METABELIAN SL($n,\mathbb{C}$) REPRESENTATIONS OF KNOT GROUPS II: FIXED POINTS

HANS U. BODEN AND STEFAN FRIEDL

Abstract. Given a knot K in an integral homology sphere Σ with exterior N_K, there is a natural action of the cyclic group $\mathbb{Z}/n$ on the space of SL($n,\mathbb{C}$) representations of the knot group $\pi_1(N_K)$, and this induces an action on the SL($n,\mathbb{C}$) character variety. We identify the fixed points of this action in terms of characters of metabelian representations, and we apply this to show that the twisted Alexander polynomial $\Delta^\alpha_{K,1}(t)$ associated to an irreducible metabelian SL($n,\mathbb{C}$) representation α is actually a polynomial in t^n.

1. Introduction

Suppose K is a knot. Throughout this paper we will always understand this to mean that K is an oriented simple closed curve in an integral homology 3-sphere Σ. We write $N_K = \Sigma^3 \setminus \tau(K)$, where $\tau(K)$ denotes an open tubular neighborhood of K.

The study of metabelian representations and metabelian quotients of knot groups goes back to the pioneering work of Neuwirth [Ne65], de Rham [dRh68], Burde [Bu67] and Fox [Fo70] (see also [BZ03, Section 14]). The theory was further developed by many authors, including Hartley [Ha79, Ha83], Livingston [Li95], Letsche [Le00], Lin [Lin01], Nagasato [Na07] and Jebali [Je08]. In [BF08] we proved a classification theorem for irreducible metabelian representations, and in this paper we continue our study of metabelian representations of knot groups.

We begin by introducing some terminology. Given a topological space M, let $R_n(M)$ be the space of SL($n,\mathbb{C}$) representations of $\pi_1(M)$ and $X_n(M)$ the associated character variety. We use ξ_α to denote the character of the representation $\alpha\colon \pi_1(M) \to \mathrm{SL}(n,\mathbb{C})$. We will often make use of the important fact that two irreducible representations determine the same character if and only if they are conjugate (see [LM85, Corollary 1.33]).

Now suppose K is a knot. There is an action of the group $\mathbb{Z}/n$ on the representation variety $R_n(N_K)$ given by twisting by the n–th roots of unity $\omega^k = e^{2\pi i k/n} \in U(1)$. (This is a special case of the more general twisting operation described in [LM85,

2000 *Mathematics Subject Classification.* Primary: 57M25, Secondary: 20C15.

Key words and phrases. Metabelian representation, knot group, character variety, group action, fixed point.

The first named author was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

Ch. 5].) More precisely, we write $\mathbb{Z}/n = \langle \sigma \mid \sigma^n = 1 \rangle$ and set $(\sigma \cdot \alpha)(g) = \omega^{\varepsilon(g)}\alpha(g)$ for each $g \in \pi_1(N_K)$, where $\varepsilon\colon \pi_1(N_K) \to H_1(N_K) = \mathbb{Z}$ is determined by the given orientation of the knot.

This constructs an action of $\mathbb{Z}/n$ on $R_n(N_K)$ which, in turn, descends to an action on the character variety $X_n(N_K)$. Our main result identifies the fixed points of $\mathbb{Z}/n$ in $X_n^*(N_K)$, the irreducible characters, as those associated to metabelian representations.

Theorem 1. *The character ξ_α of an irreducible representation $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ is fixed under the $\mathbb{Z}/n$ action if and only if α is metabelian.*

In proving this result, we actually characterize the entire fixed point set $X_n(N_K)^{\mathbb{Z}/n}$ in terms of characters ξ_α of the metabelian representations $\alpha = \alpha_{(n,\chi)}$ described in Subsection 2.3 (see Theorem 4). When $n = 2$, it turns out that every metabelian $\mathrm{SL}(2, \mathbb{C})$ representation is dihedral and in this case Theorem 1 was first proved by F. Nagasato and Y. Yamaguchi (cf. [NY08, Proposition 4.8]).

As an application of Theorem 1, we prove a result about the twisted Alexander polynomials associated to metabelian representations. This result was first shown by C. Herald, P. Kirk and C. Livingston in [HKL08] using completely different methods. Our approach is elementary and quite natural, and it is explained in Section 3.2, where we apply it to give an answer to a question raised by Hirasawa and Murasugi in [HM09].

