Comments on Problem sessions 3 and 4

Problem session 3:

We discussed the completeness of the reals: every bounded increasing (or decreasing) sequence of real numbers has a limit. This is the most critical defining property of the reals and gets used in many **Putnam** problems when one needs to see that a limit exists. We looked specifically at the problem

$$\lim_{n\to\infty} 1/n + \ldots + 1/2n$$

Each term in this sequence is positive so the sequence is bounded below by 0. The sequence decreases since $1/n > 1/(2n+1) + 1/(2n+2)$ and so this limit exists by the completeness of the reals.

We also did A2 and B1 from last year's exam: Both questions are good examples of how the examiners try to confuse you with too much information. In A2, after experimenting with some functions for a little while, one would conjecture that the only functions which satisfy the given criteria are of the form $f(x) = ax + b$. To prove this, one only needs the given property for $n = 1, 2$.

B1 requires a little additional knowledge; the solution I suggested was to use the Cauchy-Schwartz inequality. Here is a proof that uses less heavy-sounding linear algebra (but is still using the same idea). Suppose we have a sequence of real numbers $a_1, a_2, \ldots$ such that

$$\sum_i a_i^n = n$$

for all n. Again there is too much information here; let's use only this information for $n = 2, 3, 4$. We do need to know that if u, v are two vectors in R^n then $u \cdot v = |u||v|\cos(\theta)$ where θ is the angle between u and v. In fact, all I want to use about this fact is that in R^n,

$$|u \cdot v| \leq |u||v|$$

As those who know what the Cauchy-Schwarz inequality is, they will recognize this as that inequality in the special case of R^n. Now we go back to our original sequence and choose N big enough so that

$$\sum_{i=1}^{N} a_i^3 \geq 2.9$$

Then if we think about the sequences $u = (a_1, a_2, \ldots, a_N)$ and $v = (a_1^2, a_2^2, \ldots, a_N^2)$ we should have $(u \cdot v)^2 = (\sum_{i=1}^N a_i^3)^2 \geq 2.9^2$ and at the same time $\leq |u|^2|v|^2 \leq 2 \cdot 4 = 8$ which is a contradiction.

Problem session 4:

For Matt's warm-up problem, the total circumference is π times the height of the triangle. To see this, draw a series of diameters of all the circles perpendicular to the base and you get the height of the triangle.

For B1 from 2008, it is easy enough to construct a circle with two rational points on its perimeter but with non-rational centre. One needs to see that if you have three rational points on the perimeter, then the centre itself is rational. To see this, construct two chords of the circle using the rational points. The slopes of these lines is rational and the midpoint of the chords is rational all because the points we start with have rational coordinates. This means that the lines passing through the midpoints at right angles to the chords also have rational slope and y-intercept. Then centre is the intersection of these two lines and hence is rational.

For A2 from 2007, there was a gap in the proof I presented. One can restrict to a quadrilateral with one point on each branch of the two hyperbolas; the issue was why was this figure symmetric about the origin. Matt has provided a proof of this which I will link to the Putnam training website (it has a pretty picture). Once we have it is symmetric about the origin, one was left computing the area of the quadrilateral with corners at $A = (a, 1/a), B = (-b, 1/b), C = (-a, -1/a)$ and $D = (b, -1/b)$ where $a, b > 0$. This is a parallelogram and hence its area can be computed by looking at the length of the cross-product $CB \times CD$ or equivalently, computing the determinant of

$$\begin{pmatrix} a+b & 1/a - 1/b \\ a-b & 1/a + 1/b \end{pmatrix}$$

which is $2(a/b + b/a)$. If we let $\lambda = a/b$ then we are trying to minimize $2(\lambda + 1/\lambda)$ which an easy calculation shows occurs when $\lambda = 1$. So the minimal area is obtained whenever $a = b$ and is 4.

For B3 from 2006, the key thing to do here is figure out what the answer should be. To do this, if we let a_n be the sought after number for n points, compute enough values of a_n to make a conjecture. It was suggested at the session that a good conjecture is $a_{n+1} = a_n + n$. If this is true then the correct answer is $a_n = \binom{n}{2} + 1$. We'll discuss ways of proving this this Friday.