THE $SU(2)$ CASSON-LIN INVARIANT OF THE HOPF LINK

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Abstract. We compute the $SU(2)$ Casson-Lin invariant for the Hopf link and determine the sign in the formula of Harper and Saveliev relating this invariant to the linking number.

The Casson–Lin invariant $h(K)$ was defined for knots K by X.-S. Lin [6] as a signed count of conjugacy classes of irreducible $SU(2)$ representations of the knot group $G_K = \pi_1(S^3 \smallsetminus K)$ with traceless meridional image, and Corollary 2.10 of [6] shows that $h(K) = \operatorname{sign}(K)/2$, one half the knot signature. E. Harper and N. Saveliev introduced the Casson–Lin invariant $h_2(L)$ of 2-component links in [2], which they defined as a signed count of certain projective $SU(2)$ representations of the link group $G_L = \pi_1(S^3 \smallsetminus L)$. They showed that $h_2(L) = \pm lk(\ell_1, \ell_2)$, the linking number of $L = \ell_1 \cup \ell_2$, up to an overall sign. Harper and Saveliev [3] also show that $h_2(L)$ can be regarded as an Euler characteristic associated to a certain $SU(2)$ instanton Floer homology theory, defined by Kronheimer and Mrowka [5].

The purpose of this note is to determine the sign in the formula of Harper and Saveliev, establishing the following.

Theorem 1. *If $L = \ell_1 \cup \ell_2$ is an oriented 2-component link in S^3, then its Casson-Lin invariant satisfies $h_2(L) = -lk(\ell_1, \ell_2)$.*

We remark that the braid approach in [2] is close in spirit to Lin's original definition, and it shows that $h_2(L)$ is an invariant of *oriented* links, because the Alexander and Markov theorems hold for oriented links, see Theorems 2.3 and 2.8 of [4]. The sign of the invariant $h_2(L)$ depends not only on the choice of orientation on the braid, but also on the more subtle choice of identification of geometric braids with elements in the abstract braid group B_n, viewed as a subgroup of $\operatorname{Aut}(F_n)$. Here we follow Conventions 1.13 of [4] in making this choice.

Note that extensions of the Casson-Lin invariants to $SU(N)$ and to oriented links L in S^3 with at least two components are presented in [1], where as before they are defined by counting certain projective $SU(N)$ representations of the link group G_L.

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The rest of this paper is devoted to proving Theorem 1. The proof of Proposition 5.7 in [2] shows that the sign in the above formula is independent of L. (See also the proof of Theorem 2 of [2] and the general discussion of Section 5 of [2].) Thus Theorem 1 will follow from a single computation.

To that end, we will determine the Casson-Lin invariant for the right-handed Hopf link. Since there is just one irreducible projective $SU(2)$ representation of the link group, up to conjugation, it suffices to determine the sign associated to this one point.

We identify

$$SU(2) = \{x + yi + zj + wk \mid |x|^2 + |y|^2 + |z|^2 + |w|^2 = 1\}$$

with the group of unit quaternions and consider the conjugacy class

$$C_i = \{yi + zj + wk \mid |y|^2 + |z|^2 + |w|^2 = 1\} \subset SU(2)$$

of purely imaginary unit quaternions. Notice that C_i is diffeomorphic to S^2 and coincides with the set of $SU(2)$ matrices of trace zero.

Let L be an oriented link in S^3, represented as the closure of an n-strand braid $\sigma \in B_n$. We follow Conventions 1.13 on p.17 of [4] for writing geometric braids σ as words in the standard generators $\sigma_1, \ldots, \sigma_{n-1}$. In particular, braids are oriented from top to bottom and σ_i denotes a right-handed crossing in which the $(i+1)$-st strand crosses over the i-th strand. The braid group B_n gives a faithful right action on the free group F_n on n generators, and here we follow the conventions in [1] for associating an automorphism of F_n to a given braid $\sigma \in B_n$, which we write as $x_i \mapsto x_i^\sigma$ for $i = 1, \ldots, n$. To be precise, to each braid group generator σ_i we associate the map $\sigma_i \colon F_n \to F_n$ given by

$$\begin{array}{lll} x_i & \mapsto & x_{i+1} \\ x_{i+1} & \mapsto & (x_{i+1})^{-1}\, x_i\, x_{i+1} \\ x_j & \mapsto & x_j, \quad j \neq i, i+1, \end{array}$$

and this is a right action, i.e. if $\sigma, \sigma' \in B_n$ are two braids, then $(x_i)^{\sigma\sigma'} = (x_i^\sigma)^{\sigma'}$ for all $1 \leq i \leq n$. Note that each braid $\sigma \in B_n$ fixes the product $x_1 \cdots x_n$.