Acknowledgments. The authors would like to thank Steven Boyer, Christopher Herald, Michael Heusener, Paul Kirk, Charles Livingston, Andrew Nicas and Adam Sikora for generously sharing their knowledge, wisdom, and insight. We would also like to thank Fumikazu Nagasato and Yoshikazu Yamaguchi for communicating the results of their paper to us.

2. The classification of metabelian representations of knot groups

In this section we recall some results from [BF08] regarding the classification of metabelian representations of knot groups.

2.1. **Preliminaries.** Given a group π, we shall write $\pi^{(n)}$ for the n–th term of the derived series of π. These subgroups are defined inductively by setting $\pi^{(0)} = \pi$ and $\pi^{(i+1)} = [\pi^{(i)}, \pi^{(i)}]$. The group π is called *metabelian* if $\pi^{(2)} = \{e\}$.

Suppose V is a finite dimensional vector space over $\mathbb{C}$. A representation $\varrho\colon \pi \to \mathrm{Aut}(V)$ is called *metabelian* if ϱ factors through $\pi/\pi^{(2)}$. The representation ϱ is called *reducible* if there exists a proper subspace $U \subset V$ invariant under $\varrho(\gamma)$ for all $\gamma \in \pi$. Otherwise ϱ is called *irreducible* or *simple*. If ϱ is the direct sum of simple representations, then ϱ is called *semisimple*.

Two representations $\varrho_1\colon \pi \to \mathrm{Aut}(V)$ and $\varrho_2\colon \pi \to \mathrm{Aut}(W)$ are called *isomorphic* if there exists an isomorphism $\phi\colon V \to W$ such that $\phi^{-1} \circ \varrho_1(g) \circ \phi = \varrho_2(g)$ for all $g \in \pi$.

2.2. **Metabelian quotients of knot groups.** Let $K \subset \Sigma^3$ be a knot in an integral homology 3-sphere. In the following we denote by $\widetilde{N}_K$ the infinite cyclic cover of N_K corresponding to the abelianization $\pi_1(N_K) \to H_1(N_K) \cong \mathbb{Z}$. Therefore $\pi_1(\widetilde{N}_K) = \pi_1(N_K)^{(1)}$ and

$$H_1(N_K; \mathbb{Z}[t^{\pm 1}]) = H_1(\widetilde{N}_K) \cong \pi_1(N_K)^{(1)}/\pi_1(N_K)^{(2)}.$$

The $\mathbb{Z}[t^{\pm 1}]$–module structure is given on the right hand side by $t^n \cdot g := \mu^{-n} g \mu^n$, where μ is a meridian of K.

For a knot K, we set $\pi := \pi_1(N_K)$ and consider the short exact sequence

$$1 \to \pi^{(1)}/\pi^{(2)} \to \pi/\pi^{(2)} \to \pi/\pi^{(1)} \to 1.$$

Since $\pi/\pi^{(1)} = H_1(N_K) \cong \mathbb{Z}$, this sequence splits and we get isomorphisms

$$\begin{array}{ccccc} \pi/\pi^{(2)} & \cong & \pi/\pi^{(1)} \ltimes \pi^{(1)}/\pi^{(2)} & \cong & \mathbb{Z} \ltimes \pi^{(1)}/\pi^{(2)} \cong \mathbb{Z} \ltimes H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \\ g & \mapsto & (\mu^{\varepsilon(g)}, \mu^{-\varepsilon(g)} g) & \mapsto & (\varepsilon(g), \mu^{-\varepsilon(g)} g), \end{array}$$

where the semidirect products are taken with respect to the $\mathbb{Z}$ actions defined by letting $n \in \mathbb{Z}$ act by conjugation by μ^n on $\pi^{(1)}/\pi^{(2)}$ and by multiplication by t^n on $H_1(N_K; \mathbb{Z}[t^{\pm 1}])$.

2.3. **Irreducible metabelian** SL$(n,\mathbb{C})$ **representations of knot groups.** Let K be a knot. We write $H = H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. The discussion of the previous section shows that irreducible metabelian SL$(n,\mathbb{C})$ representations of $\pi_1(N_K)$ correspond precisely to the irreducible SL$(n,\mathbb{C})$ representations of $\mathbb{Z} \ltimes H$.

