lambda, \pi, g_s, \tau_s)$. In either case, if the government expenditure ratio is at an equilibrium value $\overline{g}_e$ compatible with equilibrium values for the remaining variables, then the government debt ratio converges exponentially fast, with rate $\mu(\overline{\pi}) - r$, to the equilibrium value

$$\overline{d}_g = \begin{cases} \dfrac{\overline{g}_e + \overline{g}_b + \overline{g}_s - \overline{\tau}_b - \overline{\tau}_s}{\mu(\overline{\pi}) - r} & \text{if } r < \mu(\overline{\pi}) \\ +\infty & \text{if } r > \mu(\overline{\pi}), \text{ or } r = \mu(\overline{\pi}) \text{ and } \overline{g}_e + \overline{g}_b + \overline{g}_s > \overline{\tau}_b + \overline{\tau}_s \\ 0 & \text{if } r = \mu(\overline{\pi}) \text{ and } \overline{g}_e + \overline{g}_b + \overline{g}_s < \overline{\tau}_b + \overline{\tau}_s \end{cases} \quad (40)$$

As before, the behaviour of households is fully accommodating in the sense that, given the investment function $\kappa(\pi)$ and the government expenditure ratio g_e, consumption by households is determined by the identity

$$C = Y - I - G = (1 - \kappa(\pi) - g_e)Y. \quad (41)$$

3. Equilibrium analysis

3.1. Keen model without government

As shown in Grasselli and Costa Lima (2012), there are three relevant equilibria for (14). The first one is given by $\overline{\omega}_0 = \overline{\lambda}_0 = 0$ and $\overline{d}_0$ solving the equation

$$d\left[r + \delta - \frac{\kappa(1 - rd)}{\nu}\right] = 1 - \kappa(1 - rd). \quad (42)$$

This is locally unstable for typical parameter values and corresponds to the economically uninteresting case of a crashed economy with finite debt.

The second one, hereafter called the "good equilibrium" is given by

$$\begin{aligned} \overline{\omega}_1 &= 1 - \overline{\pi}_1 - \frac{r(\kappa(\overline{\pi}_1) - \overline{\pi}_1)}{\alpha + \beta} \\ \overline{\lambda}_1 &= \Phi^{-1}(\alpha) \\ \overline{d}_1 &= \frac{\nu(\alpha + \beta + \delta) - \overline{\pi}_1}{\alpha + \beta} \end{aligned} \quad (43)$$

with

$$\overline{\pi}_1 = \kappa^{-1}(\nu(\alpha + \beta + \delta)). \quad (44)$$

The necessary and sufficient condition for its local stability is

$$r\left[\frac{\kappa'(\overline{\pi}_1)}{\nu}\left(\overline{\pi}_1 - \kappa(\overline{\pi}_1) + \nu(\alpha + \beta)\right) - (\alpha + \beta)\right] > 0, \quad (45)$$

which is satisfied by a wide range of parameter values.

The third equilibrium, henceforth referred to as the "bad equilibrium", is defined by

$$\begin{aligned} \overline{\omega}_2 &= 0 \\ \overline{\lambda}_2 &= 0 \\ \overline{d}_2 &\to +\infty \end{aligned} \quad (46)$$

and is locally stable if and only if

$$\mu(-\infty) = \frac{\kappa(-\infty)}{\nu} - \delta < r. \quad (47)$$

Observe that (47) is easily satisfied, since $\kappa(-\infty)$ is the rate of investment when capitalists face large negative profits and can be safely assumed to be very small. In what follows, we argue that government intervention in the form of spending and taxation is an effective way to prevent the system from reaching this undesirable equilibrium of vanishing employment, vanishing wages, and exploding private debt.

3.2. Finite-valued equilibria with government

The hyperplanes $g_s = 0$ (no stimulative spending) and $\tau_s = 0$ (no stimulative taxation) are invariant manifolds for (32), indicating that if the initial value for either g_s or τ_s is positive (or negative), the corresponding solution is entirely contained in that quadrant. Typically, $g_s > 0$ and $\tau_s \leq 0$, as the government attempts to stimulate the economy with a mixture of subsidies and tax cuts, although one could have $g_s \leq 0$ and/or $\tau_s > 0$ in the case of austerity measures intended to reduce the government deficit (as a naive attempt to decrease government debt) when the economy performs badly.

To find the first equilibrium, let

$$\begin{aligned} \overline{\lambda}_1 &= \Phi^{-1}(\alpha) \\ \overline{\pi}_1 &= \mu^{-1}(\alpha + \beta) \end{aligned} \quad (48)$$

so that $\dot{\omega} = \dot{\lambda} = 0$. Discarding the structural coincidences $\Gamma_s(\overline{\lambda}_1) = \alpha + \beta$ or $\Theta_s(\overline{\pi}_1) = \alpha + \beta$, the only way to obtain $\dot{g}_s = \dot{\tau}_s = 0$ is to set $\overline{g}_{s1} = \overline{\tau}_{s1} = 0$. This leads us to

$$\overline{\omega}_1 = 1 - \overline{\pi}_1 - \frac{r(\nu(\alpha + \beta + \delta) - \overline{\pi}_1)}{\alpha + \beta} + \frac{\Gamma_b(\overline{\lambda}_1) - \Theta_b(\overline{\pi}_1)}{\alpha + \beta} \quad (49)$$

as the only way to obtain $\dot{\pi} = 0$. This defines what we call the "good equilibrium" for (32), that is, an equilibrium characterized by finite values for all variables and non-zero wage share.

As shown in Appendix A, all remaining finite-valued equilibria (32) have the wage share equal to zero. To summarize, discarding equilibria whose existence depend on structurally unstable coincidences in the choice of parameter values, the finite-valued equilibria for system (32) are given by

$$(\omega,\lambda,g_s,\tau_s,\pi)=\begin{cases}(\overline{\omega}_1,\overline{\lambda}_1,0,0,\overline{\pi}_1)\\(0,\overline{\lambda}_2,\overline{g}_{s2},0,\overline{\pi}_1)\\(0,\overline{\lambda}_3,0,0,\overline{\pi}_1)\\(0,0,0,0,\overline{\pi}_4)\\(0,0,0,\overline{\tau}_{s5},\overline{\pi}_5)\\(0,0,\overline{g}_{s6},0,\overline{\pi}_6),\end{cases}\tag{50}$$

where the expressions for $\overline{\lambda}_2,\overline{g}_{s2},\overline{\lambda}_3,\overline{\pi}_4,\overline{\tau}_{s5},\overline{\pi}_5,\overline{g}_{s6},\overline{\pi}_6$ can be found in Appendix A.

Once the system (32) converges to an equilibrium $(\overline{\omega},\overline{\lambda},\overline{g}_s,\overline{\tau}_s,\overline{\pi})$, the dependent variables g_b,τ_b,d_g converge to their corresponding equilibrium values in (34). Similarly, if government expenditure converges to an equilibrium $\overline{g}_e$ compatible with the equilibrium values for the other variables, then government debt converges to the equilibrium value in (40).

We present the local stability analysis of these equilibria in Appendix B. As expected, the necessary and sufficient conditions for stability of the good equilibrium are more complicated than the simple condition (45) in the absence of government, but are easy to achieve for a wide range of parameter values. Most importantly, all the other finite-value equilibria can be made unstable fairly easily, which is a positive result, since they all correspond to the undesirable situation where the wage share is equal to zero.

More specifically, the third equilibrium is unstable whenever the good equilibrium is stable, whereas the last three equilibria are unstable whenever $\Gamma_s(0)>\alpha+\beta$, a condition that will reappear in connection with the persistence results established later. The local stability conditions for the second equilibrium are also easy to be violated, as shown in the examples in our last section.

3.3. Infinite-valued equilibria

Our original motivation to introduce a government sector was to prevent the economy from reaching the bad equilibrium (46) in the Keen model without government. Because this equilibrium is characterized by infinitely negative profits caused by explosive private debt, we focus on the cases where $\pi\to-\infty$.

Making the change of variable $u=e^{\pi}$, we obtain the system

$$\begin{aligned}\dot\omega&=\omega[\Phi(\lambda)-\alpha]\\\dot\lambda&=\lambda[\mu(\log u)-\alpha-\beta]\\\dot g_s&=g_s[\Gamma_s(\lambda)-\mu(\log u)]\\\dot\tau_s&=\tau_s[\Theta_s(\log u)-\mu(\log u)]\\\dot u&=u\left[-\omega(\Phi(\lambda)-\alpha)-r(\nu\mu(\log u)+\nu\delta-\log u)\right.\\&\quad+(1-\omega-\log u)\mu(\log u)+\Gamma_b(\lambda)-\Theta_b(\log u)\\&\quad\left.+g_s\Gamma_s(\lambda)-\tau_s\Theta_s(\log u)\right]\end{aligned}\tag{51}$$

The Jacobian matrix for this system is

$$\begin{bmatrix}\Phi(\lambda)-\alpha & \omega\Phi'(\alpha) & 0 & 0 & 0\\0 & \mu(\log u)-\alpha-\beta & 0 & 0 & \dfrac{\lambda\mu'(\log u)}{u}\\0 & g_s\Gamma_s'(\lambda) & \Gamma_s(\lambda)-\mu(\log u) & 0 & \dfrac{-g_s\mu'(\log u)}{u}\\0 & 0 & 0 & \Theta_s(\log u)-\mu(\log u) & \dfrac{\tau_s[\Theta_s'(\log u)-\mu'(\log u)]}{u}\\J_{5,1}(\lambda,u) & J_{5,2}(\lambda,g_s,u) & u\Gamma_s(\lambda) & -u\Theta_s(\log u) & J_{5,5}(\omega,\lambda,g_s,\tau_s,u)\end{bmatrix},$$

where

$$\begin{aligned}&J_{5,1}(u,\lambda)=-u(\Phi(\lambda)-\alpha+\mu(\log u))\\&J_{5,2}(u,\lambda,g_s)=\ u(g_s\Gamma_s'(\lambda)+\Gamma_b'(\lambda)-\omega\Phi'(\lambda))\\&J_{5,5}(\omega,\lambda,g_s,\tau_s,u)=-\log u[\mu(\log u)-r+\mu'(\log u)]\\&+\Gamma_b(\lambda)-\Theta_b(\log u)+r\left[1-\nu\mu(\log u)-\nu\delta-\nu\mu'(\log u)\right]\\&+g_s\Gamma_s(\lambda)-\tau_s\Theta_s(\log u)-\omega\left[\Phi(\lambda)-\alpha+\mu(\log u)\right.\\&\left.+\mu'(\log u)\right]+\mu'(\log u)-\Theta_b'(\log u)-\tau_s\Theta_s'(\log u)\end{aligned}$$

Recall that we assumed in (22) that the investment function κ has a horizontal asymptote $\kappa(-\infty)$ at $\pi\to-\infty$, implying that the growth rate μ defined in (15) also has a horizontal asymptote $\mu(-\infty)$. Similarly, assume that the taxation functions Θ_b and Θ_s satisfy

$$\lim_{\pi\to-\infty}\Theta_b(\pi)=\Theta_b(-\infty)\tag{52}$$

$$\lim_{\pi\to-\infty}\Theta_s(\pi)=\Theta_s(-\infty)<\mu(-\infty)\tag{53}$$

We then see that $(\omega,\lambda,g_s,\tau s,u)=(0,0,0,0,0)$ is an equilibrium point for (51), since all terms inside square brackets in the right-hand side of (51) approach constants as $u\to0^+$, with the exception of $\log u$, for which we have that $u\log u\to0$. Assuming further that

$$\Gamma_b,\Gamma_s\in C^1[0,1],\tag{54}$$

$$\lim_{\pi\to-\infty}\pi^2\Theta_s'(\pi)<\infty,\tag{55}$$

we have that the Jacobian at this equilibrium becomes a lower-triangular matrix and local stability is guaranteed if, in addition to the standard requirements (19), (22), (47), and the new condition (53), we impose that

$$\Gamma_s(0)<\mu(-\infty).\tag{56}$$

That is, the bad equilibrium $(\omega,\lambda,g_s,\tau_s,u)=(0,0,0,0,0)$ fails to be locally stable whenever condition (56) is violated, that is, whenever the rate of increase of government subsidies at zero employment is greater than the growth rate of the economy at infinitely negative profits. Notice that this is very easy to achieve in practice, since $\mu(-\infty)$ is in general very small. This constitutes another positive result regarding government intervention.