A standard application of the Seifert-Van Kampen theorem shows that the link complement $S^3 \smallsetminus L$ has fundamental group

$$\pi_1(S^3 \smallsetminus L) = \langle x_1, \ldots, x_n \mid x_i^\sigma = x_i,\ i = 1, \ldots, n\rangle.$$

We can therefore identify representations in $\operatorname{Hom}(\pi_1(S^3 \smallsetminus L), SU(2))$ with fixed points in $\operatorname{Hom}(F_n, SU(2))$ under the induced action of the braid σ. We further identify $\operatorname{Hom}(F_n, SU(2))$ with $SU(2)^n$ by associating to a homomorphism ϱ the n-tuple $(X_1, \ldots, X_n) = (\varrho(x_1), \ldots, \varrho_1(x_n))$. Note that $\sigma \colon SU(2)^n \to SU(2)^n$ is equivariant with respect to conjugation, so that fixed points come in whole orbits.

If $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ is an n-tuple with $\varepsilon_i = \pm 1$ and $\varepsilon_1 \cdots \varepsilon_n = 1$, then projective $SU(2)$ representations can be identified with fixed points in $\operatorname{Hom}(F_n, SU(2))$ under the induced action of $\varepsilon\sigma$, which also preserves the product $X_1 \cdots X_n$ and is conjugation equivariant. The Casson-Lin invariant $h_2(L)$ is then defined as a signed count of orbits of fixed points of $\varepsilon\sigma$ for a suitably chosen n-tuple $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$. The choice is made so that the resulting projective representations ϱ all have $w_2(\operatorname{Ad}\varrho) \neq 0$, meaning that the representations $\operatorname{Ad}\varrho$ do not lift to $SU(2)$ representations. It has the consequence that for all fixed points ϱ of $\varepsilon\sigma$, each $\varrho(x_i)$ is a traceless $SU(2)$ element.

We therefore restrict our attention to the subset of traceless representations, which are elements $\varrho \in \operatorname{Hom}(F_n, SU(2))$ with $\varrho(x_j) \in C_i$ for $j = 1, \dots, n$. Define $f\colon C_i^n \times C_i^n \to SU(2)$ by setting

$$f(X_1, \dots, X_n, Y_1, \dots, Y_n) = (X_1 \cdots X_n)(Y_1 \cdots Y_n)^{-1}.$$

We obtain an orientation on $f^{-1}(1)$ by applying the base-fiber rule, using the product orientation on $C_i^n \times C_i^n$ and the standard orientation on the codomain of f. The quotient $f^{-1}(1)/\text{conj}$ is then oriented by another application of the base-fiber rule, using the standard orientation on $SU(2)$. This step uses the fact that, if $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ is chosen so that the associated $SO(3)$ representation $\operatorname{Ad}\varrho$ has second Stiefel-Whitney class $w_2 \neq 0$ nontrivial, then every fixed point of $\varepsilon\sigma$ in $\operatorname{Hom}(F_n, SU(2))$ is necessarily irreducible.

We view conjugacy classes of fixed points of $\varepsilon\sigma$ as points in the intersection $\widehat{\Delta} \cap \widehat{\Gamma}_{\varepsilon\sigma}$, where $\widehat{\Delta} = \Delta/\text{conj}$ is the quotient of the diagonal $\Delta \subset C_i^n \times C_i^n$, and where $\widehat{\Gamma}_{\varepsilon\sigma} = \Gamma_{\varepsilon\sigma}/\text{conj}$ is the quotient of the graph $\Gamma_{\varepsilon\sigma}$ of $\varepsilon\sigma\colon C_i^n \to C_i^n$.

If the link L is the closure of a 2-strand braid, as it is for the Hopf link, then $\varepsilon = (-1, -1)$ is the only choice whose associated $SO(3)$ bundle has $w_2 \neq 0$. Furthermore, in this case the intersection $\widehat{\Delta} \cap \widehat{\Gamma}_{\varepsilon\sigma}$ takes place in the pillowcase $f^{-1}(1)/\text{conj}$, which is defined as the quotient

$$P = \{(a, b, c, d) \in C_i^4 \mid ab = cd\}/\text{conj}\,. \tag{1}$$

It is well known that P is homeomorphic to S^2. To see this, first conjugate so that $a = i$, then conjugate by elements of the form $e^{i\theta}$ to arrange that b lies in the (i, j)-circle. A straightforward calculation using the equation $ab = cd$ shows that d must also lie on the (i, j)-circle. Clearly c is determined by a, b, d. We thus obtain an embedded 2-torus of elements of C_i^4 satisfying $ab = cd$, parameterized by