Let $\chi\colon\ H \to \mathbb{C}^*$ be a character which factors through $H/(t^n - 1)$ and suppose $z \in S^1$ with $z^n = (-1)^{n+1}$. Then it follows from [BF08, Section 3] that, for $(j,h) \in \mathbb{Z} \ltimes H$, setting

$$\alpha_{(\chi,z)}(j,h) = \begin{pmatrix} 0 & & \dots & z \\ z & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & 0 \end{pmatrix}^j \begin{pmatrix} \chi(h) & 0 & \dots & 0 \\ 0 & \chi(th) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \chi(t^{n-1}h) \end{pmatrix}$$

defines an SL$(n,\mathbb{C})$ representation whose isomorphism type of this representation does not depend on the choice of z. In our notation we will not normally distinguish between metabelian representations of $\pi_1(N_K)$ and representations of $\mathbb{Z} \ltimes H$.

In the following we say that a character $\chi\colon\ H \to \mathbb{C}^*$ has *order* n if it factors through $H/(t^n - 1)$, but not through $H/(t^\ell - 1)$ for any $\ell < n$. Given a character $\chi\colon\ H \to \mathbb{C}^*$, let $t^i\chi$ be the character defined by $(t^i\chi)(h) = \chi(t^i h)$. Any character $\chi\colon\ H \to \mathbb{C}^*$ which factors through $H/(t^n - 1)$ must have order k for some divisor k of n. The following is a combination of [BF08, Lemma 2.2] and [BF08, Theorem 3.3].

Theorem 2. *Suppose $\chi\colon\ H \to \mathbb{C}^*$ is a character that factors through $H/(t^n - 1)$.*

(i) $\alpha_{(n,\chi)}\colon\ \mathbb{Z} \ltimes H \to$ SL$(n,\mathbb{C})$ *is irreducible if and only if the character χ has order n.*

(ii) *Given two characters* $\chi, \chi'\colon H \to \mathbb{C}^*$ *of order* n*, the representations* $\alpha_{(n,\chi)}$ *and* $\alpha_{(n,\chi')}$ *are conjugate if and only if* $\chi = t^k\chi'$ *for some* k*.*
(iii) *For any irreducible representation* $\alpha\colon \mathbb{Z} \ltimes H \to \mathrm{SL}(n,\mathbb{C})$ *there exists a character* $\chi\colon H \to \mathbb{C}^*$ *of order* n *such that* α *is conjugate to* $\alpha_{(n,\chi)}$*.*

3. Main results

3.1. **Metabelian characters as fixed points.** Set $\omega = e^{2\pi i/n}$ and recall the action of the cyclic group $\mathbb{Z}/n = \langle \sigma \mid \sigma^n = 1\rangle$ on representations $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ obtained by setting $(\sigma\cdot\alpha)(g) = \omega^{\varepsilon(g)}\alpha(g)$ for all $g \in \pi_1(N_K)$, where $\varepsilon\colon \pi_1(N_K) \to H_1(N_K) = \mathbb{Z}$.

We begin with the following lemma.

Lemma 3. *Suppose* $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ *is a representation whose associated character* $\xi_\alpha \in X_n(N_K)$ *is a fixed point of the* $\mathbb{Z}/n$ *action. Then up to conjugation, we have*

$$\alpha(\mu) = \begin{pmatrix} 0 & & \dots & z \\ z & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & 0 \end{pmatrix}, \tag{1}$$

for some (in fact any) $z \in U(1)$ *such that* $z^n = (-1)^{n+1}$*.*

Proof. Let $c(t) = \det(\alpha(\mu) - tI)$ denote the characteristic polynomial of $\alpha(\mu)$, which we can write as

$$c(t) = (-1)^n t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + 1.$$

Note that $c(t)$ is determined by the character $\xi_\alpha \in X_n(N_K)$, and so assuming ξ_α is a fixed point of the $\mathbb{Z}/n$ action, we conclude that $\alpha(\mu)$ and $\omega^k\alpha(\mu)$ have the same characteristic polynomials for all k. In particular,

$$\begin{aligned} c(t) &= \det(\alpha(\mu) - tI) \\ &= \det(\omega^{-1}\alpha(\mu) - tI) \\ &= \det(\omega^{-1}\alpha(\mu) - (\omega^{-1}\omega)tI) \\ &= \det(\omega^{-1}I)\det(\alpha(\mu) - \omega tI) \\ &= \det(\alpha(\mu) - t\omega I) = c(\omega t). \end{aligned}$$

However, $\omega^k \neq 1$ unless $n|k$, and this implies $0 = c_{n-1} = c_{n-2} = \cdots = c_1$ and $c(t) = (-1)^n t^n + 1$. In particular the matrix $\alpha(\mu)$ and the matrix appearing in Equation (1) have the same set of n distinct eigenvalues. This implies that the two matrices are conjugate. □

In order to prove Theorem 1, we establish the following more general result.