Unfortunately, this is not the only plausible equilibrium for the extended system (32) corresponding to the bad equilibrium (46) in the Keen model without government. Namely, allowing $\Gamma_s(0) \geq \mu(-\infty)$ in (51) gives rise to the possibility that $g_s \to \pm\infty$, depending on the initial condition $g_s(0)$. To investigate these other possibilities we make a second change of variables $v = 1/g_s$ (to analyze the behaviour of the system at large values of government subsidies) and $x = g_s/\pi$ (to analyze the relative size of government subsidies and firm profits), which leads to the modified system

$$\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\mu\left(\frac{1}{vx}\right) - \alpha - \beta\right] \\
\dot{v} &= v\left[\mu\left(\frac{1}{vx}\right) - \Gamma_s(\lambda)\right] \\
\dot{\tau}_s &= \tau_s\left[\Theta_s\left(\frac{1}{vx}\right) - \mu\left(\frac{1}{vx}\right)\right] \\
\dot{x} &= x\left[\Gamma_s(\lambda)(1-x) - r + vx(\omega(\Phi(\lambda)-\alpha)\right. \\
&\quad + rv\mu\left(\frac{1}{vx}\right) + rv\delta - (1-\omega)\mu\left(\frac{1}{vx}\right) + \Theta_b\left(\frac{1}{vx}\right) \\
&\quad \left. + \tau s\Theta_s\left(\frac{1}{vx}\right) - \Gamma_b(\lambda)\right)\Big].
\end{aligned} \tag{57}$$

We then see that $(\omega, \lambda, v, \tau_s, x) = (0, 0, 0^{\pm}, 0, 0^{\mp})$ are equilibria for (57) since all terms in the square brackets on the right-hand side of (57) approach constant values as $v \to 0^{\pm}$ and $x \to 0^{\mp}$. The associated Jacobian matrix for these equilibria is a lower-triangular matrix and their local stability is guaranteed by (19), (22), (52) and the new condition

$$\mu(-\infty) < \Gamma_s(0) < r. \tag{58}$$

If we assume that $\Gamma_s(0) \neq 0$, two other possible equilibria for (57) are given by

$$(\omega, \lambda, v, \tau_s, x) = \left(0, 0, 0^{\pm}, 0, \frac{\Gamma_s(0) - r}{\Gamma_s(0)}\right),$$

which are achievable provided either (i) $g_s(0) > 0$ and $\Gamma_s(0) < r$ (so that $v \to 0^+$ and $\pi \to -\infty$), or (ii) $g_s(0) < 0$ and $\Gamma_s(0) > r$ (so that $v \to 0^-$ and $\pi \to -\infty$). The associated Jacobian matrix for these equilibria is also lower-triangular and their local stability is guaranteed by (19), (22), (47), (52) and $\Gamma_s(0) > r$.

The case $\Gamma_s(0) = 0$ allows for the possible equilibria $(\omega, \lambda, v, \tau_s, x) = (0, 0, 0^{\pm}, 0, 0)$, depending on the sign of initial condition $g_s(0)$, but whose Jacobian matrix has zero as an eigenvalue, so that their local stability can never be guaranteed.

We summarize the different results for infinite-valued equilibria in the next proposition.

Proposition 1. *If, in addition to the standing assumptions (18)–(23), we have that (52)–(55) hold, then the following are the infinite-valued equilibria of (32):*

$$(\omega, \lambda, g_s, \tau_s, \pi) = (0, 0, 0, 0, -\infty) \tag{59}$$

$$(\omega, \lambda, g_s, \tau_s, \pi) = (0, 0, +\infty, 0, -\infty) \tag{60}$$

$$(\omega, \lambda, g_s, \tau_s, \pi) = (0, 0, -\infty, 0, -\infty) \tag{61}$$

Assuming that (19), (22), (47), (52) are satisfied, the stability of these equilibria depend on government subsidies g_s as follows:

(a) *When $g_s(0) > 0$ (stimulus):*
   (i) *if $\Gamma_s(0) < \mu(-\infty)$, then equilibrium (59) is locally stable, equilibrium (60) is achievable but unstable, and equilibrium (61) is unachievable.*
   (ii) *if $\mu(-\infty) < \Gamma_s(0) < r$, then equilibrium (59) is unstable, equilibrium (60) is achievable and locally stable, and equilibrium (61) is unachievable.*
   (iii) *if $r < \Gamma_s(0)$, then equilibrium (59) is unstable, equilibrium (60) is achievable and unstable, and equilibrium (61) is unachievable.*

(b) *When $g_s(0) < 0$ (austerity):*
   (i) *if $\Gamma_s(0) < \mu(-\infty)$, then equilibrium (59) is locally stable, equilibrium (60) is unachievable, and equilibrium (61) is achievable but unstable.*
   (ii) *if $\mu(-\infty) < \Gamma_s(0)$, then equilibrium (59) is unstable, equilibrium (60) is unachievable, and equilibrium (61) is achievable and locally stable.*

In other words, under a stimulus regime ($g_s(0) > 0$), any achievable equilibria with $\pi \to -\infty$ becomes unstable provided $\Gamma_s(0) > r$. On the other hand, under an austerity regime ($g_s(0) > 0$), there is no value of $\Gamma_s(0)$ that eliminates the possibility of local stability from all achievable equilibria with $\pi \to -\infty$. That is to say, austerity implies that the government cannot prevent the economy from remaining trapped in the basin of attraction of at least one of the bad equilibria, which is of course an undesirable outcome.

4. Persistence results

In this section we move beyond local equilibrium analysis to establish under which conditions government intervention can guarantee that the economy as represented by system (32) does not reach a state of permanently zero employment. In other words, we establish persistence for the system of differential equations (32) with respect to the employment λ. The relevant definitions are reviewed in Appendix C, illustrated by the simple example of the Goodwin model.

As it is typical in persistence analysis, although we are primarily interest in preventing the crisis situation characterized by $\pi \to -\infty$, our first result eliminates the possibility of exploding positive profits. This is necessary for technical reasons only, so the proof for this result is presented in Appendix D and can be skipped by the non-technical reader.

Proposition 2. *If $\tau_s(0) \geq 0$, and conditions (21) and (22) are satisfied, then the system described by (32) is $e^{-\pi}$-UWP, that is, there exists an $\varepsilon > 0$ such that $\limsup_{t\to\infty} e^{-\pi(t)} > \varepsilon$ for any initial conditions.*

Our core results are presented in the next two propositions. We first show that government intervention can achieve uniformly weak persistence of the functional e^{π}

even when the bad equilibrium for the model without government is locally stable.

Proposition 3. *Suppose that the structural conditions (18)–(23) and (28)–(29) are satisfied, along with the local stability condition (47) for the bad equilibrium of the Keen model (14) without government. Assume further that $g_s(0)>0$ and that condition (53) is satisfied. Then the model with government (32) is e^{π}-UWP if either of the following conditions is satisfied:*

(1) $\Gamma_s(0)>r$, *or*
(2) $\lambda\Gamma_b(\lambda)$ *is bounded below as* $\lambda\to 0$.

A few comments are in order before we present the proof for this result. Starting with the standing assumptions, conditions (18)–(20) are standard for the classical Goodwin (1967) model and simply guarantee that the employment rate satisfies $\lambda\in[0, 1]$. Likewise, conditions (21)–(23) are standard for the Keen (1995) model and simply guarantee that the equilibria described in Section 3.1 in the absence of government are well defined. Furthermore, condition (28) states that the rates of change of government subsidies (either base-level or stimulative) are decreasing functions of the employment rate λ, whereas condition (29) means that the rates of change of taxation (either base-level or stimulative) are increasing functions of the profit rate π. Condition (47) ensures that the *bad* equilibrium of the Keen model (14) without government is locally stable, which is the motivation to seek government intervention in the first place. The requirement $g_s(0)>0$ means that we are in the stimulus regime. This is because there is no hope of achieving persistence of e^{π} in the austerity regime, since at least one of the bad equilibria associated with $\pi\to-\infty$ will be locally stable according to the result established in Proposition 1. Finally, notice that condition (53) is always easy to be satisfied by choosing sufficiently large stimulative tax cuts when profits become infinitely negative. Next, to understand the meaning of this proposition recall that, according to the definitions of persistence in Appendix C, what is being claimed is that if either (1) or (2) above is satisfied, then there exist an $\varepsilon>0$ such that $e^{\pi(t)}>\varepsilon$ infinitely often as $t\to\infty$. In other words, although profits can get very negative, there exists an $m>0$ (possibly very large) such that $\pi(t)>-m$ infinitely often as $t\to\infty$. Our strategy to prove this is by contradiction. Namely we assume that for any given $m>0$ (no matter how large), the profit rate eventually satisfies $\pi(t)\leq -m$ and never returns to a level above $-m$. Notice that this is precisely what happens for initial conditions sufficiently close to the locally stable bad equilibrium in the Keen model without government. As a final comment, observe that both conditions (1) and (2) in the Proposition above refer to the size of government spending as the employment rate λ approaches zero. Condition (1) is a straightforward property and says that the rate of increase of stimulative subsidies at zero employment is larger than the real rate of interest. Alternatively, condition (2) requires that base government spending increases faster than $1/\lambda$ as $\lambda\to 0$. In practice, we expect condition (1) to be easier to implement, since base government spending is meant to be less reactive to short-term economic conditions than its stimulative counterpart.

Proof. We prove it by contradiction. If $\limsup_{t\to\infty}\pi(t)\leq -m$ for any given large $m>0$, there exists $t_0\geq 0$ such that $\pi(t)\leq -m$ for $t>t_0$. From the equation for $\dot\lambda$, it follows that

$$\lambda(t)\leq\lambda(t_0)e^{(t-t_0)(\mu(-m)-\alpha-\beta)},$$

for $t>t_0$. Choosing $m>0$ large enough so that $\mu(-m)<\alpha+\beta$ (recall condition (22)), we get that for any small $\varepsilon>0$, there exists $t_1>t_0$ such that $\lambda(t)<\varepsilon$ for $t>t_1$. From the equation for $\dot\omega$, this readily implies that

$$\omega(t)<\omega(t_1)e^{(t-t_1)(\Phi(\varepsilon)-\alpha)},$$

for $t>t_1$. Again, we may choose $\varepsilon>0$ sufficiently small that $\Phi(\varepsilon)<\alpha$ (recall conditions (18) and (19)). Hence, there exists $t_2>t_1>t_0$ such that $\omega(t)<\varepsilon$ for $t>t_2$. Finally, condition (53) guarantees that we can choose m large enough such that

$$\Theta_s(\pi)-\mu(\pi)<0,\quad\forall\pi\leq -m.$$

It then follows from the equation for $\dot\tau_s$ that there exists $t_3>t_2>t_1>t_0$ such that $\tau_s(t)<\varepsilon$ for $t>t_3$. In other words, we can bound ω, λ and τ_s by ε for t large enough.

At this point, we need to consider the hypothesis $\Gamma_s(0)>r$ and $\lambda\Gamma_b(\lambda)$ bounded separately. Assume first that $\Gamma_s(0)>r$. Since Γ is a decreasing function, we can immediately see from the equation for $\dot g_s$ that

$$\frac{\dot g_s}{g_s}=\Gamma_s(\lambda)-\mu(\pi)>\Gamma_s(\varepsilon)-\mu(\pi),$$

for $t>t_1$. Moreover, since $\Gamma_s(0)>r>\mu(-\infty)$ (see condition (47)), we can choose ε small enough and/or m big enough such that $\Gamma_s(\varepsilon)>\mu(-m)$. Accordingly, for any $t>s>t_1$, we have that

$$g_s(t)>g_s(s)e^{(t-s)[\Gamma_s(\varepsilon)-\mu(-m)]}.$$

Using the equation for $\dot\pi$ we have:

$$\begin{aligned}\dot\pi &= -\omega[\Phi(\lambda)-\alpha]-r[\nu\mu(\pi)+\nu\delta-\pi]+(1-\omega-\pi)\\ &\quad\times\mu(\pi)+\Gamma_b(\lambda)+g_s\Gamma_s(\lambda)-\Theta_b(\pi)-\tau_s\Theta_s(\pi)\\ &= -\omega[\Phi(\lambda)-\alpha]-r\kappa(\pi)+\pi(r-\mu(\pi))+(1-\omega)\mu(\pi)\\ &\quad+\Gamma_b(\lambda)+g_s\Gamma_s(\lambda)-\Theta_b(\pi)-\tau_s\Theta_s(\pi)\\ &> -r\max\{|\kappa(-\infty)|,|\kappa(-m)|\}+\pi(r-\mu(-\infty))\\ &\quad-\max\{|\mu(-\infty)|,|\mu(-m)|\}+\Gamma_b(\varepsilon)\\ &\quad+\Gamma_s(\varepsilon)g_s(t_3)e^{(t-t_3)[\Gamma_s(\varepsilon)-\mu(-m)]}-\Theta_b(-m)\\ &\quad-\varepsilon\max\{|\Theta_s(-\infty)|,|\Theta_s(-m)|\}\\ &= C+A\pi+De^{Et}\end{aligned}\tag{62}$$

where C is finite and does not depend on t, $A=r-\mu(-\infty)>0$, $D=\Gamma_s(\varepsilon)g_s(t_3)e^{-t_3(\Gamma_s(\varepsilon)-\mu(-m))}>0$ and $E=\Gamma_s(\varepsilon)-\mu(-m)>0$. Consequently, for $t>t_3$, we have that $\pi(t)>y(t)$, where $y(t)$ is the solution of

$$\dot y=C+Ay+De^{Et},\quad y(t_3)=\pi(t_3),\tag{63}$$

that is,

$$y(t) = y(t_3)e^{A(t-t_3)} + \frac{l}{A}(e^{A(t-t_3)} - 1) + \frac{D}{E-A}e^{Et_3} \times (e^{E(t-t_3)} - e^{A(t-t_3)}). \quad (64)$$

At last, since $\Gamma_s(0) > r$, we can choose ε sufficiently small and m sufficiently large such that

$$E - A = \Gamma_s(\varepsilon) - r + \mu(-\infty) - \mu(-m) > 0,$$

which leads us to conclude that e^{Et} dominates the solution $y(t)$ when $t \to \infty$, that is,

$$\lim_{t\to\infty} y(t) = \frac{D}{E-A}e^{Et} = +\infty.$$

Yet, since $\pi(t) > y(t)$ for $t > t_3$, we must have also $\pi(t) \xrightarrow{t\to\infty} +\infty$, which contradicts the fact that $\pi(t) \le -m$ for $t > t_0$.