$$g(\theta_1, \theta_2) = (i, e^{k\theta_1} i, e^{k(\theta_2 - \theta_1)} i, e^{k\theta_2} i)$$

for $\theta_1, \theta_2 \in [0, 2\pi)$, which maps onto P. It is easy to verify that this is a two-to-one submersion, except when $\theta_1, \theta_2 \in \{0, \pi\}$. This realizes P as a quotient of the torus by the hyperelliptic involution. In particular, this involution is orientation preserving, and away from the four singular points of P, we can lift all orientation questions up to the torus.

Let L be the right-handed Hopf link, which we view as the closure of the braid $\sigma = \sigma_1^2 \in B_2$, and suppose $\varepsilon = (-1, -1)$. The intersection point $\widehat{\Delta} \cap \widehat{\Gamma}_{\varepsilon\sigma}$ in question is given by the conjugacy class of $g(\pi/2, \pi/2)$, namely the point $[(i, j, i, j)] \in P$.

(This is easily verified using the action of σ_1^2 on $F_2 = \langle x, y\rangle$; see Figure 1.) Thus, in order to pin down the sign of the Casson-Lin invariant $h_2(L)$, we must determine the orientations of $\widehat{\Delta}$, $\widehat{\Gamma}_{\varepsilon\sigma}$, and P at this point.

Notice that

$$\begin{aligned} \frac{\partial}{\partial\theta_1} g(\theta_1,\theta_2) &= (0, e^{k\theta_1} j, -e^{k(\theta_2-\theta_1)} j, 0) \\ \frac{\partial}{\partial\theta_2} g(\theta_1,\theta_2) &= (0, 0, e^{k(\theta_2-\theta_1)} j, e^{k\theta_2} j). \end{aligned}$$

Evaluating at $\theta_1 = \theta_2 = \pi/2$ gives two tangent vectors $u_1 := (0, -i, -j, 0)$ and $u_2 := (0, 0, j, -i)$ to C_i^4 which span a complementary subspace in $\ker df$ to the orbit tangent space. Therefore, an ordering of these vectors determines an orientation on $P = f^{-1}(1)/\text{conj}$.

The orbit tangent space is spanned by the three tangent vectors

$$\begin{aligned} v_1 := \frac{\partial}{\partial t}\Big|_{t=0} e^{it}(i,j,i,j)e^{-it} &= (0, 2k, 0, 2k), \\ v_2 := \frac{\partial}{\partial t}\Big|_{t=0} e^{jt}(i,j,i,j)e^{-jt} &= (-2k, 0, -2k, 0), \\ v_3 := \frac{\partial}{\partial t}\Big|_{t=0} e^{kt}(i,j,i,j)e^{-kt} &= (2j, -2i, 2j, -2i). \end{aligned}$$

Then the five vectors $\{u_1, u_2, v_1, v_2, v_3\}$ form a basis for $\ker\left(df|_{(i,j,i,j)}\right) = T_{(i,j,i,j)} f^{-1}(1)$. We choose vectors $w_1 = (k, 0, 0, 0), w_2 = (0, k, 0, 0), w_3 = (j, 0, 0, 0)$ to extend this to a basis for $T_{(i,j,i,j)} C_i^4$.

The orientation conventions in the definition of $h_2(L)$ (see Section 5d of [2]) involve pulling back the orientation from $su(2) = T_1 SU(2)$ by df to obtain a co-orientation for $\ker\left(df|_{(i,j,i,j)}\right)$. With that in mind, we compute the action of df on $\{w_1, w_2, w_3\}$, namely, $df(w_1) = -j$, $df(w_2) = i$ and $df(w_3) = k$.

Notice that the ordered triple $\{df(w_1), df(w_2), df(w_3)\} = \{-j, i, k\}$ gives the same orientation as the standard basis for $su(2)$. Thus, the base-fiber rule gives the co-orientation $\{w_1, w_2, w_3\}$ on $\ker df$, so we choose the orientation $\mathcal{O}_{\ker df}$ on $\ker df$ such that $\mathcal{O}_{\{w_1,w_2,w_3\}} \oplus \mathcal{O}_{\ker df}$ agrees with the product orientation on $C_i^2 \times C_i^2$.