Theorem 4. *The fixed point set of the $\mathbb{Z}/n$ action on $X_n(N_K)$ consists of characters ξ_α of the metabelian representations $\alpha = \alpha_{(n,\chi)}$ described in Section 2.3. In other words,*

$$X_n(N_K)^{\mathbb{Z}/n} = \{\xi_\alpha \mid \alpha = \alpha_{(n,\chi)} \text{ for } \chi\colon\ H_1(N_K;\mathbb{Z}[t^{\pm 1}]) \to \mathbb{C}^*\}.$$

Notice that Theorem 1 can be viewed as the special case of Theorem 4 where α is irreducible. (Recall that irreducible representations are conjugate if and only if they define the same character.) Notice further that not every reducible metabelian representation is of the form $\alpha_{(n,\chi)}$.

Proof. We first show that if $\alpha\colon\ \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ is given as $\alpha = \alpha_{(n,\chi)}$, then $\sigma \cdot \alpha$ is conjugate to α. This of course implies that $\xi_\alpha = \xi_{\sigma\cdot\alpha}$.

Assume then that $\alpha = \alpha_{(n,\chi)}$. Then we have

$$\alpha(\mu) = \begin{pmatrix} 0 & \dots & & z \\ z & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & 0 \end{pmatrix},$$

where z satisfies $z^n = (-1)^{n+1}$. Further, $\alpha(g)$ is diagonal for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. By definition of $\sigma \cdot \alpha$, we see that

$$(\sigma\cdot\alpha)(\mu) = \omega\alpha(\mu) = \begin{pmatrix} 0 & \dots & & \omega z \\ \omega z & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \omega z & 0 \end{pmatrix}$$

and that $(\sigma\cdot\alpha)(g) = \alpha(g)$ for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. It follows easily from Theorem 2 (2) that $\sigma \cdot \alpha$ and $\alpha_{(n,\chi)}$ are conjugate; however it is easy to see this directly too. Simply take

$$P = \begin{pmatrix} 1 & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega^{n-1} \end{pmatrix},$$

and compute that $\sigma \cdot \alpha = P\alpha P^{-1}$ as claimed.

We now show the other implication, namely that each point $\xi \in X_n(N_K)^{\mathbb{Z}/n}$ in the fixed point set can be represented as the character $\xi = \xi_\alpha$ of a metabelian representation $\alpha = \alpha_{(n,\chi)}$, where $\chi\colon\ H_1(N_K;\mathbb{Z}[t^{\pm 1}]) \to \mathbb{C}^*$ is a character that factors through $H_1(N_K;\mathbb{Z}[t^{\pm 1}])/(t^n - 1)$, hence has order k for some k dividing n. (Note that Theorem 2 (1) tells us that $\alpha_{(n,\chi)}$ is irreducible if and only if χ has order n.)

By the general results on representation spaces and character varieties (see [LM85]), it follows that every point in the character variety $X_n(N_K)$ can be represented as ξ_α for some semisimple representation $\alpha\colon\ \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$. Further, two semisimple representations α_1 and α_2 determine the same character if and only if α_1 is conjugate to α_2. (This is evident from the fact that the orbits of the semisimple representations under conjugation are closed.)