Alternatively, assume now that $\lambda\Gamma_b(\lambda)$ is bounded from below as $\lambda \to 0$. We can still bound ω, λ and τ_s by ε for t large enough as before. Moreover, since $\lambda\Gamma_b(\lambda) > L$ for some positive L as $\lambda \to 0$, we now have that $\Gamma_b(\lambda) > \Gamma_b(\lambda)\lambda/\varepsilon > L/\varepsilon$. From the equation for $\dot{\pi}$ we then have

$$\begin{aligned}\dot{\pi} &= -\omega[\Phi(\lambda) - \alpha] - r[\nu\mu(\pi) + \nu\delta - \pi] \\ &\quad + (1 - \omega - \pi)\mu(\pi) + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_b(\pi) - \tau_s\Theta_s(\pi) \\ &= -\omega[\Phi(\lambda) - \alpha] - r\kappa(\pi) + \pi(r - \mu(\pi)) + (1 - \omega)\mu(\pi) \\ &\quad + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_b(\pi) - \tau_s\Theta_s(\pi) > -r\max\{|\kappa(-\infty)|, \\ &\quad |\kappa(-m)|\} + \pi(r - \mu(-\infty)) - \max\{|\mu(-\infty)|, |\mu(-m)|\} \\ &\quad + L/\varepsilon - \Theta_b(-m) - \varepsilon\max\{|\Theta_s(-\infty)|, |\Theta_s(-m)|\} \\ &= \tilde{C}(\varepsilon) + \tilde{A}\pi,\end{aligned} \quad (65)$$

where $\tilde{C}$ can be made arbitrarily large by choosing ε sufficiently small, while $\tilde{A} = r - \mu(-\infty) > 0$. Therefore, for $t > t_3$, we have that $\pi(t) \ge y(t)$, where $y(t)$ is now the solution of

$$\dot{y}(t) = \tilde{C} + \tilde{A}y, \quad y(t_3) = \pi(t_3),$$

that is,

$$y(t) = \frac{\left(\tilde{C}(\varepsilon) + \tilde{A}y(t_3)\right)e^{\tilde{A}(t-t_3)} - \tilde{C}}{\tilde{A}}.$$

We can then choose ε small enough such that $\tilde{C}(\varepsilon) + \tilde{A}y(t_3) > 0$ and hence $\lim_{t\to\infty} y(t) = +\infty$. But this implies that $\pi(t) \xrightarrow{t\to\infty} +\infty$, which again contradicts the fact that $\pi(t) \le -m$ for $t > t_0$. □

Although profits play a key role in the model, from the point of view of economic policy, arguably the most important variable in (32) is the rate of employment. Our next and final result shows that under slightly stronger conditions we can still obtain uniformly weak persistence with respect to the functional λ itself. Before stating it, define the function

$$h(x) = -r[\nu\mu(x) + \nu\delta - x] + (1 - x)\mu(x) + \Gamma_b(0) - \Theta_b(x), \quad (66)$$

and observe that it has the properties:

(i) $h(\overline{\pi}_1) = \overline{\omega}_1(\alpha + \beta) + \Gamma_b(0) - \Gamma_b(\overline{\lambda}_1) > 0$,
(ii) $\lim_{x\to\pm\infty} h(x) = -\infty$, and
(iii) $\max[h(\pi)] < +\infty$.

Proposition 4. *Suppose that the structural conditions (18)–(23) and (28)–(29) are satisfied, along with the local stability condition (47) for the bad equilibrium of the Keen model (14) without government. Assume further that $g_s(0) > 0$ and that condition (53) is satisfied. Then the system (32) is λ-UWP if either of the following four conditions is satisfied:*

(1) *$\tau_s(0) = 0$ and $\Gamma_s(0) > \max\{r, \alpha + \beta\}$, or*
(2) *$\tau_s(0) = 0$ and $\lambda\Gamma_b(\lambda)$ is bounded below as $\lambda \to 0$, or*
(3) *$\tau_s(0) = 0$, $r < \Gamma_s(0) \le \alpha + \beta$, and $h(x) > 0$ whenever $\mu(x) \in [\Gamma_s(0), \alpha + \beta]$, or*
(4) *$\Gamma_s(0) > \max\{r, \alpha + \beta\}$, $\Theta_s(-\infty) < 0$, $\Theta_s(\overline{\pi}_1) < \alpha + \beta$, and Θ_s is convex.*

Let us again make a few comments before proving this result. All the standing assumptions for this proposition are identical to those of Proposition 3. The difference in hypotheses between the two propositions rests on the alternative sufficient conditions (1)–(4). First of all, items (1)–(3) above assume from the outset that $\tau_s(0) = 0$ (no stimulative taxes), whereas Proposition 3 is valid for any value of $\tau_s(0)$. Apart from this, the condition on base-level government subsidies expressed in item (2) above is identical to that in Proposition 3. As for stimulative subsidies, condition (1) above is slightly stronger than condition (1) in Proposition 3 by requiring that $\Gamma_s(0) > \alpha + \beta$ whenever $r < \alpha + \beta$. Item (3) then offers an alternative condition when $r < \Gamma_s(0) \le \alpha + \beta$ instead. Finally, the difficult case to prove is when $\tau_s(0) > 0$ and is covered in item (4) for completeness, in view of the experience of a few European countries (e.g., France) which opted to increase taxes in the middle of a recession in an effort to decrease government deficits. It says that the result still holds in this case, provided government subsidies are as large as in item (1) and the rate of change of tax increases is convex and satisfies some bounds at specific points. As before, to understand the meaning of the result, recall that uniform weak persistence in λ means that there exists and $\epsilon > 0$ such that $\lambda(t) > \epsilon$ infinitely often at $t \to \infty$. Our strategy to establish this is again to assume the contrary, namely that for any $\varepsilon > 0$ the employment rate eventually satisfies $\lambda(t) \le \varepsilon$ and never rises above this level again, as is the case for initial conditions sufficiently close to the locally stable bad equilibrium in the Keen model without government.

Proof. We prove the result by contradiction again. If $\limsup_{t\to\infty} \lambda(t) \le \varepsilon$ for any $\varepsilon > 0$, then there exists $t_0 > 0$ such that $\lambda(t) \le \varepsilon$ for $t > t_0$. Since we can always choose ε small enough so that $\Phi(\varepsilon) - \alpha < 0$, it follows from the equation for $\dot{\omega}$ as before that there exists $t_1 > t_0$ such that $\omega(t) < \epsilon$ for all $t > t_1$.

For items (1) and (2), observe that it follows from UWP of e^{π} obtained in Proposition 3 that we can find a large $m_1 > 0$ such that $\limsup_{t\to\infty} \pi(t) > -m_1$. In addition, we have that $\liminf_{t\to\infty}\pi < \mu^{-1}(\alpha+\beta) = m_2$, since otherwise λ cannot converge to zero and there is nothing left to prove. Let $m = \max\{m_1, m_2\}$.

If $\Gamma_s(0) > \max\{r, \alpha+\beta\}$, we see from the equation for $\dot{\lambda}$ that

$$\exp\left[\int_{t_1}^{t}\mu(\pi_s)ds\right] < \frac{\varepsilon}{\lambda(t_1)}e^{(\alpha+\beta)(t-t_1)} \quad \forall t > t_1,$$

which implies that

$$g_s(t) > \frac{\lambda(t_1)g_s(t_1)}{\varepsilon}\exp\left[\left(\Gamma_s(\varepsilon)-(\alpha+\beta)\right)(t-t_1)\right] \quad \forall t > t_1$$

In other words, given any large $L > 0$, provided we choose ε sufficiently small so that $\Gamma_s(\varepsilon) > \alpha+\beta$, there exists $t_2 > t_1$ such that $g_s(t) > L$ for $t > t_2$. Alternatively, if $\lambda\Gamma_b(\lambda)$ is bounded below as $\lambda\to 0$, given any large $L > 0$, we can choose ε sufficiently small so that $\Gamma_b(\lambda) > L$ for $\lambda < \varepsilon$ (since $\Gamma_b(\lambda) > L_0/\lambda > L_0/\varepsilon$ for some $L_0 > 0$, just choose $\varepsilon \le L_0/L$).

In either case, we can find $\varepsilon > 0$ small enough and/or $t_2 > t_1$ such that

$$-\omega[\Phi(\lambda)-\alpha] - r[\nu\mu(\pi)+\nu\delta-\pi] + (1-\omega-\pi)\mu(\pi) + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_b(\pi) > \varepsilon \quad (67)$$

for all $\omega\in[0, \varepsilon]$, $\lambda\in[0, \varepsilon]$, $\pi\in[-m, m]$ and $t > t_2$. Since $\limsup\pi > -m$ and $\liminf\pi < m$, we can find $t_3 > t_2$ such that $\pi(t_3)\in(-m, m)$, from which it follows from (67) and the equation for $\dot{\pi}$ that $\dot{\pi}(t_3) > 0$. Furthermore, $\dot{\pi}(t) > 0$ for all $t > t_3$ with $\pi(t) \le m$. Hence, there exists $t_4 > t_3$ such that $\pi(t_4) = m$ and $\pi(t) > m$ for all $t > t_4$. But this contradicts the fact $\liminf\pi < m$, and UWP of λ follows.

For item (3), we can again find a sufficiently small ε and a sufficiently large $t_0 > 0$ such that $\omega(t) < \varepsilon$ and $\lambda(t) < \varepsilon$ for all $t > t_0$, and

$$-\omega[\Phi(\lambda)-\alpha] - r[\nu\mu(\pi)+\nu\delta-\pi] + (1-\omega-\pi)\mu(\pi) + \Gamma_b(\varepsilon) + g_s\Gamma_s(\varepsilon) - \Theta_b(\pi) > \varepsilon$$

for all $\omega\in[0, \varepsilon]$, $\lambda\in[0, \varepsilon]$ and π in the interval such that $\Gamma_s(0) \le \mu(\pi) \le \alpha+\beta$. We use the fact that $\Gamma_s(0) > r$, which implies $e^{\pi} - UWP$, to obtain that π does enter the interval $[-m, m]$, for some large $m \ge \mu^{-1}(\alpha+\beta)$, at some instant $t_1 > t_0$. But since $\dot{\pi}(t) > \varepsilon$ whenever $\pi(t)$ lies in the interval such that $\Gamma_s(0) \le \mu(\pi) \le \alpha+\beta$, this in turn implies that $-m < \pi < \mu^{-1}(\Gamma_s(0))$ for all $t > t_1$, because otherwise $\pi > \mu^{-1}(\alpha+\beta)$ for all large t and $\lambda(t)$ could not go to zero. However, $\mu(\pi) < \Gamma_s(0)$ for all large t implies that $g_s(t)$ can be made arbitrarily large and we have that (67) holds, which again leads to a contradiction.

The proof of item (4) is presented in Appendix E. □

5. Examples

In this section, we compare the results obtained for a Keen model without government described by (14) with the model with government described by (32) under several different scenarios. For the basic parameter values we fix the capital-to-output ratio ν, the rate of productivity growth α, the rate of population growth β, the depreciation rate δ =, and the real short-term interest rate on for private debt r at the following values

$$\nu = 3, \quad \alpha = 0.025, \quad \beta = 0.02, \quad \delta = 0.01, \quad r = 0.03.$$

In addition, for the Keen model without government we use the functions

$$\Phi(\lambda) = \frac{\phi_1}{(1-\lambda)^2} - \phi_0 \quad (68)$$

$$\kappa(\pi) = \kappa_0 + \kappa_1\arctan(\kappa_2\pi + \kappa_3) \quad (69)$$

with parameter values given in Appendix F. We can easily verify that all the structural conditions (18)–(23) are satisfied for these functions. Moreover, we have that

$$r\left[\frac{\kappa'(\overline{\pi}_1)}{\nu}\left(\overline{\pi}_1 - \kappa(\overline{\pi}_1) + \nu(\alpha+\beta)\right) - (\alpha+\beta)\right] = 0.00515 > 0 \quad (70)$$

so that (45) is satisfied and the good equilibrium

$$(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1) = (0.83667, 0.96, 0.11111) \quad (71)$$

is locally stable. Finally, observe that

$$\frac{\kappa(-\infty)}{\nu} - \delta = -0.01 < 0.03 = r \quad (72)$$

so that (47) is satisfied and the bad equilibrium $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2) = (0, 0, +\infty)$ is also locally stable. That is, for these parameters, in the absence of government intervention, the economy converges to either the good or the bad equilibrium depend on how close to them we chose the initial conditions.

For the model (32) with government, we use functions

$$\Gamma_b(\lambda) = \gamma_0(1-\lambda) \quad (73)$$

$$\Gamma_s(\lambda) = \gamma_1 - \gamma_2\lambda^{\gamma_3} \quad (74)$$

$$\Theta_b(\pi) = \theta_0 + \theta_1 e^{\theta_2\pi} \quad (75)$$

$$\Theta_s(\pi) = \theta_3 + \theta_4 e^{\theta_5\pi} \quad (76)$$

$$g_e(\pi, \lambda) = (1-\kappa(\pi))(1-\lambda)^{\gamma_4} \quad (77)$$

with parameter values given in Appendix F. Observe that we specified government expenditures directly through the function $g_e(\pi, \lambda)$ above, instead of equivalently defining it as the solution of (38).

We can again easily verify that the structural conditions (52)–(55) are satisfied. We can also verify that conditions (B.4)–(B.7) are also satisfied, so that the good equilibrium

$$(\omega, \lambda, g_s, \tau_s, \pi) = (\overline{\omega}_1, \overline{\lambda}_1, 0, 0, \overline{\pi}_1) = (0.76067, 0.96, 0, 0, 0.16)$$

is locally stable. Moreover, we can verify that the conditions for stability of the other finite-valued equilibria in (50) are easily violated for our choice of parameters, so that none of them is locally stable.

