2.2 SOLUTIONS TO HOMOGENOUS SYSTEMS WITH CONSTANT COEFFICIENTS

Let $A(t) = A$ (constant coefficients), we will consider $\begin{cases} \dot{x} = Ax & \forall \\ x(0) = x_0 & t \in \mathbb{R} \end{cases}$ and show that the form of the solution $x(t)$ is determined by the eigenvalue structure of A.

CASE I

Let A be diagonalizable, ST $P = [v_1, \dots, v_n]$ is an invertible matrix of eigenvectors.

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$$

and $P^{-1}AP = D \iff A = PDP^{-1}$

Then, the solution has the form

$$x(t) = \underbrace{PE(t)P^{-1}}_{\phi(t)} x_0, \quad E(t) = \operatorname{diag}(e^{t\lambda_1}, e^{t\lambda_2}, \dots, e^{t\lambda_n})$$

Let $z = P^{-1}x_0$

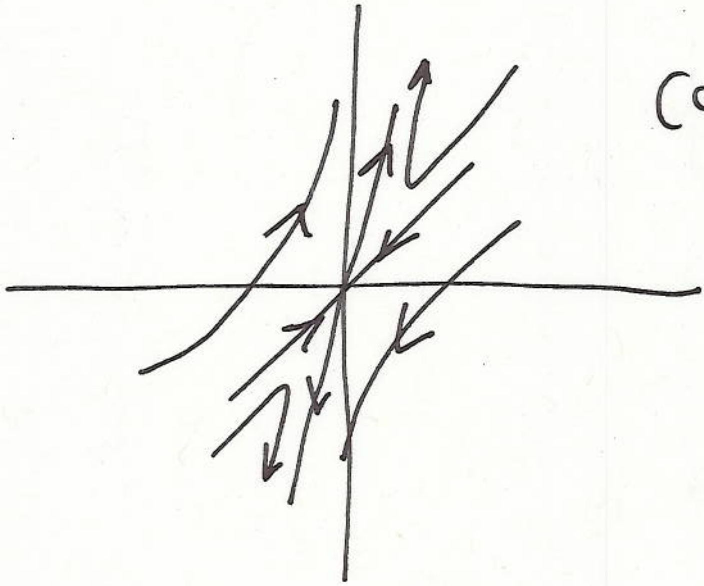
$$x(t) = PE(t)z = \sum_{j=1}^{n} z_j e^{t\lambda_j} v_j$$ — linear superposition principle

Ex $A = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$

$$\det \begin{vmatrix} -1-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 4 = 0 \Rightarrow \lambda_1 = 2, \ v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\lambda_2 = -2, \ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(t) = z_1 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + z_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$(0,0)$ is a "<u>saddle point</u>"

<u>Def</u>

Let A be matrix with constant coefficients. Then the <u>matrix exponential</u> is defined for any $t \in \mathbb{R}$ by:

$$e^{tA} = I + \sum_{k \in \mathbb{N}} \frac{t^k}{k!} A^k$$

We will show that e^{tA} makes sense $\forall\, t \in \mathbb{R}$ and that $\phi_t(x_0) = e^{tA} x_0$ for the solution of

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$$

<u>Theorem</u>

The series $\displaystyle\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$ converges absolutely and uniformly on $t \in [-\tau, \tau]$ for any $\tau > 0$.

Proof

$$\left\| \frac{t^k}{k!} A^k \right\| \leq \frac{|t|^k}{k!} \|A^k\| \leq \frac{\tau^k}{k!} \|A\|^k$$

So that

$$\left\| \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right\| \leq \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \|A\|^k = e^{\tau \|A\|} \quad \forall\, t \in [-\tau, \tau]$$

converges for any $\tau < \infty$, $\|A\| < \infty$.

By the Weierstrass M-test, the series converges absolutely and uniformly. ∎

Corollary $\quad \|e^{tA}\| \leq e^{\tau \|A\|} \quad \forall\, t \in [-\tau, \tau]$

Properties of e^{tA}

* ① If A is diagonalizable: $P^{-1} A P = D \Leftrightarrow A = P D P^{-1}$

then $P^{-1} e^{tA} P = e^{tD} = E(t) \Leftrightarrow e^{tA} = P e^{tD} P^{-1} = P E(t) P^{-1}$

Proof

$$P^{-1} e^{tA} P = \sum_{k=0}^{\infty} \frac{t^k}{k!} P^{-1} A^k P = \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k = e^{tD}$$

∎

② If A and B commute, then $e^{t(A+B)} = e^{tA} e^{tB}$

Proof

~~$e^{t(A+B)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} (A+B)^k$~~

If $AB = BA$ by binomial thm: $(A+B)^n = n! \sum_{j+k=n} \frac{A^j B^k}{j!\, k!}$

Then

$$e^{A+B} = \underbrace{\sum_{n=1}^{\infty} \sum_{j+k=n} \frac{A^j B^k}{j!\,k!} = \sum_{j=0}^{\infty} \frac{A^j}{j!} \sum_{k=0}^{\infty} \frac{B^k}{k!}}_{\text{product of two absolutely convergent series is an absolutely convergent series.}}$$

③ $e^{(t+s)A} = e^{tA} e^{sA}$ (same proof)

④ $\forall_{t \in \mathbb{R}}$ the inverse of e^{tA} exists and $(e^{tA})^{-1} = e^{-tA}$

__Proof__

Let $B = -A$, since A and B commute

$$e^{tA} \cdot e^{-tA} = e^{0} = I \Rightarrow e^{-tA} \text{ is the inverse of } e^{tA}$$

__Remark__

e^{tA} is invertible even if A is singular

__Theorem__

A unique solution of $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$ has the form $x(t) = e^{tA} x_0$

__Proof__

$$\frac{d}{dt} e^{tA} = \lim_{h \to 0} \frac{e^{(t+h)A} - e^{tA}}{h} = e^{tA} \lim_{h \to 0} \frac{e^{hA} - I}{h}$$

$$= e^{tA} \lim_{h \to 0} \left(A + \frac{h}{2!} A^2 + \frac{h^2}{3!} A^3 + \dots \right) = \underbrace{e^{tA} A = A e^{tA}}_{\text{since } A \text{ and } e^{tA} \text{ commute}} \quad \blacksquare$$

The fundamental matrix $\phi(t) = e^{tA}$ satisfies

$$\begin{cases} \dot{\phi} = A\phi \\ \phi(0) = I \end{cases}$$