The orientation on the pillowcase P is then obtained by applying the base-fiber rule a second time to the quotient (1), using $\mathcal{O}_{\ker df}$ to orient $f^{-1}(1)$ and giving the orbit tangent space the orientation induced from that on $SU(2)$ as well. We claim that the basis $\{u_1, u_2\}$ for the tangent space to the pillowcase has the opposite orientation. To see this, we note that $\{v_1, v_2, v_3\}$ is the fiber orientation for $SO(3) \to f^{-1}(1) \to P$ and compare

$$S = \{w_1, w_2, w_3, u_1, u_2, v_1, v_2, v_3\}$$

to the product orientation on $C_i^2 \times C_i^2$. Using the basis

$$\{(j,0),(k,0),(0,k),(0,i)\}$$

for $T_{(i,j)}(C_i^2)$, we see that

$$\begin{aligned}\beta &= \{(j,0,0,0),(k,0,0,0),(0,k,0,0),(0,i,0,0),\\ &\quad (0,0,j,0),(0,0,k,0),(0,0,0,k),(0,0,0,i)\}\end{aligned}$$

is an oriented basis for $T_{(i,j,i,j)}C_i^4 = T_{(i,j)}C_i^2 \times T_{(i,j)}C_i^2$ with the product orientation.[1]

Let M be the matrix expressing the vectors in S in terms of the basis β. Since

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 \end{bmatrix},$$

one easily computes that $\det M = -8$, confirming our claim that $\{u_2, u_1\}$ is a positively oriented basis for the pillowcase tangent space.

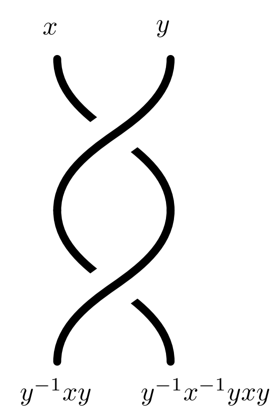

FIGURE 1. The action of $\sigma = \sigma_1^2$ on $F_2 = \langle x, y\rangle$

Recall that L is the right-handed Hopf link, which we represent as the closure of the braid $\sigma = \sigma_1^2 \in B_2$. For $\varepsilon = (-1,-1)$, as in Figure 1, one can verify that

$$\varepsilon\sigma(X,Y) = (-Y^{-1}XY, -Y^{-1}X^{-1}YXY).$$

Consider the curve $\alpha(\theta) = (i, e^{k\theta}i)$, passing through the point $(i,j) \in C_i^2$ when $\theta = \pi/2$, which is transverse to the orbit $[(i,j)]$. Then $(\alpha(\theta),\alpha(\theta))$ and $(\alpha(\theta), \varepsilon\sigma \circ$

[1]As explained in Section 5d of [2], the invariant $h_2(L)$ is independent of the choice of orientation on C_i. In fact, C_i^2 can be oriented arbitrarily provided one uses the *product* orientation on $C_i^2 \times C_i^2$.

$\alpha(\theta))$ are curves in Δ and $\Gamma_{\varepsilon\sigma}$, respectively, and both are necessarily transverse to the orbit in C_i^4/conj. Therefore, we can compare the orientations induced by the parameteriziations $[(\alpha(\theta), \alpha(\theta))]$ and $[(\alpha(\theta), \varepsilon\sigma \circ \alpha(\theta))]$ of $\widehat{\Delta}$ and $\widehat{\Gamma}_{\varepsilon\sigma}$ to the pillowcase orientation determined above, namely $\{u_2, u_1\}$. The velocity vectors for the paths $(\alpha(\theta), \alpha(\theta)) = (i, e^{k\theta}i, i, e^{k\theta}i)$ and $(\alpha(\theta), \varepsilon\sigma\circ\alpha(\theta)) = (i, e^{k\theta}i, -e^{2k\theta}i, -e^{3k\theta}i)$ at $\theta = \pi/2$ are given by $(0, -i, 0, -i) = u_1 + u_2$ and $(0, -i, 2j, -3i) = u_1 + 3u_2$, respectively.

The Casson-Lin invariant is defined as the intersection number $h_2(L) = \langle \widehat{\Delta},\ \widehat{\Gamma}_{\varepsilon\sigma}\rangle$, and in our case the sign of the unique intersection point in $\widehat{\Delta} \cap \widehat{\Gamma}_{\varepsilon\sigma}$ is determined by comparing the orientation of $\{u_1 + u_2, u_1 + 3u_2\}$ with $\{u_2, u_1\}$. Since the change of basis matrix $\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ has negative determinant, it follows that $h_2(L) = -1$, and this completes the proof of the theorem. □

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