As we have seen in Proposition 1, the stability of the infinite-valued equilibria in the presence of government intervention depends crucially on the parameter $\Gamma_s(0) = \gamma_1$

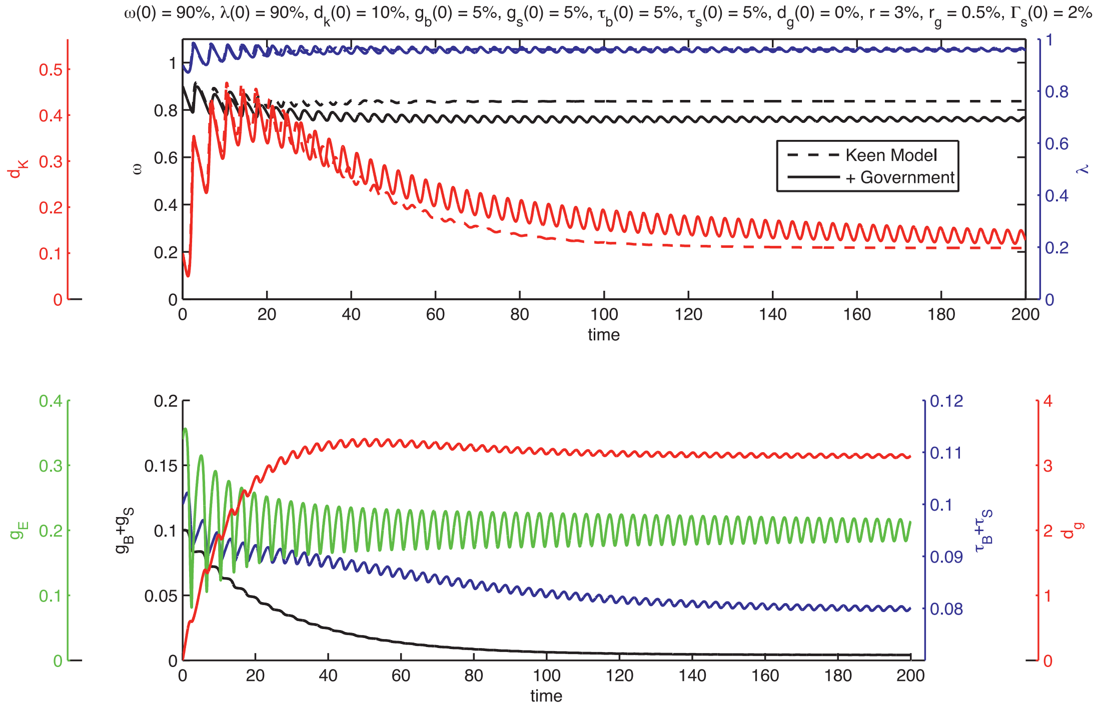

Fig. 1. Solution to the Keen model with and without a timid government for good initial conditions.

corresponding to the maximum value of the discretionary subsidy function above. In what follows we set

$$\gamma_1 = \begin{cases} 0.02 & \text{for a timid government,} \\ 0.20 & \text{for a responsive government.} \end{cases} \tag{78}$$

It then follows from item (a) of Proposition 1 that in a stimulus regime, namely for initial conditions with $g_s(0) > 0$, equilibrium (59) is unstable in either case, whereas equilibrium (60) is stable in the case of a timid government but unstable in the case of a responsive government. On

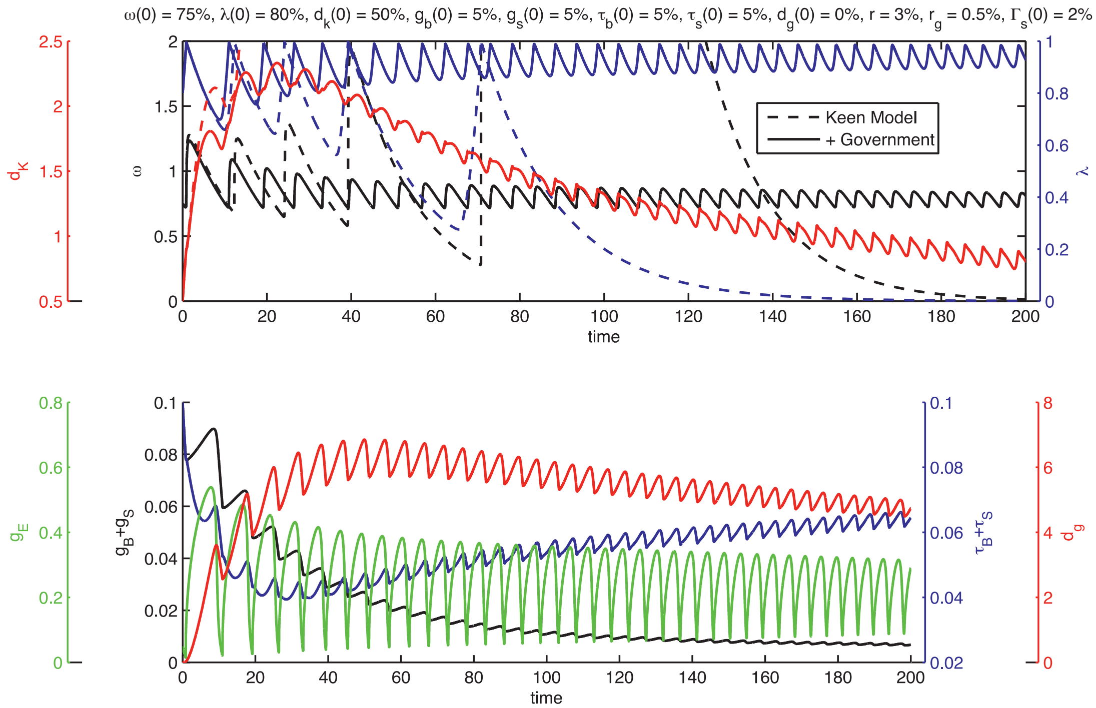

Fig. 2. Solution to the Keen model with and without a timid government with bad initial conditions.

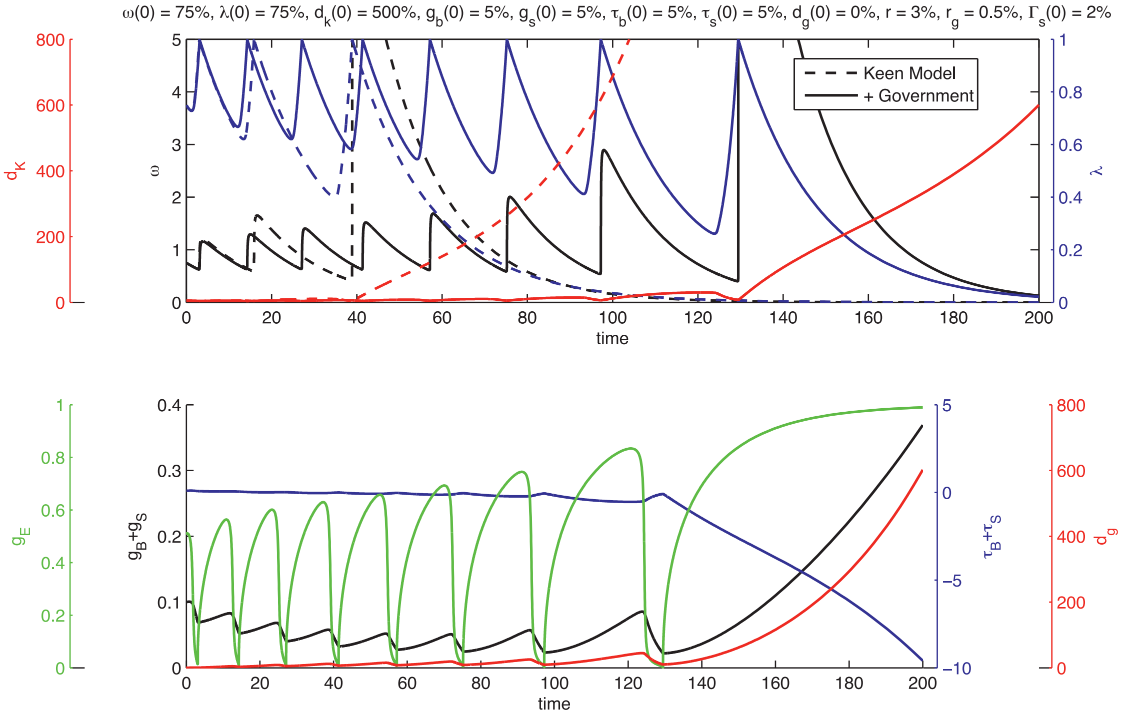

Fig. 3. Solution to the Keen model with and without a timid government with extremely bad initial conditions.

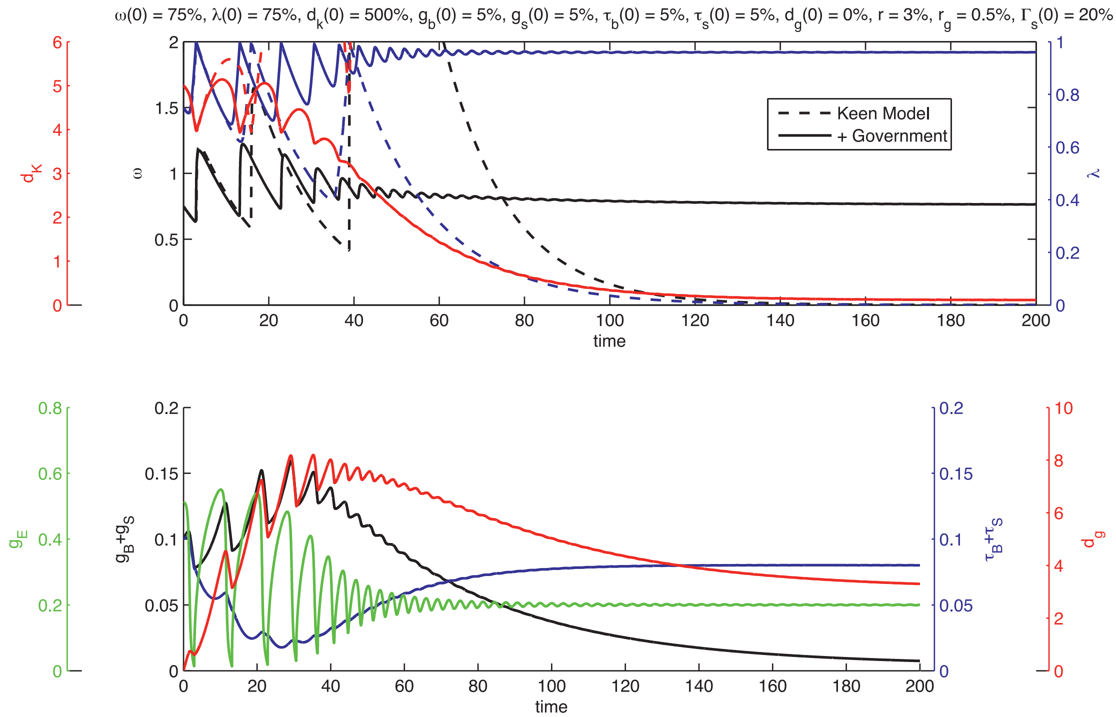

Fig. 4. Solution to the Keen model with and without a responsive government with extremely bad initial conditions.

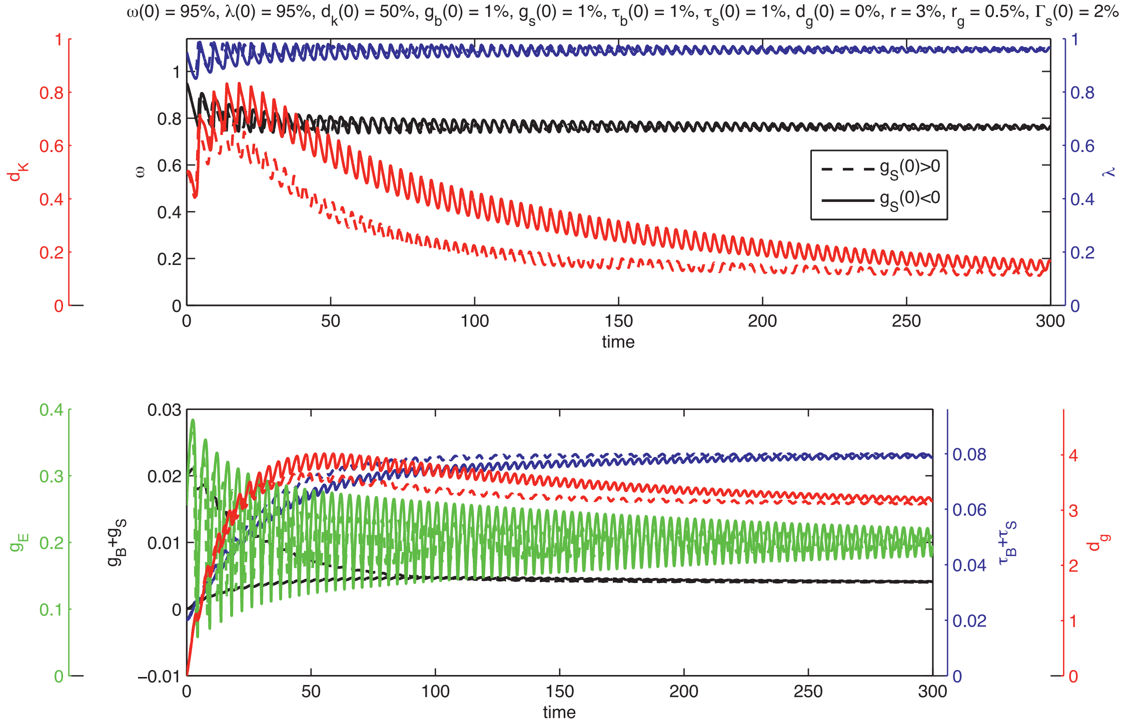

Fig. 5. Solution to the Keen model, starting close the good equilibrium point, with positive (stimulus) and negative (austerity) government spending.

