Unitary Representations of Brieskorn Spheres

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Abstract

In this article, we commence an investigation of the SU(N) representation space of Seifert fibered homology spheres $\Sigma(a_1, \ldots, a_n)$. Under mild assumptions (e.g. if N is prime), then Theorem 3.1 implies that any closed connected component of irreducible SU(N) representations of $\Sigma(a_1, \ldots, a_n)$ is homeomorphic to a component of SU(N) representations of an associated genus zero Fuchsian group. The latter representation spaces can be studied using the general correspondence between representations of Fuchsian groups and the moduli of parabolic bundles given by Mehta and Seshadri. For example, the inductive procedure of Atiyah-Bott-Nitsure determines the cohomology of this moduli space and it follows that the odd dimensional cohomology groups of any component of irreducible SU(N) representations of $\Sigma(a_1, \ldots, a_n)$ vanish. In particular, any irreducible component of the SU(3) representation space of a Brieskorn spheres $\Sigma(p, q, r)$ is either a point or a two sphere. By repeated application of the inductive procedure, the precise number of points and two spheres in this representation space is determined. Specific results for the Brieskorn spheres with $p = 2$ are given, where the representation space is a collection of points. In the last section, the SU(N) spectral flow of irreducible representations of Seifert fibered homology spheres is shown to be *even*. This gives a calculation of the leading term in a gauge-theoretic definition of the generalized Casson invariants.

1 Introduction

There is a rich and elegant theory of representations of finite groups. Up to conjugation, there are only finitely many distinct irreducible representations in any given rank, and the collection of all irreducible representations satisfy a famous arithmetic relation [17].

Suppose that (p, q, r) are pairwise relatively prime and let $\Sigma(p, q, r)$ be a Brieskorn sphere, that is, the link of the singularity of the variety $x^p + y^q + z^r = 0$ in $\mathbb{C}^3$. The rank two representation theory of the groups $\pi_1\Sigma(p, q, r)$ shares many properties with that of finite groups. In particular, up to conjugation, there are only finitely many irreducible representations of rank two. Counting the number of these representations immediately yields Casson's invariant. This follows from the observation of Fintushel and Stern [8] that the SU(2) spectral flow of any Seifert fibered homology sphere is even and the characterization of Casson's invariant as the Euler characteristic for Floer homology [18]. In the general case of a Seifert fibered homology sphere $\Sigma(a_1, \ldots, a_n)$ (the link of the singularity of complete intersection of complex dimension 2 in $\mathbb{C}^n$), the SU(2) representation space is not discrete but has components of dimension $2m$ for each $0 \leq m \leq n-3$. The perturbation argument of §4 of [8] along with the results in [12] show that Casson's invariant of $\Sigma(a_1, \ldots, a_n)$ is just the Euler characteristic of its SU(2) representation space.

In this paper, we consider the problem of describing in rather general terms the SU(N) representation space of Seifert fibered homology spheres $\Sigma(a_1, \ldots, a_n)$. For example, it is shown that the odd dimensional homology of any connected component of irreducible SU(N) representations vanishes. This is done by interpreting it as the moduli of parabolic bundles over the Riemann sphere. Since any component of irreducible SU(3) representations of a Brieskorn sphere has dimension ≤ 2, it follows that it is either a point or a two sphere. Restricting our attention further to the Brieskorn spheres $\Sigma(2, p, q)$, the SU(3) representation space is just a discrete set of points.

By the *leading term* of the SU(3) Casson invariant, we mean $\sum_\rho (-1)^{SF(\Theta,\rho)}$ where the sum is taken over ρ an irreducible representation in a (possibly perturbed) representation space. The other term is a sum of Maslov indices over the reducibles, which is more subtle to define and is not discussed here (cf. [7]). For ρ an irreducible SU(N) representation of a Seifert fibered homology sphere, we prove that $SF(\Theta, \rho)$ is always even. Thus, for the Brieskorn spheres $\Sigma(2, p, q)$, the calculation of the leading term of the SU(3) Casson invariant is reduced to the problem of counting the number of irreducible representations. The formula we adopt for the generalized Casson invariants suggests that there may be an SU(N) Floer homology for 3-manifolds, but this is beyond the scope of this paper.

We now briefly outline the contents of each section. In §2, we derive some general results about reducible representations of perfect groups and, more specifically, fundamental groups of Seifert fibered homology spheres $\Sigma(a_1, \ldots a_n)$. In §3, we prove that if all but one of the Seifert numbers $a_1, \ldots, a_n$ are relatively prime to N, then any connected component of irreducible SU(N) representations of $\Sigma(a_1, \ldots a_n)$ is homeomorphic to a component of a SU(N) representations of the genus zero Fuchsian group $T(Na_1, \ldots a_n)$. Proposition 3.2 gives a formula for the dimension of a given component of the SU(N) representation space of T, from which it follows that the SU(3) representation spaces of Brieskorn spheres have dimension ≤ 2.

In §4, the moduli of parabolic bundles is introduced and the inductive procedure of Atiyah-Bott-Nitsure is discussed. The necessary definitions are given in §4.1 and the inductive procedure is outlined in general terms in §4.2. In short, given a parabolic bundle E, the Harder Narasimhan filtration of E, together with an intersection matrix I_p defined for each parabolic point p, determines a stratification on the space of holomorphic structures on E which is equivariantly perfect with respect to the gauge group $\mathcal{P}$ of parabolic automorphisms of E [16]. In particular, one can deduce the $\mathcal{P}$-equivariant cohomology of the top stratum of semistable bundles by knowing the equivariant cohomology of each lower stratum. In the case semistable = stable, then the moduli is the quotient of this top stratum by the gauge group $\mathcal{P}$. If, in addition, the parabolic structure is nontrivial, then Proposition 4.8 proves that the cohomology of the moduli space is torsion free and in fact the tensor product of the equivariant cohomology of the semistable bundles with the cohomology of the classifying space of the isotropy group (just BU(1)). A corollary is that if the underlying curve is $\mathbb{CP}^1$, then the odd dimensional cohomology of the moduli vanishes. In particular, any moduli of complex dimension 1 is isomorphic to $\mathbb{CP}^1$. In §4.3, two specific examples are presented for rank three parabolic bundles over $\mathbb{CP}^1$.