Given $\xi \in X_n(N_K)^{\mathbb{Z}/n}$, we can therefore suppose that $\xi = \xi_\alpha$ for some semisimple representation α. Clearly $\sigma \cdot \alpha$ is also semisimple, and since $\xi_\alpha = \xi_{\sigma\cdot\alpha}$, we conclude from the above that α and $\sigma \cdot \alpha$ are conjugate representations. This means that there exists a matrix $A \in \mathrm{SL}(n, \mathbb{C})$ such that $A\alpha A^{-1} = \sigma \cdot \alpha$, in other words, for all $g \in \pi_1(N_K)$, we have

$$A\alpha(g)A^{-1} = \omega^{\varepsilon(g)}\alpha(g). \tag{2}$$

Lemma 3 implies $\alpha(\mu)$ is conjugate to the matrix in Equation (1). It is convenient to conjugate α so that $\alpha(\mu)$ is diagonal, meaning that

$$\alpha(\mu) = \begin{pmatrix} z & & & 0 \\ & \omega z & & \\ & & \ddots & \\ 0 & & & \omega^{n-1}z \end{pmatrix},$$

where z satisfies $z^n = (-1)^{n+1}$.

We now apply Equation (2) to the meridian to conclude that

$$A\alpha(\mu) = \omega\alpha(\mu)A,$$

which implies $A = (a_{ij})$ satisfies $a_{ij} = 0$ unless $j = i + 1 \mod (n)$. Thus, we see that

$$A = \begin{pmatrix} 0 & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda_{n-1} \\ \lambda_n & 0 & \dots & 0 & 0 \end{pmatrix}$$

for some $\lambda_1, \dots, \lambda_n$ satisfying $\lambda_1 \cdots \lambda_n = (-1)^{n+1}$.

It is completely straightforward to see that the characteristic polynomial of A is given by

$$\det(A - tI) = (-1)^n(t^n - (-1)^{n+1}).$$

From this, we conclude that A has as its eigenvalues the n distinct n–th roots of $(-1)^{n+1}$. In particular, the subset of $\mathrm{SL}(n, \mathbb{C})$ of matrices that commute with A is just a copy of the unique maximal torus $T_A \cong (\mathbb{C}^*)^{n-1}$ containing A.

For any $g \in [\pi_1(N_K), \pi_1(N_K)]$, we have $\alpha(g) = (\sigma \cdot \alpha)(g)$. Thus it follows that $A\alpha(g)A^{-1} = \alpha(g)$, and this implies that $\alpha(g) \in T_A$ for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. This shows that the restriction of α to the commutator subgroup $[\pi_1(N_K), \pi_1(N_K)]$ is abelian, and we conclude from this that α is indeed metabelian. Notice that this, and an application of Theorem 2 (iii), completes the proof in the case α is irreducible.

In the general case, it follows from the discussion in Section 2.2 that α factors through $\mathbb{Z} \ltimes H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. Let $H = H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. Given a character $\chi\colon\ H \to \mathbb{C}^*$

we define the associated weight space V_χ by setting

$$V_\chi = \{v \in \mathbb{C}^n \mid \chi(h) \cdot v = \alpha(h)v \text{ for all } h \in H\}.$$

Recall that $A \cdot \alpha(h) \cdot A^{-1} = \alpha(h)$ for any $h \in H$. It is straightforward so show that A restricts to an automorphism of V_χ. Since H is abelian there exists at least one character $\chi\colon H \to \mathbb{C}^*$ such that V_χ is non–trivial. For any i we denote by $t^i\chi$ the character given by $(t^i\chi)(h) = \chi(t^ih), h \in H$.

Note that A has n distinct eigenvalues and therefore is diagonalizable. Since A restricts to an automorphism of V_χ, there is an eigenvector v of A which lies in V_χ. Let λ be the corresponding eigenvalue. By the proof of [BF08, Theorem 2.3], the map $\alpha(\mu)$ induces an isomorphism $V_\chi \to V_{t\chi}$. We now calculate

$$A \cdot \alpha(\mu)v = (A\alpha(\mu)A^{-1}) \cdot Av = \omega\alpha(\mu) \cdot \lambda v = \lambda\omega \cdot \alpha(\mu)v,$$

i.e. $\alpha(\mu)v \in V_{t\chi}$ is an eigenvector of A with eigenvalue $\omega\lambda$.

Iterating this argument, we see that $\alpha(\mu)^iv$ lies in $V_{t^i\chi}$ and is an eigenvector of A with eigenvalue $\omega^i\lambda$. Since ω is a primitive n–th root of unity, the eigenvalues $\lambda, \omega\lambda, \dots, \omega^{n-1}\lambda$ are all distinct, and this implies that the corresponding eigenvectors $v, \alpha(\mu)v, \dots, \alpha(\mu)^{n-1}v$ form a basis for $\mathbb{C}^n$.