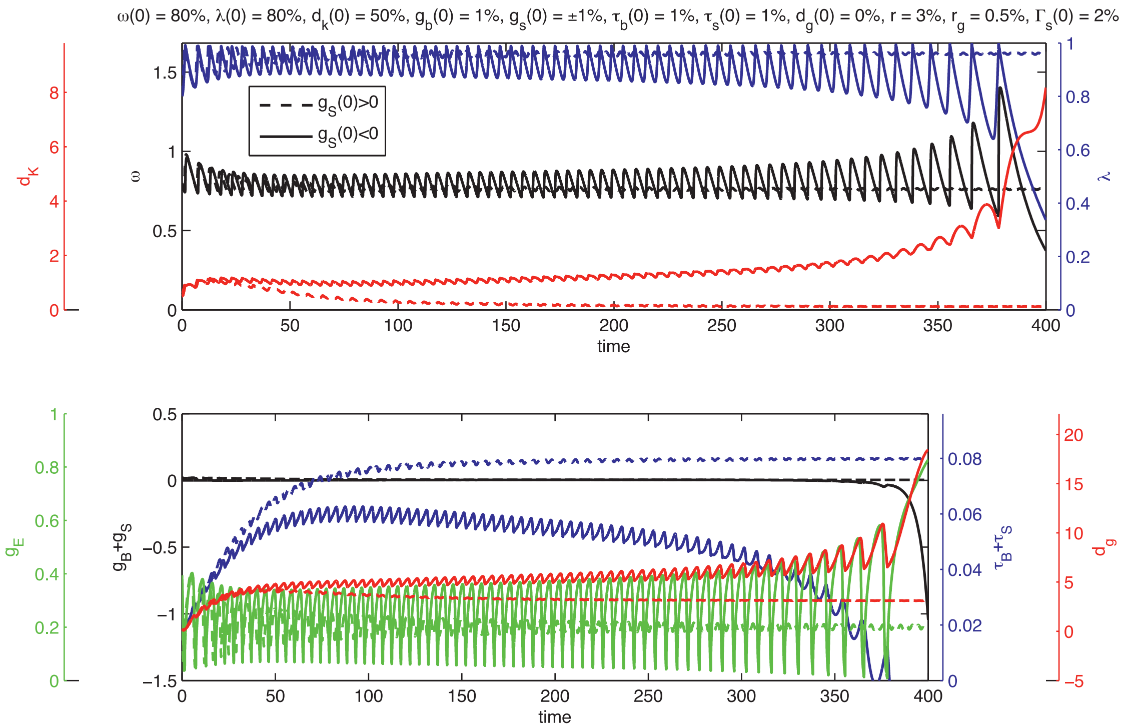

Fig. 6. Solution to the Keen model, starting far from the good equilibrium point, with positive (stimulus) and negative (austerity) government spending.

the other hand, it follows from item (b) that in an austerity regime, that is for initial conditions with $g_s(0) < 0$, equilibrium (61) is locally stable in either case.

Moving to the persistence results in Section 4, observe that condition (1) of Proposition 3 is satisfied in the case of a responsive government, but that neither conditions in this proposition are satisfied in the case of a timid government. As a result, provided $g_s(0) > 0$, the responsive government above ensures uniformly weakly persistence with respect to e^{π}, but the timid government does not.

Similarly, we can verify that condition (4) of Proposition 4 is satisfied by our responsive government even when $\tau_s(0) > 0$, but none of the conditions in this proposition are satisfied by the timid government. Consequently, provided $g_s(0) > 0$, the responsive government above ensures uniformly weakly persistence with respect to λ, but the timid government does not.

We illustrate these results in the next six figures. Choosing benign initial conditions, that is to say, high wage share (90% of GDP), high employment rate (90%), and low private debt (10% of GDP), we see in Fig. 1 that the economy eventually converges to the corresponding good equilibrium with or without government intervention, even in the case of a timid government.

As we move to worse initial conditions, that is lower wage share (75% of GDP), lower employment rate (80%), and higher private debt (50% of GDP), we see in Fig. 2 that the "free economy" represented by the model without government eventually collapses to the bad equilibrium of zero wage share, zero employment and infinite private debt, whereas the model with a timid government is more robust and eventually converges to the good equilibrium.

A timid government, however, is not capable of saving the economy from a crash if the initial conditions are too extreme, for example a low wage share (75% of GDP), low employment rate (75%) and extremely high level of private debt (500% of GDP), as shown in Fig. 3. On the other hand, a responsive government, effectively brings the economy from the severe crisis induced by this extremely bad initial conditions, as shown in Fig. 4).

Additionally, the effects of austerity measures are exemplified in Figs. 5 and 6. For a healthy initial state, we see that the transient period suffers from the negative spending, compared to a positive stimulus, without any long term consequences. Once we push the initial state further away from the good equilibrium, we can immediately verify the disastrous consequences of austerity: the government focuses so much on reducing public debt that it throws the economy into recession.

6. Concluding remarks

We proposed a macroeconomic model in which government intervention has a clear positive effect in preventing a crises characterized by collapsing employment rates. In the absence of government, the dynamics of the model is primarily driven by the investment decisions of capitalists based on profit levels: high profits lead to high investment and economic expansion financed by increasing private debt levels. This can lead either to an equilibrium with a finite private debt to output ratio or to another equilibrium where this ratio becomes infinite while the wage share and the employment rate both collapse to zero. Government intervention prevents this outcome by putting a floor under profit levels.

The model without government is essentially that proposed in Keen (1995) and further analyzed in Grasselli and Costa Lima (2012). However, the extended model with government presented in Keen (1995) does not differentiate between direct subsidies to firms and government expenditures in goods and services. The distinction is important because the former affects the profit share π in (30), whereas the latter influences it only directly through a increased output. This was later partially remedied in Keen (1998), where government spending is restricted to subsidies, resulting in government debt not fully reflecting all government transactions. By contrast, we explicitly model both government subsidies in (26) and expenditures in (37). Expectedly, because of their direct appearance in the profit Eq. (30), government subsidies to firms play a far more important role in the model than government expenditures on goods and services. Somewhat unexpectedly, the actual functional form of government expenditures can be very general without altering the results at all: the only requirement in (37) is that is cannot depend explicitly on the level of government debt. This relative unimportance of government consumption is partially explained by the fact that, in this formulation of the model, consumption by households is an accommodating variable determined by equation (41). It is likely that, in more general formulations with an independent specification of household saving propensity, government consumption plays a more important role in maintaining aggregate demand, with the overall role of government intervention in preventing a collapse in profits remaining the same as in this paper.

Our first positive result is the local stability conditions for all finite-valued equilibria associated with zero wage shares in (50) are easily violated. But this was also the case for both the Keen model without government and for its predecessor, the Goodwin model. The next result, however, is truly novel: all equilibria with zero wage share and employment arising from infinitely negative profit shares can be made either unstable or unachievable by moderately high government subsidies at very low employment. This is in contrast with the situation without government, where the corresponding bad equilibrium is locally stable for typical parameter values.

The core persistence results of Section 4 are much stronger: government intervention, in the form of responsive enough subsidy and taxation policies, prevent the economy from remaining permanently at arbitrarily low levels of employment regardless of the initial conditions of the system. It may be that stabilizing an unstable economy is too tall an order for the government sector, but destabilizing a stable crises is perfectly possible.

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Appendix A. Other finite-valued equilibria with government

1. Take $\overline{\omega}_2 = 0$ and $\overline{\pi}_2 = \overline{\pi}_1$ so that $\dot{\omega} = \dot{\lambda} = 0$. In this case, discarding the structural coincidence $\Theta_s(\overline{\pi}_1) = \alpha + \beta$, the only way to obtain $\dot{\tau}_s = 0$ is to set $\overline{\tau}_{s2} = 0$. For the remaining variables we define

$$\overline{\lambda}_2 = \Gamma_s^{-1}(\alpha + \beta) \tag{A.1}$$

so that $\dot{g}_s = 0$ and

$$\overline{g}_{s2} = \frac{\Theta_b(\overline{\pi}_1) - \Gamma_b(\overline{\lambda}_2)}{\alpha + \beta} + \frac{r(\nu\mu(\overline{\pi}_1) + \nu\delta - \overline{\pi}_1)}{\alpha + \beta} - (1 - \overline{\pi}_1) \tag{A.2}$$

so that $\dot{\pi} = 0$.

2. Take $\overline{\omega}_3 = \overline{\tau}_{s3} = 0$ and $\overline{\pi}_3 = \overline{\pi}_1$ and so that $\dot{\omega} = \dot{\lambda} = \dot{\tau}_s = 0$ as before. In addition take $\overline{g}_{s3} = 0$ so that $\dot{g}_s = 0$. To obtain $\dot{\pi} = 0$ define

$$\overline{\lambda}_3 = \Gamma_b^{-1}\left(r\left(\nu\mu(\overline{\pi}_1) + \nu\delta - \overline{\pi}_1\right) - (1 - \overline{\pi}_1)\left(\alpha + \beta\right) + \Theta_b(\overline{\pi}_1)\right). \tag{A.3}$$

3. Take $\overline{\omega}_4 = \overline{\lambda}_4 = \overline{g}_{s4} = \overline{\tau}_{s4} = 0$ so that $\dot{\omega} = \dot{\lambda} = \dot{g}_s = \dot{\tau}_s = 0$. To obtain $\dot{\pi} = 0$ define $\overline{\pi}_4$ as the solution of

$$-r(\nu\mu(\pi) + \nu\delta - \pi) + (1 - \pi)\mu(\pi) + \Gamma_b(0) - \Theta_b(\pi) = 0. \tag{A.4}$$

4. Take $\overline{\omega}_5 = \overline{\lambda}_5 = \overline{g}_{s5} = 0$ so that $\dot{\omega} = \dot{\lambda} = \dot{g}_s = 0$. To obtain $\dot{\tau}_s = 0$ define $\overline{\pi}_5$ as the solution of

$$\Theta_s(\pi) - \mu(\pi) = 0. \tag{A.5}$$

Finally, to obtain $\dot{\pi} = 0$ set

$$\overline{\tau}_{s5} = \frac{-r(\nu\mu(\overline{\pi}_5) + \nu\delta - \overline{\pi}_5) + (1 - \overline{\pi}_5)\Theta_s(\overline{\pi}_5) + \Gamma_b(0) - \Theta_b(\overline{\pi}_5)}{\Theta_s(\overline{\pi}_5)}. \tag{A.6}$$

5. Take $\overline{\omega}_6 = \overline{\lambda}_6 = 0$ so that $\dot{\omega} = \dot{\lambda} = 0$. To obtain $\dot{g}_s = 0$, define

$$\overline{\pi}_6 = \mu^{-1}\left(\Gamma_s(0)\right). \tag{A.7}$$

Provided we discard again the structural coincidence $\Theta_s(\overline{\pi}_6) = \Gamma_s(0)$, this means that to obtain $\dot{\tau}_s = 0$ we must set $\overline{\tau}_{s6} = 0$. For the remaining variable we take

$$\overline{g}_{s6} = \frac{r(\nu\mu(\overline{\pi}_6) + \nu\delta - \overline{\pi}_6) - (1 - \overline{\pi}_6)\Gamma_s(0) - \Gamma_b(0) + \Theta_b(\overline{\pi}_6)}{\Gamma_s(0)} \tag{A.8}$$

so that $\dot{\pi} = 0$.