In §5 we return to the study of representations of Brieskorn spheres. §5.1 applies the theory from the previous sections to study the SU(3) representation spaces of Brieskorn spheres. Specific results are listed in tables at the end of §5.1. Then in §5.2 the method of [8] is generalized to compute the SU(N) spectral flow of irreducible representations of Seifert fibered homology spheres. It is proved that the SU(N) spectral flow of any irreducible representation of a Seifert fibred homology sphere is even. This shows that the results in §5.1 determine the leading term of the generalized SU(3) Casson invariant for these Brieskorn spheres.

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2 Reducible Representations

Let G denote a compact Lie group with center Z. Define the space of representations of a finitely presented group π into G, denoted $Rep(\pi, G)$, to be the set of homomorphisms $\rho : \pi \to G$, with the usual (compact-open) topology. The presentation

$$\pi = \langle x_1, \ldots, x_n \mid w_1, \ldots, w_m \rangle,$$

describes $Rep(\pi, G)$ as an algebraic variety in $G \times \overset{n}{\cdots} \times G$ by identifying $\rho \in Rep(\pi, G)$ with the images of the generators $X_i = \rho(x_i)$. Thus

$$Rep(\pi, G) \cong \{X_1, \ldots, X_n \mid X_i \in G, w_j(X_1, \ldots, X_n) = 1\}.$$

By the representation space, denoted $R(\pi, G)$, we just mean the space of representations modulo the G action by conjugation, i.e.

$$R(\pi, G) = Rep(\pi, G)/conj.$$

Definition 2.1 *(i) For any $S \subset G$, define $Z(S) = \{\gamma \in G \mid \gamma s = s\gamma$ for all $s \in S\}$.*
(ii) Let $Rep^*(\pi, G) = \{\rho \in Rep(\pi, G) \mid Z(im(\rho)) = Z(G)\}$ be the subset of irreducible representations. Since Rep^* is invariant under conjugation, we may define $R^* = Rep^*/conj$.

Lemma 2.2 *Suppose that $H_1(\pi) = 0$, i.e. $\pi = [\pi, \pi]$.*
(i) If $\rho \in Rep(\pi, U(n))$ is reducible, then by conjugating if necessary, we have

$$im(\rho) \subset \mathrm{SU}(n_1) \times \cdots \times \mathrm{SU}(n_k).$$

(ii) In particular, if $\rho \in Rep(\pi, U(3))$ is reducible, then $im(\rho) \subset \mathrm{SU}(2)$.

proof: First observe that $\rho \in Rep(\pi, U(n)) \Rightarrow im(\rho) \subset \mathrm{SU}(n)$. This follows from the observation that $H_1(\pi) = 0 \Rightarrow \det(\rho) : \pi \to \mathrm{U}(1)$ is trivial.

Suppose that ρ is reducible. This means that, after conjugating, we may assume $im(\rho) \subset U(n_1) \times \cdots \times U(n_k)$. Now (i) follows by applying the initial observation to each component $U(n_i)$, and part (ii) is an immediate consequence of (i). ♠

Henceforth in this section, unless stated otherwise, $\Sigma = \Sigma(a_1, \ldots, a_n)$ will denote a Seifert fibered homology sphere. Its fundamental group has a standard presentation

$$\pi_1\Sigma = \langle x_1, \ldots, x_n, h \mid h \text{ central}, \; x_i^{a_i} = h^{-b_i}, x_1 \cdots x_n = h^{-b_0} \rangle. \tag{1}$$

Here, the b_i are not unique but must satisfy

$$a(-b_0 + \sum_{i=1}^{n} \frac{b_i}{a_i}) = 1 \tag{2}$$

where $a = a_1 \cdots a_n$. Recalling that the center of SU(N) is isomorphic to $\mathbb{Z}_N$ and adopting the notation for manifolds Σ, $R(\Sigma, G) \equiv R(\pi_1\Sigma, G)$, we have

Lemma 2.3 *(i) If $\rho \in R^*(\Sigma, \mathrm{SU(N)})$, then $\rho(h) \in \mathbb{Z}_N$.*
(ii) If $\rho \in R(\Sigma, \mathrm{SU(N)})$ and $\rho(x_i) \in \mathbb{Z}_N$ for $1 \leq i \leq n-2$, then ρ is trivial.
(iii) $R(\Sigma(p, q, r), \mathrm{SU}(2))$ consists of a finite collection of points.
(iv) If $\rho \in R^*(\Sigma(2, p, q), \mathrm{SU}(2))$, then $\rho(h) = -I$.

proof: The presentation (1) implies that $\rho(h) \in Z(im(\rho))$ and (i) follows from the definition of irreducibility. To see (ii), suppose that $\rho \in R(\Sigma, \mathrm{SU(N)})$ is a representation with $\rho(h), \rho(x_1), \ldots, \rho(x_{n-2}) \in \mathbb{Z}_N$. Then the last relation of (1) implies that $\rho(x_{n-1})\rho(x_n) \in \mathbb{Z}_N$, from which it follows that $\rho(x_{n-1})$ and $\rho(x_n)$ commute. But that implies ρ is abelian, so ρ is trivial. Statement (iii) follows from [8] (or equivalently, from the dimension count of Proposition 3.2). To prove (iv), first note that $\rho(h) = \pm I$ since h is central. If $\rho(h) = 1$, then ρ factors through to give a representation of the triangle group

$$T(2, p, q) = \langle x_1, x_2, x_3 \mid x_1^2 = x_2^p = x_3^q = x_1x_2x_3 = 1 \rangle.$$

It is left as an exercise to see that there are no nontrivial SU(2) representations of the above triangle group (using $X_1^2 = 1$). ♠

Suppose that $\rho_1 \in R(\Sigma, \mathrm{SU(N)})$ is a reducible representation which is the endpoint of a path of irreducible representations ρ_t. Since $\rho_t(h) \in \mathbb{Z}_N$ and is continuous in t, it is constant. The following proposition now follows from (iii) and (iv) of the previous lemma.