Let m be the order of χ, i.e. m is the minimal number such that $\chi = t^m\chi$. By the above we see that $\mathbb{C}^n$ is generated by $V_\chi, V_{t\chi}, \dots, V_{t^m\chi}$. Since the characters $\chi, t\chi, \dots, t^m\chi$ are pairwise distinct, it follows that $\mathbb{C}^n$ is given as the direct sum $V_\chi \oplus V_{t\chi} \oplus \dots \oplus V_{t^{m-1}\chi}$.

We write $k = \dim_{\mathbb{C}}(V_\chi)$ and note that $n = km$. We note further that $\alpha(\mu)^m$ has eigenvalues given by the set

$$\{z^m, z^m e^{2\pi i/k}, \dots, z^m e^{2\pi i(k-1)/k}\}, \tag{3}$$

and each eigenvalue has multiplicity m. Clearly $\alpha(\mu)^m$ restricts to an automorphism of $V_{t^i\chi}$ for $i = 0, \dots, m-1$, and equally clearly we see that the restrictions all give conjugate representations. This implies that the restriction of $\alpha(\mu)^m$ to V_χ has eigenvalues in the set (3) above, each occurring with multiplicity 1. In particular we can find a basis $\{v_1, \dots, v_k\}$ for V_χ in which the matrix of $\alpha(\mu)^m$ has the form

$$\alpha(\mu^m) = \begin{pmatrix} 0 & \dots & & z^m \\ z^m & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z^m & 0 \end{pmatrix}.$$

It is now straightforward to verify that with respect to the ordered basis

$$\left\{ \begin{array}{cccc} v_1, & z^{-1}\alpha(\mu)v_1, & \dots, & z^{-(m-1)}\alpha(\mu)^{m-1}v_1, \\ v_2, & z^{-1}\alpha(\mu)v_2, & \dots, & z^{-(m-1)}\alpha(\mu)^{m-1}v_2, \\ \vdots & \vdots & \dots & \vdots \\ v_k, & z^{-1}\alpha(\mu)w_k, & \dots, & z^{-(m-1)}\alpha(\mu)^{m-1}v_k \end{array} \right\},$$

α is given by $\alpha(n,\chi)$. □

3.2. **Application to twisted Alexander polynomials.** As an application, we now prove the following result regarding twisted Alexander polynomials of knots corresponding to metabelian representations. In the following, we use $\Delta^{\alpha}_{K,i}(t)$ to denote the i–th twisted Alexander polynomial for a given representation $\alpha\colon \pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$ as presented in [FV09].

Proposition 5. *Let α be a metabelian representation of the form $\alpha=\alpha_{(n,\chi)}\colon \pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$. Then*

$$\Delta^{\alpha}_{K,0}(t)=\begin{cases} 1-t^n, & \text{if } \chi \text{ is trivial,}\\ 1, & \text{otherwise.}\end{cases}$$

Furthermore the twisted Alexander polynomial $\Delta^{\alpha}_{K,1}(t)$ is actually a polynomial in t^n.

Remark 6. In their paper [HKL08], C. Herald, P. Kirk, and C. Livingston prove the same result using an entirely different approach (cf. p. 10 of [HKL08]). We also point out that Proposition 5 gives a positive answer to Conjecture A from a recent paper by M. Hirasawa and K. Murasugi (see [HM09]).

Proof. The proof of the first statement is not difficult. It is immediate when χ is trivial, and it follows by a direct calculation when χ is non–trivial.

We now turn to the proof of the second statement. For $\theta\in U(1)$ and any representation $\beta:\pi_1(N_K)\to \mathrm{GL}(n,\mathbb{C})$, define the θ-twist of β to be the representation sending $g\in\pi_1(N_K)$ to $\theta^{\varepsilon(g)}\beta(g)$, where $\varepsilon\colon \pi_1(N_K)\to\mathbb{Z}$ is determined by the orientation of K. We denote the newly obtained representation by $\beta_\theta\colon \pi_1(N_K)\to \mathrm{GL}(n,\mathbb{C})$. Note that in case $\alpha:\pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$ and $\theta=e^{2\pi ik/n}$ is an n-th root of unity, α_θ is again an $\mathrm{SL}(n,\mathbb{C})$ representation. The proof of the proposition relies on the formula