Appendix B. Local stability of finite-value equilibria with government

We begin our local stability analysis by determining the Jacobian matrix for the system (32):

$$\begin{bmatrix} \Phi(\lambda) - \alpha & \omega\Phi'(\lambda) & 0 & 0 & 0 \\ 0 & \mu(\pi) - \alpha - \beta & 0 & 0 & \lambda\mu'(\pi) \\ 0 & g_s\Gamma_s'(\lambda) & \Gamma_s(\lambda) - \mu(\pi) & 0 & -g_s\mu'(\pi) \\ 0 & 0 & 0 & \Theta_s(\pi) - \mu(\pi) & -\tau_s\mu'(\pi) \\ \alpha - \Phi(\lambda) - \mu(\pi) & \begin{matrix} -\omega\Phi'(\lambda) + \Gamma_b'(\lambda) \\ +g_s\Gamma_s'(\lambda) \end{matrix} & \Gamma_s(\lambda) & -\Theta_s(\pi) & \begin{matrix} r - \mu(\pi) \\ +\mu'(\pi)(1 - \omega - \pi - r\nu) \\ -\left(\Theta_b'(\pi) + \tau_s\Theta_s'(\pi)\right) \end{matrix} \end{bmatrix} \tag{B.1}$$

Returning to the equilibria defined in (50), we have the following cases:

1. Defining the constant

$$K = r + \mu'(\overline{\pi}_1)(1 - \overline{\pi}_1 - r\nu) - \left(\alpha + \beta\right) - \Theta_b'(\overline{\pi}_1) \tag{B.2}$$

the characteristic polynomial for the Jacobian matrix (B.1) at the good equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, 0, 0, \overline{\pi}_1)$ can be written as

$$\begin{aligned}\overline{p}_1(y) = {} & \left[-y^3 + y^2(K - \overline{\omega}_1\mu'(\overline{\pi}_1)) + y\overline{\lambda}_1\mu'(\overline{\pi}_1)\right. \\ & \left.\left(\Gamma_b'(\overline{\lambda}_1) - \overline{\omega}_1\Phi'(\overline{\lambda}_1)\right) - (\alpha + \beta)\overline{\lambda}_1\mu'(\overline{\pi}_1)\overline{\omega}_1\Phi'(\overline{\lambda}_1)\right] \\ & \times \left(\Gamma_s(\overline{\lambda}_1) - (\alpha + \beta) - y\right)\left(\Theta_s(\overline{\pi}_1) - (\alpha + \beta) - y\right)\end{aligned} \tag{B.3}$$

This equilibrium will be locally stable if and only if the polynomial (B.3) has only roots with negative real part. We can identify two of the real roots to be $\Gamma_s(\overline{\lambda}_1) - (\alpha + \beta)$ and $\Theta_s(\overline{\pi}_1) - (\alpha + \beta)$. The Routh-Hurwitz criterion gives us the remaining necessary and sufficient conditions for stability:

$$\Gamma_s(\overline{\lambda}_1) < \alpha + \beta \tag{B.4}$$

$$\Theta_s(\overline{\pi}_1) < \alpha + \beta \tag{B.5}$$

$$\overline{\omega}_1 > 0 \tag{B.6}$$

$$\frac{\overline{\omega}_1\mu'(\overline{\pi}_1) - K}{\alpha + \beta} > \frac{\overline{\omega}_1\Phi'(\overline{\lambda}_1)}{\overline{\omega}_1\Phi'(\overline{\lambda}_1) - \Gamma_b'(\overline{\lambda}_1)} \tag{B.7}$$

2. The characteristic polynomial at the equilibrium $(0, \overline{\lambda}_2, \overline{g}_{s2}, 0, \overline{\pi}_1)$ is

$$\begin{aligned}\overline{p}_2(y) = {} & \left(\Phi(\overline{\lambda}_2) - \alpha - y\right)(\Theta_s\left(\overline{\pi}_1\right) - \left(\alpha + \beta\right) - y)\{-y^3 + Ky^2 \\ & + \mu'(\overline{\pi}_1)\left[\overline{\lambda}_2\left(\Gamma_b'(\overline{\lambda}_2) + \overline{g}_{s2}\Gamma_s'(\overline{\lambda}_2)\right) - \overline{g}_{s2}\left(\alpha + \beta\right)\right]y \\ & + \left(\alpha + \beta\right)\overline{\lambda}_2\overline{g}_{s2}\mu'(\overline{\pi}_1)\Gamma_s'(\overline{\lambda}_2)\}\end{aligned} \tag{B.8}$$

It follows that this equilibrium is locally stable if and only if the following conditions are satisfied:

$$\Phi(\bar{\lambda}_2) < \alpha \tag{B.9}$$

$$\Theta_s(\bar{\pi}_1) < \alpha + \beta \tag{B.10}$$

$$K < 0 \tag{B.11}$$

$$\bar{g}_{s2}\left[(\alpha+\beta) - \bar{\lambda}_2\Gamma_{s'}(\bar{\lambda}_2)\right] - \bar{\lambda}_2\Gamma_{b'}(\bar{\lambda}_2) > (\alpha+\beta) \times \bar{\lambda}_2\bar{g}_{s2}\Gamma_{s'}(\bar{\lambda}_2)/K \tag{B.12}$$

It is noteworthy that this equilibrium will only be attainable if $0 < \bar{\lambda}_2 = \Gamma_s^{-1}(\alpha+\beta) < 1$, for which it is necessary and sufficient to have $\Gamma_s(0) > \alpha+\beta > \Gamma_s(1)$.

On a different note, if we assume that the good equilibrium is stable, then not only we have $\Theta_s(\bar{\pi}_1) < \alpha+\beta$ but also $\Gamma_s(\bar{\lambda}_1) < \alpha+\beta = \Gamma_s(\bar{\lambda}_2)$, which shows us that $\bar{\lambda}_1 > \bar{\lambda}_2$ since Γ_s is a decreasing function. Since Φ is an increasing function, we have that $\alpha = \Phi(\bar{\lambda}_1) > \Phi(\bar{\lambda}_2)$, so the first two conditions (B.9) and (B.10) for stability of this equilibrium are satisfied.

3. The characteristic polynomial at the equilibrium $(0, \bar{\lambda}_3, 0, 0, \bar{\pi}_1)$ is

$$\bar{p}_3(y) = \left(\Phi(\bar{\lambda}_3) - \alpha - y\right)\left(\Theta_s(\bar{\pi}_1) - (\alpha+\beta) - y\right) \times \left(\Gamma_s(\bar{\lambda}_3) - (\alpha+\beta) - y\right)\left(y^2 - K_3 y - \bar{\lambda}_3\mu'(\bar{\pi}_1)\right. \Gamma_{b'}(\lambda_3)) \tag{B.13}$$

Accordingly, local stability is guaranteed if and only if the following conditions are satisfied:

$$\Phi(\bar{\lambda}_3) < \alpha \tag{B.14}$$

$$\Theta_s(\bar{\pi}_1) < \alpha+\beta \tag{B.15}$$

$$\Gamma_s(\bar{\lambda}_3) < \alpha+\beta \tag{B.16}$$

$$K < 0 \tag{B.17}$$

Recalling that the employment level for this equilibrium is

$$\bar{\lambda}_3 = \Gamma_b^{-1}\left(\Gamma_b(\bar{\lambda}_1) - (\alpha+\beta)\bar{\omega}_1\right) > \bar{\lambda}_1,$$

we have that

$$\Phi(\bar{\lambda}_3) > \Phi(\bar{\lambda}_1) = \alpha,$$

which shows that this equilibrium is locally unstable whenever the good equilibrium is stable.

4. The Jacobian matrix at the equilibrium $(0, 0, 0, 0, \bar{\pi}_4)$ is a lower-triangular matrix, so we can identify its eigenvalues at the diagonal and conclude that local stability is equivalent to the following conditions:

$$\mu(\bar{\pi}_4) < \alpha+\beta \tag{B.18}$$

$$\Gamma_s(0) < \mu(\bar{\pi}_4) \tag{B.19}$$

$$\Theta_s(\bar{\pi}_4) < \mu(\bar{\pi}_4) \tag{B.20}$$

$$r + \mu'(\bar{\pi}_4)(1 - \bar{\pi}_4 - r\nu) < \mu(\bar{\pi}_4) + \Theta_{b'}(\bar{\pi}_4) \tag{B.21}$$

These inequalities can only be satisfied simultaneously if $\Gamma_s(0) < \alpha+\beta$.

5. The characteristic polynomial at the equilibrium $(0, 0, 0, \bar{\tau}_{s5}, \bar{\pi}_5)$ is

$$\begin{aligned}\bar{p}_5(y) = &\left[y^2 - y\,(r + \mu'(\bar{\pi}_5)(1 - \bar{\pi}_5 - r\nu)\right.\\ &- \Theta_s(\bar{\pi}_5) - \Theta_{b'}(\bar{\pi}_5) - \bar{g}_{T_5}\Theta_{s'}(\bar{\pi}_5))\\ &\left.-\bar{g}_{T_5}\Theta_s(\bar{\pi}_5)\mu'(\bar{\pi}_5)\right] \times \left(\Phi(0) - \alpha - y)(\Theta_s(\bar{\pi}_5)\right.\\ &\left.-(\alpha+\beta) - y\right)\left(\Gamma_s(0) - \Theta_s(\bar{\pi}_5) - y\right),\end{aligned} \tag{B.22}$$

from which we can derive the necessary and sufficient conditions for local stability:

$$\Theta_s(\bar{\pi}_5) < \alpha+\beta \tag{B.23}$$

$$\Gamma_s(0) < \Theta_s(\bar{\pi}_5) \tag{B.24}$$

$$r + \mu'(\bar{\pi}_5)(1 - \bar{\pi}_5 - r\nu) < \Theta_s(\bar{\pi}_5) + \Theta_{b'}(\bar{\pi}_5) + \bar{\tau}_{s5}\Theta_{s'}(\bar{\pi}_5) \tag{B.25}$$

$$\bar{\tau}_{s5}\Theta_s(\bar{\pi}_5) < 0 \tag{B.26}$$

Since $\tau_s(0) \geq 0$, this equilibrium can only be attained if $\bar{\tau}_{s5} > 0$. In that case, we need $\Theta_s(\bar{\pi}_5) < 0$, for it to be locally stable, which would then force $\Gamma_s(0)$ to be negative. Since this is not economically meaningful, we can conclude that this equilibrium will always be locally unstable to all effects and purposes.

6. The characteristic polynomial at the equilibrium $(0, 0, \bar{g}_{s6}, 0, \bar{\pi}_6)$ is

$$\begin{aligned}\bar{p}_6(y) = &\left[y^2 - y\,(r + \mu'(\bar{\pi}_6)(1 - \bar{\pi}_6 - r\nu) - \Gamma_s(0)\right.\\ &\left.- \Theta_{b'}(\bar{\pi}_6)\right) y + \Gamma_s(0)\bar{g}_{S_6}\mu'(\bar{\pi}_6)\Big]\\ &\times (\Phi(0) - \alpha - y)\left(\Gamma_s(0) - (\alpha+\beta) - y\right)\\ &\left(\Theta_s(\bar{\pi}_6) - \Gamma_s(0) - y\right)\end{aligned} \tag{B.27}$$

Therefore, this equilibrium is locally stable if and only if the following conditions are satisfied:

$$\Gamma_s(0) < \alpha+\beta \tag{B.28}$$

$$\Theta_s(\bar{\pi}_6) < \Gamma_s(0) \tag{B.29}$$

$$r + \mu'(\bar{\pi}_6)(1 - \bar{\pi}_6 - r\nu) < \Gamma_s(0) + \Theta_{b'}(\bar{\pi}_6) \tag{B.30}$$

$$\bar{g}_{s6}\Gamma_s(0) > 0 \tag{B.31}$$

Appendix C. Persistence definitions and examples

We start with a few standard definitions. Let $\Phi(t, x) : \mathbb{R}^+ \times X \to X$ be the semiflow generated by a differential system with initial values $x \in X$. For a nonnegative functional ρ from X to $\mathbb{R}^+$, we say

- Φ is ρ – uniformly strongly persistent (USP) if there exists an $\varepsilon > 0$ such that $\liminf_{t\to\infty} \rho(\Phi(t, x)) > \varepsilon$ for any $x \in X$ with $\rho(x) > 0$.
- Φ is ρ – uniformly weakly persistent (UWP) if there exists an $\varepsilon > 0$ such that $\limsup_{t\to\infty} \rho(\Phi(t, x)) > \varepsilon$ for any $x \in X$ with $\rho(x) > 0$.
- Φ is ρ – strongly persistent (SP) if $\liminf_{t\to\infty} \rho(\Phi(t, x)) > 0$ for any $x \in X$ with $\rho(x) > 0$.

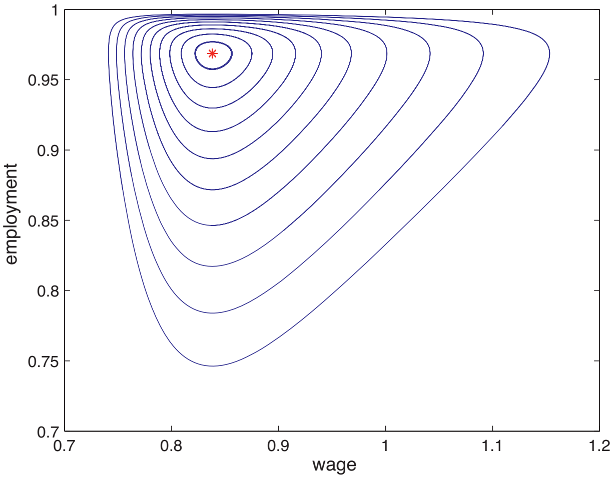


Fig. C7. Closed orbits in a Goodwin model.

- Φ is ρ – weakly persistent (WP) if $\limsup_{t\to\infty}\rho(\Phi(t,x)) > 0$ for any $x \in X$ with $\rho(x) > 0$.

As an example, consider the well known Goodwin (1967) obtained as a special case of (14) with $\kappa(x) = x$ and $d = 0$:

$$\begin{aligned} \dot{\omega} &= \omega\,[\Phi(\lambda) - \alpha] \\ \dot{\lambda} &= \lambda\left[\frac{1-\omega}{\nu} - \delta - (\alpha+\beta)\right] \end{aligned} \tag{C.1}$$

As it is well known (see Grasselli and Costa Lima, 2012 and references therein), the solution of (C.1) passing through the initial condition (ω_0, λ_0) satisfies the equation

$$\begin{aligned} &\left(\frac{1}{\nu} - \alpha - \beta - \delta\right)\log\frac{\omega}{\omega_0} - \frac{1}{\nu}(\omega - \omega_0) \\ &= -\alpha\log\frac{\lambda}{\lambda_0} + \int_{\lambda_0}^{\lambda}\frac{\Phi(s)}{s}ds. \end{aligned} \tag{C.2}$$

The closed periodic orbits implied by this equation are shown in Fig. C7. Recalling that $\pi = 1 - \omega$ for this model, observe that ω remains bounded on each orbit, so that $\liminf_{t\to\infty}\exp(1-\omega) > 0$ and the system is e^{π} – strongly persistent. However, since the bound on ω can be made arbitrarily large by changing the initial conditions, we see that the system is not e^{π} – uniformly strongly persistent. Finally, we see in Fig. C7 that ω becomes smaller than the equilibrium value

$$\overline{\omega} = 1 - \nu(\alpha + \beta + \delta) \tag{C.3}$$

infinitely often, regardless of the initial conditions. Therefore, taking $\varepsilon < \exp(1-\overline{\omega})$ shows that the system is e^{π} – uniformly weakly persistent. Exactly the same arguments show that the Goodwin model (C.1) is λ–SP, λ–UWP, but not λ–USP.