Proposition 2.4 *If $\rho \in R(\Sigma(p, q, r), \mathrm{SU}(3))$ with $\rho(h) = diag(1, -1, -1)$ up to conjugation, then ρ is an isolated reducible representation. In particular, every reducible representation $\rho \in R(\Sigma(2, p, q), \mathrm{SU}(3))$ is isolated.*

3 Unitary Representations of Fuchsian Groups

In this section, a general result relating the SU(N) representation space of Seifert-fibered homology spheres to representations of a certain Fuchsian group is proved. For technical reasons, we shall assume that N is relatively prime to all but one of the Seifert numbers. We conclude by giving a formula for the dimension of this representation space.

Fix SU(N) and let $\Sigma = \Sigma(a_1, \ldots, a_n)$ be any Seifert-fibered homology sphere whose Seifert numbers a_i are relatively prime to N for $i > 1$. (Up to reordering, this will always hold provided N is prime.) Using the notation of the presentation (1) for the group $\pi_1\Sigma$, Define

$$Rep_c(\Sigma, \mathrm{SU(N)}) = \{\rho \in R(\Sigma, \mathrm{SU(3)}) \mid \rho(h) = \mathbb{Z}_N\}$$

and

$$R_c(\Sigma, \mathrm{SU(N)}) = Rep_c(\Sigma, \mathrm{SU(N)})/conj.$$

Notice that $R^*(\Sigma, \mathrm{SU(N)}) \subseteq R_c(\Sigma, \mathrm{SU(N)})$ by Lemma 2.3.

Now the quotient of Σ by the natural circle action is an orbifold X of dimension 2 with genus 0 and n cone points of cone angles $2\pi/a_i$. Its orbifold fundamental group is just $\pi_1\Sigma/\langle h = 1\rangle$ and has the presentation as the genus zero Fuchsian group

$$T(a_1, \ldots, a_n) = \langle y_1, \ldots, y_n \mid y_i^{a_i} = 1, y_1 \cdots y_n = 1\rangle.$$

We wish to relate the SU(N) representation spaces of Σ and its quotient because representations of $T(a_1, \ldots, a_n)$ can be studied with stable parabolic bundles. In order to accurately relate the two representation spaces, we must multiply the order of the first cone point by N.

So consider the group $T = T(Na_1, a_2, \ldots, a_n)$. Denote by $\overline{T}$ the quotient

$$\overline{T} = T/\langle y_1^{a_1} \text{ is central } \rangle.$$

The quotient map identifies $R(\overline{T}, \mathrm{SU(N)})$ with a submanifold of $R(T, \mathrm{SU(N)})$ which we denote $R_c(T, \mathrm{SU(N)})$. Then we have

Theorem 3.1 $R_c(\Sigma, \mathrm{SU(N)}) \cong R_c(T, \mathrm{SU(N)})$.

proof: First, in the presentation (1), we may choose b_i divisible by N for $i \neq 1$ [15]. In this way, we see that $\overline{T}$ is also a quotient of $\pi_1(\Sigma)$, namely

$$\overline{T} = \pi_1(\Sigma)/\langle h^N = 1\rangle.$$

The quotient map defines a continuous one-to-one map

$$\Phi : R(\overline{T}, \mathrm{SU(N)}) \to R(\Sigma, \mathrm{SU(N)}).$$

To see that Φ is surjective on R_c, consider $\rho \in Rep_c(\Sigma, \mathrm{SU(N)})$. Since $\rho(h)^N = I$, we see that ρ factors through to $\overline{T}$. It is immediate that Φ^{-1}, defined on R_c, is continuous. ♠

Before we present the next proposition, which describes the Zariski tangent space of an irreducible SU(N) representation of $\Sigma(a_1, \ldots, a_n)$, we develop some notation.

Let W be the mapping cylinder of the Seifert fibration $\Sigma(a_1, \ldots, a_n) \to S^2$. View this map as an orbifold circle bundle over X, the orbifold with underlying space S^2 and with n cone points of orders $a_1, \ldots, a_n$. Thus W is an orbifold with singularities which are cones on lens spaces and it is well known that $\pi_1^{orb} X \cong \pi_1^{orb} W \cong T(a_1, \ldots, a_n)$. Furthermore, if $\rho \in R^*(\Sigma(a_1, \ldots, a_n), \mathrm{SU(N)})$, then

$$\mathrm{H}^1(\Sigma(a_1, \ldots, a_n), \mathfrak{ad}(\rho)) \cong \mathrm{H}^1(W, \mathfrak{ad}(\rho)) \cong \mathrm{H}^1(X, \mathfrak{ad}(\rho)),$$

which follows by interpreting each term as group cohomology. Since ρ is irreducible, $\mathrm{H}^0(X, \mathfrak{ad}(\rho)) = 0$. By Poincaré duality (using an ad-invariant inner product of $\mathfrak{su}(\mathrm{N})$ on the fibers of $\mathfrak{ad}(\rho)$), we see also $\mathrm{H}^2(X, \mathfrak{ad}(\rho)) = 0$. We could use the Fox differential calculus to compute $\mathrm{H}^1(X, \mathfrak{ad}(\rho))$, but there is a short-cut which exploits the irreducibility of ρ and involves only counting the dimensions of the conjugacy classes of each $\rho(y_i)$.