$$\Delta^{\beta_\theta}_{K,1}(t)=\Delta^{\beta}_{K,1}(\theta t). \tag{4}$$

This formula is well-known and follows directly from the definition of the twisted Alexander polynomial. Equation (4) combines with Theorem 1 to complete the proof, as we now explain. Take $\omega=e^{2\pi i/n}$. If $\alpha=\alpha_{(n,\chi)}$ is metabelian, then Theorem 1 shows that its conjugacy class is fixed under the $\mathbb{Z}/n$ action. In particular, since α and α_ω are conjugate, Equation (4) shows that

$$\Delta^{\alpha}_{K,1}(t)=\Delta^{\alpha_\omega}_{K,1}(t)=\Delta^{\alpha}_{K,1}(\omega t).$$

Expanding $\Delta^{\alpha}_{K,1}(t)=\sum a_it^i$ and using the fact that $t^k=(\omega t)^k$ if and only if k is a multiple of n, this shows that $a_k=0$ unless k is a multiple of n and this completes the proof. □

References

[BF08] H. U. Boden and S. Friedl, *Metabelian* SL$(n, \mathbb{C})$ *representations of knot groups,* Pacific J. Math. **238** (2008), 7–25.

[Bu67] G. Burde, *Darstellungen von Knotengruppen*, Math. Ann. 173 (1967) 24–33

[Bu90] G. Burde, Knots, *$SU(2)$-representation spaces for two-bridge knot groups,* Math. Ann. **288** (1990), no. 1, 103–119

[BZ03] G. Burde and H. Zieschang, *Knots*, Second edition. de Gruyter Studies in Mathematics, 5. Walter de Gruyter & Co., Berlin, 2003.

[dRh68] G. de Rham, *Introduction aux polynômes d'un nœud*, Enseignement Math. (2) 13, 187–194 (1968).

[Fo70] R. H. Fox, *Metacyclic invariants of knots and links*, Canad. J. Math. 22 1970 193–201.

[FV09] S. Friedl and S. Vidussi, *A survey of twisted Alexander polynomials*, 2009 preprint, math.GT/0905.0591.

[Ha79] R. Hartley, *Metabelian representations of knot groups*, Pacific J. Math. 82 (1979), no. 1, 93–104.

[Ha83] R. Hartley, *Lifting group homomorphisms*, Pacific J. Math. 105 (1983), no. 2, 311–320.

[HKL08] C. Herald, P. Kirk, and C. Livingston, *Metabelian representations, twisted Alexander polynomials, knot slicing, and mutation*, 2008 preprint, to appear in Math. Z. math.GT/0804.1355.

[HM09] M. Hirasawa and K. Murasugi, *Twisted Alexander polynomials of 2-bridge knots associated to metabelian representations*, 2009 preprint math.GT 0903.1689

[Je08] H. Jebali, *Module d'Alexander et représentations métabéliennes*, Annales de la faculté des sciences de Toulouse Sér. 6, 17 no. 4 (2008), p. 751-764.

[Le00] C. Letsche, *An obstruction to slicing knots using the eta invariant*, Math. Proc. Cambridge Philos. Soc. 128 (2000), no. 2, 301–319.

[Lin01] X. S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin. (Engl. Ser.) 17 (2001), no. 3, 361–380.

[Li95] C. Livingston, *Lifting representations of knot groups*, J. Knot Theory Ramifications 4 (1995), no. 2, 225–234.

[LM85] A. Lubotzky and A. R. Magid, *Varieties of representations of finitely generated groups*, Memoirs of the Amer. Math. Soc. Vol. 58, No. 336 (1985).

[Na07] F. Nagasato, *Finiteness of a section of the* SL$(2, \mathbb{C})$*–character variety of knot groups*, Kobe J. Math., **24** no. 2 (2007) 125–136.

[NY08] F. Nagasato and Y. Yamaguchi, *On the geometry of a certain slice of the character variety of a knot group*, 2008 preprint, math.GT/0807.0714.

[Ne65] L. P. Neuwirth, *Knot groups*, Annals of Mathematics Studies, No. 56 Princeton University Press, Princeton, N.J. (1965)

Department of Mathematics, McMaster University, Hamilton, Ontario
E-mail address: boden@mcmaster.ca

University of Warwick, Coventry, UK
E-mail address: sfriedl@gmail.com