For the Keen model without government intervention defined in (14) the situation is less satisfactory. Whenever the conditions for local stability of the bad equilibrium (46) are satisfied, we cannot have either λ or e^{π} persistence of any form, since initial conditions sufficiently close to the bad equilibrium will necessarily lead to $\lambda = e^{\pi} = 0$ asymptotically.

Appendix D. Proof of Proposition 2

Showing this consists of demonstrating

$$\liminf_{t\to\infty}\pi(t) < m$$

for some $m \in \mathbb{R}$. We are going to show this by contradiction, so assume that $\liminf \pi > m$ for any m, as large (and positive) as we want. We can then find a t_0 such that $\pi(t) > m$ for all $t \geq t_0$.

First, we can then bound employment from below since for $t \geq t_0$ we have

$$\frac{\dot{\lambda}}{\lambda} = \mu(\pi) - \alpha - \beta \geq \mu(m) - \alpha - \beta$$

which is positive for m large enough. That means that $\lambda(t) > \lambda(t_0)\exp\left[(\mu(m) - \alpha - \beta)(t - t_0)\right]$ for all $t > t_0$.

Consequently, there exists $t_1 > t_0$ for which $\Phi(\lambda(t_1)) > \alpha$ and thus

$$\dot{\omega}/\omega = \Phi(\lambda) - \alpha$$

will be positive. We then have that

$$\omega(t) \geq \omega(t_1)\exp\left[\Phi(\lambda(t_1)) - \alpha\right]$$

Next, for $t \geq t_1$, the government spending dynamics satisfy

$$\dot{g}_s/g_s = \Gamma_s(\lambda) - \mu(\pi) \leq \Gamma_s(\lambda(t_1)) - \mu(m)$$

which can be made negative for m large enough. Consequently,

$$|g_s(t)| \leq |g_s(t_1)|\exp\left[(\Gamma_s(\lambda(t_1)) - \mu(m))(t - t_1)\right]$$

for all $t > t_0$.

Finally, one can choose m big enough such that $\kappa(m) \geq 0$, $\theta_b(m) \geq 0$, $\theta_s(m) \geq 0$, and $\mu(m) > r$ (possible because of (22)), allowing us to find the following bound for $\dot{\pi}$, valid for all $t > t_1$:

$$\begin{aligned} \dot{\pi} &= -\omega[\Phi(\lambda) - \alpha] - r(\kappa(\pi) - \pi) + (1 - \omega - \pi)\mu(\pi) \\ &+ \Gamma_b(\lambda) - \Theta_b(\pi) + g_s\Gamma_s(\lambda) - \tau_s\Theta_s(\pi) \leq \pi\,[r - \mu(m)] \\ &- \omega(t_1)e^{[\Phi(\lambda(t_1)) - \alpha](t - t_1)}[\Phi(\lambda(t_1)) - \alpha] + C_m, \end{aligned} \tag{D.1}$$

where $C_m = \Gamma_b(\lambda(t_0)) + (g_s(t_0))^+\Gamma_s(\lambda(t_0))$ is a positive constant. Consequently, Gronwall's inequality gives the following bound, valid for any $t > t_0$

$$\begin{aligned} \pi(t) &\leq \pi(t_1)e^{-(\mu(m) - r)(t - t_1)} + \frac{C_m}{\mu(m) - r} \\ &\times \left(1 - e^{-[\mu(m) - r](t - t_1)}\right) \\ &- \frac{\omega(t_1)[\Phi(\lambda(t_1)) - \alpha]}{[\Phi(\lambda(t_1)) - \alpha] + [\mu(m) - r]} \\ &\times \left(e^{[\Phi(\lambda(t_1)) - \alpha](t - t_1)} - e^{-[\mu(m) - r](t - t_1)}\right) \end{aligned} \tag{D.2}$$

From (20), we can choose t_1 appropriately such that $\Phi(\lambda(t_1))-\alpha\geq\mu(+\infty)-r$ and thus the RHS of (D.2) converges to $-\infty$ as t increases, which provides us with a contradiction.

Appendix E. Proof of Proposition 4 – item (4)

For item (4) of Proposition 4, let $\tau_s(0)>0$, since otherwise this reduces to item (1) and there is nothing to prove. We start by defining $v=\tau_s/g_s$ and observing that

$$\frac{\dot v}{v}=\Theta_s(\pi)-\Gamma_s(\lambda).$$

We can write $\dot\pi$ in terms of v and g (defined in (66)) as

$$\dot\pi=-\omega[\Phi(\lambda)-\alpha]-\omega\mu(\pi)+g(\pi)+\Gamma_b(\lambda)-\Gamma_b(0)+g_s\left[\Gamma_s(\lambda)-v\Theta_s(\pi)\right] \tag{E.1}$$

Let us now choose ε small enough such that $\Phi(\varepsilon)<\alpha$, $\Gamma_s(\varepsilon)>\alpha+\beta$ and

$$\Gamma_s(\varepsilon)\frac{\Gamma_s(\varepsilon)-2\varepsilon}{\Gamma_s(0)+2\varepsilon}>\Theta_s(\overline{\pi}_1), \tag{E.2}$$

which is possible because by hypothesis $\Theta_s(\overline{\pi}_1)<\alpha+\beta<\Gamma_s(0)$.

There must then exists some $t_0>0$ such that $\lambda(t)\leq\varepsilon$ and $\omega(t)\leq\varepsilon$ for all $t>t_0$. From UWP of e^{π}, we can find $m>0$ large enough such that $\limsup\pi>-m$ and $\liminf\pi<m$. Let us choose m large enough such that $-m<\Theta_s^{-1}(0)$ and

$$\frac{\Gamma_s(\varepsilon)-2\varepsilon}{\Gamma_s(0)+2\varepsilon}\Theta_s(m)>\Gamma_s(0).$$

Using the equations for $\dot\lambda$ and $\dot g_s$, it is straightforward to see that

$$\varepsilon g_s(t)>g_s(t_0)\lambda(t_0)e^{[\Gamma_s(\varepsilon)-(\alpha+\beta)](t-t_0)}\quad\forall t>t_0, \tag{E.3}$$

which grows exponentially since $\Gamma_s(\varepsilon)>\alpha+\beta$. Accordingly, we can find $t_1>t_0$ such that:

(i) $\varepsilon g_s(t)>\varepsilon[\alpha-\Phi(0)-\mu(-\infty)]+\max_{\pi\in\mathbb{R}}[g(\pi)]$ and
(ii) $\varepsilon g_s(t)>\varepsilon\mu(m)+\max_{\pi\in[-m,m]}|g(\pi)|+\Gamma_b(0)-\Gamma_b(\varepsilon)$ and
(iii) $\varepsilon g_s(t)>\frac{\Gamma_s(0)^2}{4\Theta_{s'}(-m)}$ and
(iv) $\varepsilon g_s(t)>\frac{\Theta_s(m)[\Theta_s(m)-\Gamma_s(\varepsilon)]}{\Theta_{s'}(-m)}$

for all $t>t_1$. As a result, $\dot\pi$ can be globally bounded from above by

$$\dot\pi<\varepsilon[\alpha-\Phi(0)-\mu(-\infty)]+\max[g(\pi)]+g_s[\Gamma_s(0)-v\Theta_s(\pi)] \tag{E.4}$$

$$<g_s\left[\varepsilon+\Gamma_s(0)-v\Theta_s(\pi)\right] \tag{E.5}$$

for all $t>t_1$. In addition, we have that $\dot\pi$ can be locally bounded from below by

$$\dot\pi>-\varepsilon\mu(m)-\max_{\pi\in[-m,m]}|g(\pi)|+\Gamma_b(\varepsilon)-\Gamma_b(0)+g_s[\Gamma_s(\varepsilon)-v\Theta_s(\pi)]>g_s\left[-\varepsilon+\Gamma_s(\varepsilon)-v\Theta_s(\pi)\right] \tag{E.6}$$

for all $t>t_1$ such that $\pi(t)\in[-m,m]$. We can therefore conclude that, for $t>t_1$ and $\pi(t)\in[-m,m]$, if $v\Theta_s(\pi)\geq\Gamma_s(0)+2\varepsilon$, then $\dot\pi\leq-\varepsilon g_s$ and if $v\Theta_s(\pi)\leq\Gamma_s(\varepsilon)-2\varepsilon$, then $\dot\pi\geq\varepsilon g_s$.

Moreover, we can globally bound $\dot v$ from both sides as

$$\Theta_s(\pi)-\Gamma_s(0)<\frac{\dot v}{v}<\Theta_s(\pi)-\Gamma_s(\varepsilon), \tag{E.7}$$

so that $\pi<\Theta^{-1}(\Gamma_s(\varepsilon))$ implies $\dot v<0$, whereas $\pi>\Theta^{-1}(\Gamma_s(0))$ implies $\dot v>0$.

Observe further that $\liminf\pi\geq\Theta_s^{-1}(0)$, because when $\pi\in[-m,\Theta_s^{-1}(0)]$ the lower bound for $\dot\pi$ becomes strictly positive for $t>t_1$, forcing π to grow higher than $\Theta_s^{-1}(0)$. We can therefore assume, without loss of generality, that $0\leq\Theta_s(\overline{\pi}_1)\leq\Theta_s(m)$, since otherwise we would be done ($\overline{\pi}_1=\mu^{-1}(\alpha+\beta)$ would be smaller than the lower bound of the $\liminf\pi$ and λ could not go to zero).

We shall now define the following regions, contained in $[\theta_s^{-1}(0),m]\times\mathbb{R}^+$ (see Fig. E8):

- $V:=\left\{(\pi,v)\in[\Theta_s^{-1}(0),m]\times\left[\frac{\Gamma_s(0)+2\varepsilon}{\Theta_s(m)},+\infty\right):\Gamma_s(\varepsilon)-2\varepsilon\leq v\Theta_s(\pi)\leq\Gamma_s(0)+2\varepsilon\right\}$;
- $S:=\left\{(\pi,v)\in[\Theta_s^{-1}(0),m]\times\left[\frac{\Gamma_s(\varepsilon)-2\varepsilon}{\Gamma_s(0)},\frac{\Gamma_s(0)+2\varepsilon}{\Gamma_2(\varepsilon)}\right]:\Gamma_s(\varepsilon)-2\varepsilon\leq v\Theta_s(\pi)\leq\Gamma_s(0)+2\varepsilon\right\}$;
- $P:=\left(\Theta_s^{-1}(\Gamma_s(0)),m\right]\times\left(0,\frac{\Gamma_s(0)+2\varepsilon}{\Theta_s(m)}\right)$

With the bounds on $\dot\pi$ and $\dot v$ obtained above, one can observe the following (valid for $t>t_1$):

(i) The flow through $v=(\Gamma_s(0)+2\varepsilon)/\Theta_s(\pi)$ goes inwards the region V. To see this, define the outward normal vector

$$\vec n_u:=\left\{\begin{array}{l}[\Gamma_s(0)+2\varepsilon]\,\Theta_{s'}(\pi)\\ \Theta_s^2(\pi)\end{array}\right\}$$

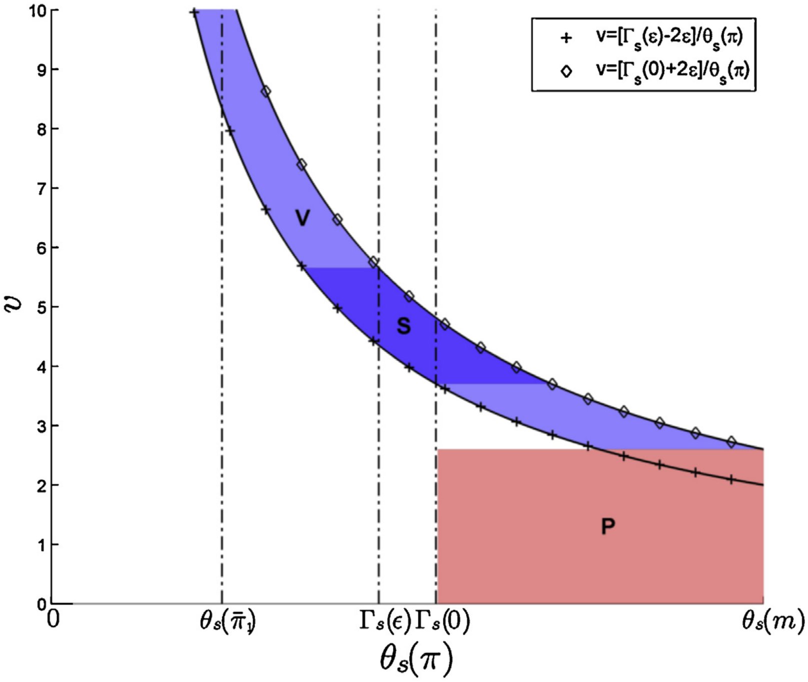


Fig. E8. The part of the plane $(\Theta_s(\pi)\times v)$ where one can see the invariant regions V and S. Every solution that enters V eventually makes it to S and never leaves it. The region P is not invariant. Yet, solutions that enter it must eventually leave it and enter the basin of attraction of S, either directly, or after spending some time on $(\pi, v)\in[m,\infty)\times\mathbb{R}^+$.