First some general remarks about elements in SU(N). Any $Y \in \mathrm{SU(N)}$ is conjugate to a diagonal matrix $\exp(diag(\theta_1, \ldots, \theta_N))$ where $0 \leq \theta_1 \leq \cdots \leq \theta_N < 1$. Since $\det(Y) = 1$, we have $\sum_1^N \theta_i \in \mathbb{Z}$. Further, if $Y^a = I$, then $\theta_j = l_j/a$ for integers $0 \leq l_j < a$. Given a, there are only finitely many conjugacy classes of a^{th} roots of unity in SU(N). One can list them by their diagonal representatives $\exp(diag(l_1/a, \ldots, l_N/a))$ where $0 \leq l_1/a \leq \cdots \leq l_N/a < 1$.

The conjugacy class of Y, denoted $C(Y)$, is just SU(N) modulo Γ_Y, the isotropy subgroup of Y. Writing $\Theta = \{\eta_1, \ldots, \eta_s\}$, where $0 \leq \eta_1 < \cdots < \eta_s < 1$ are listed without multiplicity, and defining m_j to be the multiplicity of η_j (i.e. the dimension of the $e^{2\pi i \eta_j}$ eigenspace of Y), then $\Gamma_Y \cong \mathrm{S}(\mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_s))$ and it follows that

$$\dim C(Y) = N^2 - \sum_{i=1}^{s} m_i^2.$$

We can apply these considerations to the representations of the genus zero Fuchsian groups by identifying $\rho \in Rep(T(a_1, \ldots, a_n), \mathrm{SU(N)})$ with the points

$$(Y_1, \ldots, Y_n) \in \mathrm{SU(N)} \times \overset{n}{\cdots} \times \mathrm{SU(N)}$$

satisfying $Y_i^{a_i} = I$ and $Y_1 \cdots Y_N = I$. (Obviously, $Y_i = \rho(y_i)$.) Choose α_i a diagonal a_i^{th} root of unity in SU(N) as above and set $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$. Define

$$Rep_{\vec{\alpha}} = \{\rho \in Rep(T(a_1, \ldots, a_n), \mathrm{SU(N)}) \mid \rho(y_i) \in C(\alpha_i)\}.$$

So, $Rep_{\vec{\alpha}} = \mu^{-1}(I)$ where

$$\mu : C(\alpha_1) \times \cdots \times C(\alpha_n) \to \mathrm{SU(N)}$$

is just the n-fold product $\mu(Y_1, \ldots, Y_n) = Y_1 \cdots Y_n$. If $I \notin im(\mu)$ then $Rep_{\vec{\alpha}} = \emptyset$. Otherwise $Rep_{\vec{\alpha}}$ is connected (this will follow from the correspondence between representation spaces and moduli of semistable parabolic bundles described in the next section), and a standard argument shows that for $\rho \in Rep^*_{\vec{\alpha}}$, $d\mu_\rho$ is surjective, i.e. irreducible representations are regular points of μ. Thus, letting $d_i = \dim C(\alpha_i)$, we find that

$$\dim Rep^*_{\vec{\alpha}} = \sum_{i=1}^{n} d_i \; - \dim \mathrm{SU(N)}.$$

Defining

$$R_{\vec{\alpha}} = Rep_{\vec{\alpha}}/conj,$$

then since the action of conjugation on the irreducibles is a free $\mathrm{PU}(N)$ action, we have shown

Proposition 3.2 *If* $\rho \in R^*_{\vec{\alpha}}$, *then* $\mathrm{H}^1(X, \mathfrak{ad}\rho) = \sum_1^n d_i \; - 2\dim \mathrm{SU(N)}$.

Remark: The above proposition holds for Fuchsian groups of genus g and one gets the formula $\dim R^*_{\vec{\alpha}} = (2g-2)\dim \mathrm{SU(N)} + \sum_1^n d_i$.

For example, in SU(3), because $d_i \in \{0, 4, 6\}$, there are only two possibilities for a nonempty representation space. The first possibility is if each $d_i = 6$ so that $\dim R_{\vec{\alpha}} = 2$ (in fact, it will follow later that $R_{\vec{\alpha}} \approx S^2$). The second possibility is if $d_1 = 4$ and $d_2 = 6 = d_3$, in which case $\dim R_{\vec{\alpha}} = 0$ in which case $R_{\vec{\alpha}}$ is a point.

Because there are only finitely many conjugacy classes of a^{th} roots of unity in SU(N) for a given integer a,

$$R(T(a_1, \ldots, a_n), \mathrm{SU(N)}) = \coprod_{\vec{\alpha}} R_{\vec{\alpha}},$$

and by Theorem 4.1 of [4],

$$R_{\vec{\alpha}} \cong \mathfrak{N}_{\vec{\alpha}}, \tag{3}$$

where $\mathfrak{N}_{\vec{\alpha}}$ denotes the moduli space of semistable parabolic bundles over $\mathbb{CP}^1$ with parabolic degree 0. There are n parabolic points $p_1, \ldots, p_n$ and the parabolic structure at p_i is determined by α_i (see §4 of [4]). Connectedness of $R_{\vec{\alpha}}$ now follows from Proposition 2.8 of [16] where it is proved that $\mathfrak{N}_{\vec{\alpha}}$ is connected (provided it is nonempty).

4 The Moduli Space of Parabolic Bundles

4.1 Definitions

We now introduce the moduli space of semistable parabolic bundles. Let M be a closed surface of genus g with a set $P = \{p_1, \ldots, p_n\}$ of n distinct points on M. Suppose that E is a $\mathbb{C}^N$ bundle over M.