and notice that the flow going through the curve obeys

$$\begin{aligned}\vec{n}_u\cdot\begin{Bmatrix}\dot{\pi}\\ \dot{v}\end{Bmatrix} &= [\Gamma_s(0)+2\varepsilon]\,\Theta_{s'}(\pi)\dot{\pi}+\Theta_s^2(\pi)v\left[\Theta_s(\pi)-\Gamma_s(\lambda)\right]\\ &= [\Gamma_s(0)+2\varepsilon]\,\Theta_{s'}(\pi)\dot{\pi}+\Theta_s^2(\pi)\frac{\Gamma_s(0)+2\varepsilon}{\Theta_s(\pi)}\left[\Theta_s(\pi)-\Gamma_s(\lambda)\right]\\ &= [\Gamma_s(0)+2\varepsilon]\left\{\Theta_{s'}(\pi)\dot{\pi}+\Theta_s(\pi)\left[\Theta_s(\pi)-\Gamma_s(\lambda)\right]\right\}\\ &\le [\Gamma_s(0)+2\varepsilon]\left\{-\varepsilon\Theta_{s'}(\pi)g_s+\Theta_s(\pi)\left[\Theta_s(\pi)-\Gamma_s(\varepsilon)\right]\right\}\\ &< [\Gamma_s(0)+2\varepsilon]\left\{-\Theta_{s'}(\pi)\frac{\Theta_s(m)\left[\Theta_s(m)-\Gamma_s(\varepsilon)\right]}{\Theta_{s'}(-m)}+\Theta_s(m)\left[\Theta_s(m)-\Gamma_s(\varepsilon)\right]\right\}<0\end{aligned}\tag{E.8}$$

(ii) the flow through $v=(\Gamma_s(\varepsilon)-2\varepsilon)/\Theta_s(\pi)$ also goes inwards the region V. To see this, define the outward normal vector

$$\vec{n}_l := -\begin{Bmatrix}[\Gamma_s(\varepsilon)-2\varepsilon]\,\Theta_{s'}(\pi)\\ \Theta_s^2(\pi)\end{Bmatrix}$$

which yields

$$\begin{aligned}\vec{n}_l\cdot\begin{Bmatrix}\dot{\pi}\\ \dot{v}\end{Bmatrix} &= -[\Gamma_s(\varepsilon)-2\varepsilon]\left\{\Theta_{s'}(\pi)\dot{\pi}+\Theta_s(\pi)\left[\Theta_s(\pi)-\Gamma_s(\lambda)\right]\right\}\\ &\le -[\Gamma_s(\varepsilon)-2\varepsilon]\left\{\varepsilon\Theta_{s'}(\pi)g_s+\Theta_s(\pi)\left[\Theta_s(\pi)-\Gamma_s(0)\right]\right\}\\ &< -[\Gamma_s(\varepsilon)-2\varepsilon]\left\{\Theta_{s'}(\pi)\frac{\Gamma_s^2(0)}{4\Theta_{s'}(-m)}-\frac{\Gamma_s^2(0)}{4}\right\}<0\end{aligned}\tag{E.9}$$

where we have bounded $\Theta_s(\pi)\left[\Theta_s(\pi)-\Gamma_s(0)\right]$ by realizing that it is a quadratic polynomial like $y=x[x-\Gamma_s(0)]$ on $x\in[0,\ \Theta_s(m)]$, with minimum $y=-\Gamma_s^2(0)/4$.

(iii) the flow through the top side of P goes up. This is simply due to the fact that if $(\pi,\nu)\in P$, then $\pi>\Theta_s^{-1}(\Gamma_s(0))$, which implies that $\dot{\nu}>0$.

(iv) the flow through the left side of P goes inside P. To see this, notice that for $\pi=\Theta_s^{-1}(\Gamma_s(0))$ and $\nu<(\Gamma_s(0)+2\varepsilon)/\Theta_s(m)$, we have that $\nu\Theta_s(\pi)<((\Gamma_s(0)+2\varepsilon)/\Theta_s(m))\Gamma_s(0)<\Gamma_s(\varepsilon)-2\varepsilon$, hence $\dot{\pi}>0$.

(v) once $(\pi,\nu)\in V$, there exists some $t_2>t_1$ for which $(\pi,\nu)\in S$. One can be convinced of this from the fact that if $(\pi,\nu)\in V\setminus S$, then it must be either that $\pi<\Theta_s^{-1}(\Gamma_s(\varepsilon))$, in which case $\dot{\nu}<0$, or that $\pi>\Theta_s^{-1}(\Gamma_s(0))$, and hence $\dot{\nu}>0$. Either case, $\dot{\nu}$ drives the solution towards S.

Finally, the last argument goes as follows. Once π enters $[\theta_s^{-1}(0),m]$ (at time, say, $\hat{t}$), there exists some $t_2>t_1$ for which $\pi(t)>\overline{\pi}_1$ for all $t>t_2$. To see this, observe that if $(\pi(\hat{t}),\nu(\hat{t}))$ is above the curve $\nu=(\Gamma_s(0)+2\varepsilon)/\Theta_s(\pi)$, then it must eventually enter the region V, which then drives it to S at some future moment. If, however, $(\pi(\hat{t}),\nu(\hat{t}))$ starts below the curve $\nu=(\Gamma_s(\varepsilon)-2\varepsilon)/\Theta_s(\pi)$, then it might move to V, or P. If (π,ν) enters V, we are done, as we know that it will eventually enter S and stay away from $\overline{\pi}_1$. If, however, it enters either P, we are done as well, since from that region the solution can either:

(i) leave P through its top side, entering the region of attraction of S, or
(ii) leave P through its right side, so π becomes bigger than m, while ν continues growing. From there, the solution must return to $[\theta_s^{-1}(0),m]$ at some later time, at which it might return to P, or enter the region of attraction of V, eventually leading it to S.

In other words, every solution must eventually converge to the region S, where $\pi>\overline{\pi}_1$, which contradicts the fact that $\lambda\to 0$. Notice that it is crucial to this proof to have an unbounded region V, so we can guarantee that solutions entering $[\theta_s^{-1}(0),m]$ from the right, with ν bigger than $(\Gamma_s(0)+2\varepsilon)/\Theta_s(\pi)$ will eventually enter the band and find their way to the region S. Hence, the importance of having $\Theta_s(-\infty)<0$. If this was not the case, we would not be able to eliminate cyclic solutions starting from the region P, exiting to $(m,+\infty)\times\mathbb{R}^+$, returning to $[\theta_s^{-1}(0),m]$ above the band, completely avoiding the region V, escaping to $(-\infty,m)\times\mathbb{R}^+$, returning to $[\theta_s^{-1}(0),m]$ under V and then return to P, which would not serve as a contradiction.

Appendix F. Parameters

We chose the parameters of the functions Φ and κ according to the following constraints:

$$\overline{\lambda}_1=0.96 \tag{F.1}$$

$$\overline{\pi}_1=0.16 \tag{F.2}$$

$$\Phi(0)=\min_{0\le\lambda\le 1}\Phi(\lambda)=-0.04 \tag{F.3}$$

$$\lim_{\pi\to-\infty}\kappa(\pi)=\kappa(-\infty)=0 \tag{F.4}$$

$$\lim_{\pi\to+\infty}\kappa(\pi)=\kappa(+\infty)=1 \tag{F.5}$$

$$\kappa\prime(\overline{\pi}_1)=5. \tag{F.6}$$

It is easy to see that these choices lead to

$$\phi_0=\frac{\alpha(1-\overline{\lambda}_1)^2-\Phi(0)}{1-(1-\overline{\lambda}_1)^2}=0.040104 \tag{F.7}$$

$$\phi_1=\phi_0+\Phi(0)=0.00010417 \tag{F.8}$$

$$\kappa_1=\frac{1}{\pi}(\kappa(+\infty)-\kappa(-\infty))=0.31831 \tag{F.9}$$

$$\kappa_0=\kappa(+\infty)-\kappa_1\frac{\pi}{2}=0.5 \tag{F.10}$$

$$\kappa_2=\frac{\kappa\prime(\overline{\pi}_1)}{\kappa_1}\left[1+\tan\left(\frac{\kappa(\overline{\pi}_1)-\kappa_0}{\kappa_1}\right)^2\right]=63.989 \tag{F.11}$$

$$\kappa_3=\tan\left(\frac{\kappa(\overline{\pi}_1)-\kappa_0}{\kappa_1}\right)-\kappa_2\overline{\pi}_1=-11.991 \tag{F.12}$$

The government functions η_b, η_s, η_e and θ_b, θ_s were calibrated to satisfy the following

$$g_e(\overline{\pi}_1,\overline{\lambda}_1)=0.20 \tag{F.13}$$

$$\overline{g}_{b1}=0.004 \tag{F.14}$$

$$\overline{\tau}_{b1}=0.08 \tag{F.15}$$

$$\Gamma_s(0)=\begin{matrix}0.02 & \text{for a timid government,}\\ 0.20 & \text{for a responsive government}\end{matrix} \tag{F.16}$$

$$\Gamma_s(\overline{\lambda}_1)=\frac{1}{2}\min\left\{\alpha+\beta,\Gamma_s(0)\right\} \tag{F.17}$$

$$\Gamma_s\prime(\overline{\lambda}_1)=-0.5 \tag{F.18}$$

$$\lim_{\pi\to-\infty}\Theta_b(\pi)=-0.20 \tag{F.19}$$

$$\Theta_b(0)=0 \tag{F.20}$$

$$\Theta_s(\overline{\pi}_1)=\frac{1}{2}(\alpha+\beta) \tag{F.21}$$

$$\lim_{\pi\to-\infty}\Theta_s(x)=-0.20 \tag{F.22}$$

$$\Theta_s(0)=0 \tag{F.23}$$

which leads to the following set of parameters

$$\gamma_0=\frac{(\alpha+\beta)\overline{g}_{b1}}{1-\overline{\lambda}_1}=0.0045 \tag{F.24}$$

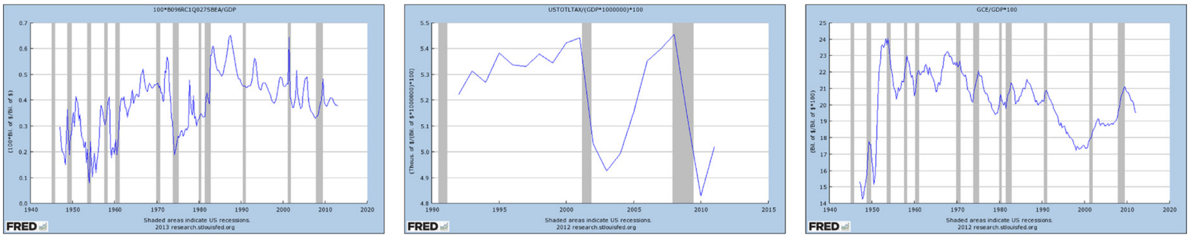
$$\theta_0=-0.20 \tag{F.25}$$

$$\theta_1=-\theta_0=0.20 \tag{F.26}$$

$$\theta_2=\frac{1}{\overline{\pi}_1}\log\left(\frac{\overline{\tau}_{b1}(\alpha+\beta)-\theta_0}{\theta_1}\right)=0.1115 \tag{F.27}$$

$$\theta_3=-0.20= \tag{F.28}$$

$$\theta_4=-\theta_3=0.20 \tag{F.29}$$



(a) (b) (c)

Fig. F9. US government subsidies (a), taxation (b), and expenditure (c), as percentage of GDP.

Source: FRED.

$$\theta_5 = \frac{1}{\overline{\pi}_1} \log\left(\frac{(\alpha+\beta)/2-\theta_3}{\theta_4}\right) = 0.66631 \qquad \text{(F.30)}$$

and, in the case of a responsive government,

$$\gamma_1 = \Gamma_s(0) = 0.20 \qquad \text{(F.31)}$$

$$\gamma_3 = \frac{\Gamma_{s'}(\overline{\lambda}_1)\overline{\lambda}_1}{\Gamma_s(0)-\Gamma_s(\overline{\lambda}_1)} = 2.7042 \qquad \text{(F.32)}$$

$$\gamma_2 = \left(\gamma_1 - \Gamma_s(\overline{\lambda}_1)\right)\overline{\lambda}_1^{-\gamma_3} = 0.19822 \qquad \text{(F.33)}$$

or, in the case of a timid government,

$$\gamma_1 = \Gamma_s(0) = 0.02 \qquad \text{(F.34)}$$

$$\gamma_3 = \frac{\Gamma_{s'}(\overline{\lambda}_1)\overline{\lambda}_1}{\Gamma_s(0)-\Gamma_s(\overline{\lambda}_1)} = 48 \qquad \text{(F.35)}$$

$$\gamma_2 = \left(\gamma_1 - \Gamma_s(\overline{\lambda}_1)\right)\overline{\lambda}_1^{-\gamma_3} = 0.070955 \qquad \text{(F.36)}$$

The values of $\overline{g}_{b1}$ and $g_e(\overline{\pi}_1, \overline{\lambda}_1)$ were chosen according to the historical average of government subsidies and expenditure in the United States, as seen in Fig. F9. We chose the value of $\overline{\tau}_{b1}$ slightly higher than the historical average of government taxation as we believe that the dataset available illustrates a period of extremely low taxation.

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