Definition 4.1 *A topological parabolic structure in E is a collection of weighted flags in the fibers of E above each $p \in P$, i.e.*

$$E_p = F_1 \supset F_2 \supset \cdots \supset F_s \supset 0$$
$$0 \le \eta_1 < \eta_2 < \cdots < \eta_s < 1.$$

Each flag makes a local contribution to the parabolic degree, defined as

$$k = \sum_{i=1}^{s} m_i \eta_i,$$

where $m_i = \dim(F_i) - \dim(F_{i+1})$. Of course, all the structure associated with the flag depends on the parabolic point in question. When we want to emphasize this dependence, we shall write $F_i(p), \eta_i(p), s(p), m_i(p)$, and $k(p)$. The parabolic degree and slope of E are defined by

$$\mathbf{pd}(E) = \deg\, E + \sum_{p \in P} k(p),$$

$$\mu(E) = \frac{\mathbf{pd}(E)}{\mathrm{rank}\ E}.$$

Let $\mathcal{C}$ denote the space of all holomorphic structures on E.

Definition 4.2 *A given $d'' \in \mathcal{C}$ is called stable (semistable) if, for every proper holomorphic subbundle E', we have $\mu(E') < \mu(E)$ (respectively, $\mu(E') \leq \mu(E)$). Let $\mathcal{C}_s \subseteq \mathcal{C}_{ss}$ denote the subspaces of stable and semistable bundles.*

To construct the moduli space of semistable parabolic bundles, consider the gauge group $\mathcal{G}^{\mathbb{C}}$ of bundle automorphisms of E lying over M. Then $\mathcal{G}^{\mathbb{C}}$ acts on $\mathcal{C}$ but does not preserve $\mathcal{C}_{ss}$. We must instead consider the subgroup $\mathcal{P}$ of bundle automorphisms which preserve the flag structures, i.e.

$$\mathcal{P} = \{g \in \mathcal{G}^{\mathbb{C}} \mid g(F_j(p)) = F_j(p) \text{ over each parabolic point } p \in P\}.$$

Then $\mathcal{P}$ acts on $\mathcal{C}_{ss}$ and we can define the moduli $\mathfrak{N} = \mathcal{C}_{ss} /\!\!/ \mathcal{P}$. We adopt a notation reminiscent of the Mumford quotient (i.e. the double slash) to indicate the moduli is not simply the quotient by the group action, but rather further identifications need to be made in order to obtain a reasonable space (cf. §2 of [5] for details). We warn the reader that the moduli $\mathfrak{N}$ is not, strictly speaking, a Mumford quotient because the group $\mathcal{P}$ is not generally reductive. In this paper, we shall assume that $\mathcal{C}_s = \mathcal{C}_{ss}$, in which case the moduli space $\mathfrak{N} = \mathcal{C}_s/\mathcal{P}$ is an honest quotient and our notation should cause no confusion.

For each $d'' \in \mathcal{C}$, the parabolic bundle (E, d'') has a canonical filtration by parabolic subbundles

$$0 \subset E_1 \subset \cdots \subset E_r = E \tag{4}$$

with semistable quotients $D_i = E_i/E_{i-1}$ whose slopes $\mu_i = \mu(D_i)$ satisfy $\mu_i > \mu_{i+1}$. Letting $n_i = \text{rank } D_i$ and $\mathbf{pd}_i = \mathbf{pd}(D_i)$, then the Harder-Narasimhan type λ of (E, d'') is the polygon in $\mathbb{R}^2$ with vertices

$$(0,0), (n_1, \mathbf{pd}_1), \ldots, (n_r, \mathbf{pd}_r).$$

The *length* of λ, denoted $|\lambda|$, is just the integer r. Notice that $d'' \in \mathcal{C}_{ss} \Leftrightarrow |\lambda| = 1$. Unfortunately, setting

$$\mathcal{C}_\lambda = \{d'' \mid (E, d'') \text{ has type } \lambda\}$$

does not provide a nice $\mathcal{P}$-equivariant stratification for the simple reason that $\mathcal{C}_\lambda$ is generally disconnected. To get the desired stratification on $\mathcal{C}$, we need the more refined notion of compound type introduced by Nitsure [16]. The extra information to keep track of is how each subbundle E_i in the filtration intersects each flag $F_j(p)$, given by a matrix-valued function I_p of P defined as follows.

Definition 4.3 (Intersection Matrix) *For $p \in P$, set $d_{i,j} = \dim(E_i \cap F_j)$ and define the $r \times s(p)$ matrix I_p by*

$$I_p(i,j) = d_{i,j} - d_{i-1,j} - d_{i,j+1} + d_{i-1,j+1}.$$

(This is just the symmetric difference of (d_{ij}), but looks strange because the flags are descending while the filtration is ascending.)

Definition 4.4 (Compound Type) *Define the compound type of the parabolic bundle (E, d'') to be the pair (λ, I), where λ is the Harder-Narasimhan type of (E, d'') and I is the intersection matrix. Also, set*

$$\mathcal{C}_{\lambda,I} = \{d'' \mid (E, d'') \text{ has compound type } (\lambda, I)\},$$

the stratum of holomorphic structures with compound type (λ, I).

4.2 The Inductive Procedure of Atiyah-Bott-Nitsure

We now describe the inductive procedure of Atiyah and Bott, as modified by Nitsure, for the computation of the $\mathcal{P}$-equivariant cohomology of the space of semistable parabolic bundles. Because this calculation is both technical and key to our description of the representation spaces of Brieskorn spheres, the material is presented in a complete and self-contained way.

The idea is to relate the cohomology of $\mathfrak{N}$ in the case $\mathcal{C}_s = \mathcal{C}_{ss}$ to the $\mathcal{P}$-equivariant cohomology of $\mathcal{C}_{ss}$. There is a subtle issue due to the fact that $\mathcal{P}$ does not act freely on $\mathcal{C}_{ss}$, which is addressed in Proposition 4.8. In any case, one can calculate the $\mathcal{P}$-equivariant cohomology of $\mathcal{C}_{ss}$ using the $\mathcal{P}$-equivariantly perfect stratification on $\mathcal{C}$ determined by the compound type of the parabolic bundle [16]. After reviewing this calculation for moduli spaces of bundles of arbitrary rank, in the next section we give specific examples for rank 3 bundles over the Riemann sphere, which is the principal case of interest in this paper.

Nitsure proved that each stratum $\mathcal{C}_{\lambda,I}$, if nonempty, is connected, and that the stratification induced on $\mathcal{C}$ is $\mathcal{P}$-equivariantly perfect. This allows one to compute the $\mathcal{P}$-equivariant cohomology of $\mathcal{C}_{ss}$, the top stratum, by knowing the equivariant cohomology of all the other strata.

More precisely, because

$$\mathcal{C} = \mathcal{C}_{ss} \cup \bigcup_{\lambda,I} \mathcal{C}_{\lambda,I}$$

is a $\mathcal{P}$-equivariant, perfect stratification, it follows that

$$\tilde{P}_t(\mathcal{C}_{ss}) = \tilde{P}_t(\mathcal{C}) - \sum_{\lambda,I} t^{2d_{\lambda,I}} \tilde{P}_t(\mathcal{C}_{\lambda,I}) \tag{5}$$

where $d_{\lambda,I}$ denotes the complex codimension of the $\mathcal{C}_{\lambda,I}$ and $\tilde{P}_t$ refers to the equivariant Poincaré polynomial (cf. Proposition 3.8 of [16]). Each unstable stratum has equivariant cohomology isomorphic to the tensor product of the equivariant cohomologies of semistable strata of lower dimension (see Propositions 7.12 of [1] and 3.4 of [16]). In terms of the equivariant Poincaré polynomials, this means that

$$\tilde{P}_t(\mathcal{C}_{\lambda,I}) = \prod_{i=1}^{r} \tilde{P}_t(\mathcal{C}_{ss}(D_i)), \tag{6}$$

where for each $\tilde{P}_t$, we mean equivariant cohomology using the appropriate gauge group $\mathcal{P}(D_i)$ of parabolic automorphisms of D_i. Assuming by induction that $\tilde{P}_t(\mathcal{C}_{\lambda,I})$ are all known, to determine $\tilde{P}_t(\mathcal{C}_{ss})$, we need to find $\tilde{P}_t(\mathcal{C}) = P_t(\mathrm{B}\mathcal{P})$ and then we need to enumerate all the unstable strata which occur and to compute their codimensions.

Proposition 4.5 *The equivariant Poincaré polynomial of* $\mathrm{B}\mathcal{P}$ *is given by*

$$P_t(\mathrm{B}\mathcal{P}) = \left((1-t^{2N}) \prod_{k=1}^{N} \frac{(1+t^{2k-1})^{2g}}{(1-t^{2k})^2} \right) \left(\prod_{p \in P} \prod_{k=1}^{s(p)-1} \prod_{j=M_k(p)+1}^{M_{k+1}(p)} \frac{1-t^{2j}}{1-t^{2(j-M_k(p))}} \right).$$

proof: Let $\mathcal{F}$ denote the flag variety and consider the fibration

$$\mathcal{P} \hookrightarrow \mathcal{G}^{\mathbb{C}} \to \mathcal{F}. \tag{7}$$

On the level of classifying spaces, this gives

$$\mathcal{F} \to \mathrm{B}\mathcal{P} \to \mathrm{B}\mathcal{G}^{\mathbb{C}}.$$

As explained in §6.4 of [4], this last fibration is cohomologically trivial, so its Poincaré polynomial is given by

$$P_t(\mathrm{B}\mathcal{P}) = P_t(\mathcal{F})P_t(\mathrm{B}\mathcal{G}^{\mathbb{C}}). \tag{8}$$

Furthermore, it is shown in Theorem 2.15 of [1] that $\mathrm{B}\mathcal{G}^{\mathbb{C}}$ is torsion free with Poincaré polynomial

$$P_t(\mathrm{B}\mathcal{G}^{\mathbb{C}}) = (1-t^{2N}) \prod_{k=1}^{N} \frac{(1+t^{2k-1})^{2g}}{(1-t^{2k})^2}. \tag{9}$$

The cohomology of the flag variety is also torsion free and well understood. First,

$$\mathcal{F} = \prod_{p \in P} \mathcal{F}_p,$$

where each

$$\mathcal{F}_p = U(N)/U(m_1(p)) \times \cdots \times U(m_s(p)).$$

Recall $m_j(p)$ is the multiplicity $\dim(F_j(p)) - \dim(F_{j+1}(p))$. Set

$$M_k(p) = \sum_{i=1}^{k} m_i(p).$$

Each flag is cohomologically a product of Grassmanians, and suppressing dependence on p, Proposition 23.2 of [6] gives

$$P_t(\mathcal{F}_p) = \prod_{k=1}^{s-1} \prod_{j=M_k+1}^{M_{k+1}} \frac{1-t^{2j}}{1-t^{2(j-M_k)}}. \tag{10}$$

This completes the proof. ♠

The next proposition gives a formula for the codimension $d_{\lambda,I}$ of an unstable stratum $\mathcal{C}_{\lambda,I}$. Before stating it, we introduce some notation. Let $\mathfrak{End}E$ and $\mathfrak{ParEnd}E$ denote the sheaves of germs of endomorphisms of E and parabolic endomorphisms of E, respectively. Also, let $\mathfrak{End}''E = \mathfrak{End}E/\mathfrak{End}'E$ and $\mathfrak{ParEnd}''E = \mathfrak{ParEnd}E/\mathfrak{ParEnd}'E$ be the quotient sheaves by the subsheaves $\mathfrak{End}'E$ and $\mathfrak{ParEnd}'E$ which preserve the Harder-Narasimhan filtration (4) of E. Then there is a short exact sequence

$$0 \to \mathfrak{ParEnd}''E \to \mathfrak{End}''E \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is a skyscraper sheaf supported on the set P of parabolic points.

Proposition 4.6 (Nitsure)

$$
\begin{aligned}
d_{\lambda,I} &= \sum_{i>j}(n_i d_j - n_j d_i) + ((\textit{rank } E)^2 - \sum_{i\le j} n_i n_j)(g-1+n) \\
&\quad + \sum_{p\in P}\left(\sum_{\substack{i\le j\\ l\le k}} I_p(i,k)I_p(j,l) - \sum_{i\le j} m_i(p)m_j(p)\right).
\end{aligned}
$$

proof: See the proof of Proposition 1.17 in [16] for the details. Briefly,

$$d_{\lambda,I} = h^1(M, \mathrm{ParEnd}''E) = -\chi(\mathrm{End}''E) + \chi(\mathcal{Q}). \tag{11}$$

Writing $n_i = \text{rank } D_i$ and $d_i = \deg D_i$ for the rank and degree of the semistable quotient D_i in the Harder-Narasimhan filtration (4) of E and using Riemann-Roch, one can compute as in 7.16 of [1] to get

$$\chi(\mathfrak{End}''E) = \sum_{i>j}(n_i d_j - n_j d_i) + n_i n_j (g-1).$$

Since $\mathcal{Q}$ is a skyscraper sheaf supported on P, the following formula for the rank of $\mathcal{Q}$ at p determines $\chi(\mathcal{Q})$.

$$\text{rank } (\mathcal{Q}_p) = (\text{rank } E)^2 - \sum_{i\le j} n_i n_j - \sum_{i\le j} m_i m_j + \sum_{\substack{i\le j\\ l\le k}} I_p(i,k)I_p(j,l). \tag{12}$$

There is a slight discrepancy between our formula for $\chi(\mathcal{Q})$ and that given by Nitsure, here we correct two typographical errors. ♠

There is a alternative description of $\chi(\mathcal{Q})$ which is useful for computations. For the sake of argument, suppose that there is only one parabolic point and that the quotients D_i of the Harder-Narasimhan filtration are all line bundles (i.e. $|\lambda| = N$). Assume further that the flag over p is full, i.e. that $s = N$. Then the following proposition gives a simple combinatorial description of $\chi(\mathcal{Q})$.

Proposition 4.7 *Under these assumptions, I_p is a permutation matrix and $\chi(\mathcal{Q})$ is the minimal number of adjacent row transpositions necessary to obtain the identity matrix from I_p.*

proof: Let R_i denote the i^{th} row vector of I_p. Then R_i is given by a standard basis vector e_{j_i}. It is clear from the formula for $\chi(\mathcal{Q})$ that a transposition of two adjacent rows $R_i = e_{j_i}$ and $R_{i+1} = e_{j_{i+1}}$ either increases or decreases $\chi(\mathcal{Q})$ by 1 depending on whether $j_i < j_{i+1}$ or $j_i > j_{i+1}$. Equally clear is the fact that if I_p is the identity, then $\chi(\mathcal{Q}) = 0$. Since I_p is a permutation matrix, some sequence of adjacent row transpositions will give the identity, and a minimal sequence will all decrease $\chi(\mathcal{Q})$ by 1, proving the proposition. ♠

Remark: First of all, this proposition generalizes in the obvous way to the case of more than one parabolic point. Also, it holds with column transpositions replacing row transpositions, and then generalizes to the cases $|\lambda| = N$ and arbitrary s using rows, and $s = N$ and arbitrary $|\lambda|$ using columns.

The final step in the inductive procedure is to enumerate all the unstable strata that occur. For genus $g \geq 2$, this is fairly straightforward, while for genus $g = 0$, it is complicated by the fact that certain lower rank moduli spaces may be empty (see the remark at the end of §6.3 of [4]), and consequently not all strata which are present for higher genus appear in the genus zero case. The specific examples of the following section show how to deal with this issue.

Having completed the inductive procedure and deduced the $\mathcal{P}$-equivariant cohomology of $\mathcal{C}_{ss}$, we are still left with the problem of relating this to the cohomology of the moduli $\mathfrak{N}$. The technique is to use the fact that for a free action of a group G on a manifold M with quotient N, the G-equivariant cohomology of M coincides with the cohomology of N. In our case, the problem is that, even with the assumption that $\mathcal{C}_{ss} = \mathcal{C}_s$, the $\mathcal{P}$ action is not free, because the subgroup $\mathbb{C}^*$ of constant central bundle automorphisms acts trivially. However, stable bundles are *simple*, i.e. for any $d'' \in \mathcal{C}_s$, the isotropy group of d'' is precisely $\mathbb{C}^*$. Consider the fibration

$$\mathbb{C}^* \to \mathcal{P} \to \overline{\mathcal{P}} \tag{13}$$

It follows that the group $\overline{\mathcal{P}} = \mathcal{P}/\mathbb{C}^*$ does act freely, so we would be done if we could relate the $\mathcal{P}$- and the $\overline{\mathcal{P}}$-equivariant cohomologies of $\mathcal{C}_s$. We have already seen that $\mathrm{H}^*(\mathrm{B}\mathcal{P}, \mathbb{Z})$ and $\mathrm{H}^*_{\mathcal{P}}(\mathcal{C}_{ss}, \mathbb{Z})$ are torsion free (here $\mathrm{H}^*_{\mathcal{P}}$ refers to $\mathcal{P}$-equivariant cohomology). So using Proposition 6.1 of [4] we just need to show that the fibration obtained from (13) by taking classifying spaces,

$$\mathrm{BU}(1) \xrightarrow{i} \mathrm{B}\mathcal{P} \to \mathrm{B}\overline{\mathcal{P}}, \tag{14}$$

is trivial. Proposition 6.2 of [4] proves this is the case for rank 2 parabolic bundles provided there is at least one nontrivial flag. Proposition 4.8 extends this to arbitrary rank. This is proved by noticing that because the fiber of (14) is a $K(\mathbb{Z}, 2)$, this bundle is classified by an element of

$$[\mathrm{B}\overline{\mathcal{P}}, K(\mathbb{Z}, 3)] = \mathrm{H}^3(\mathrm{B}\overline{\mathcal{P}}, \mathbb{Z}).$$

To show that (14) is trivial, it is enough to show that the induced map

$$\mathrm{H}^2(\mathrm{BU}(1), \mathbb{Z}) \xrightarrow{i} \mathrm{H}^2(\mathrm{B}\mathcal{P}, \mathbb{Z})$$

is onto. Using the Hurewitz map, this is equivalent to requiring that the image of the map

$$\pi_1 \mathrm{U}(1) \xrightarrow{i} \pi_1 \mathcal{P}$$

is a direct sum. In [1], it is observed that the image of $\pi_1 \mathrm{U}(1) \xrightarrow{i} \pi_1 \mathcal{G}^{\mathbb{C}}$ is a direct sum whenever the rank and degree of the bundle are coprime. The same for $\mathcal{P}$ follows immediately from the following elementary observation. Suppose G' is a subgroup of G, $H \overset{\phi}{\hookrightarrow} G'$ with $im(\phi)$ is a direct sum of G. Then $im(\phi)$ is also a direct sum in G'.

Using this we shall prove Proposition 4.8. Suppose that E is a topological parabolic bundle with at least one nontrivial flag

$$E_p = F_1 \supset F_2 \supset \cdots \supset F_s \supset 0.$$

Let $\mathcal{P}$ be as usual the group of bundle automorphisms preserving the flag structure. Forgetting all the parabolic structure except that at p, where we use only the flag

$$E_p = F_1 \supset F_2 \supset 0,$$

we denote by $\mathcal{P}'$ the parabolic automorphisms with respect to this new parabolic structure. Clearly, we have the inclusion $\mathcal{P} \hookrightarrow \mathcal{P}'$.

Claim: The image of the natural inclusion map $\pi_1 \mathrm{U}(1) \to \pi_1 \mathcal{P}'$ is a direct sum.

We can immediately conclude

Proposition 4.8 *Suppose that the topological parabolic bundle E has at least one nontrivial flag. Then the fibration (14) is trivial. In particular, if $\mathcal{C}_{ss} = \mathcal{C}_s$, then the moduli of stable bundles is torsion-free with cohomology*

$$H^*_{\mathcal{P}}(\mathcal{C}_{ss}, \mathbb{Z}) = \mathrm{H}^*(\mathrm{BU}(1), \mathbb{Z}) \otimes \mathrm{H}^*(\mathfrak{N}, \mathbb{Z}).$$

proof of claim: Looking at the long exact sequence of (7) in homotopy, it follows that

$$\pi_2 \mathcal{F}' \to \pi_1 \mathcal{P}' \to \pi_1 \mathcal{G}^{\mathbb{C}}$$

is short exact. By (10), it follows that $\pi_2 \mathcal{F}' = \mathbb{Z}$. Let r be the map which restricts a bundle automorphism to the point p. Replacing groups by their maximal compacts and applying r_*, we get the sequence

$$\pi_2 \mathcal{F}' \to \pi_1 U(m_1) \oplus U(m_2) \to \pi_1 U(n).$$

(Here, $m_2 = \dim F_2$ and $m_1 = n - m_2$.) The same exact argument as that which proves Proposition 6.2 of [4] finishes the proof. ♠

Corollary 4.9 *In addition to the hypotheses of the previous proposition, assume that the underlying Riemann surface is $\mathbb{CP}^1$. Then $\mathrm{H}^{2i+1}(\mathfrak{N}, \mathbb{Z}) = 0$. In particular, if $\dim(\mathfrak{N}) = 2$, then $\mathfrak{N} \cong \mathbb{CP}^1$.*

proof: This is proved by induction, the case of rank one is trivial and rank two is treated in [4]. From the previous proposition, it is enough to show that the $\mathcal{P}$-equivariant Poincaré polynomial of $\mathcal{C}_{ss}$ is actually a polynomial in t^2. Suppose inductively that this has been proved whenever rank $E < N$. This is evident in genus 0 from formulas (5) and (6) and Proposition 4.5. ♠

Remark: In light of this corollary, it is reasonable to expect that these moduli are *rational*. A proof of this will appear in [5] for moduli over $\mathbb{CP}^1$, and as a direct consequence one may conclude that these moduli admit Morse functions with critical points of only even index.

4.3 Examples

In this section, we present two examples of the inductive procedure for rank 3 bundles over the Riemann sphere. In both, $M = \mathbb{CP}^1$ with three marked points $\{p_1, p_2, p_3\}$ and E is a rank 3 topological parabolic bundle over M of degree -3.

For the first example, suppose E has parabolic structures over p_1, p_2 and p_3 given by the weights $(0, \frac{1}{3}, \frac{2}{3}), (\frac{2}{9}, \frac{1}{3}, \frac{4}{9})$ and $(\frac{9}{31}, \frac{10}{31}, \frac{12}{31})$ respectively, and consider the associated moduli space $\mathfrak{N}$. In this case, Proposition 4.5 gives

$$P_t(\mathrm{B}\mathcal{P}) = \frac{(1+t^2)(1+t^2+t^4)^2}{(1-t^2)^5}. \tag{15}$$

The other term in formula (5) is a sum which can be decomposed according to the length of λ into two sums

$$\sum_{\substack{\lambda, I \\ |\lambda|=3}} t^{2d_{\lambda,I}} \tilde{P}_t(\mathcal{C}_{\lambda,I}) + \sum_{\substack{\lambda, I \\ |\lambda|=2}} t^{2d_{\lambda,I}} \tilde{P}_t(\mathcal{C}_{\lambda,I}), \tag{16}$$

which we treat separately.

First, suppose that $|\lambda| = 3$, so E has a filtration $0 \subset E_1 \subset E_2 \subset E_3 = E$ whose quotients $L_i = E_i/E_{i-1}$ are line bundles. Note that for line bundles L, since $\mathcal{C}_{ss}(L) = \mathcal{C}(L)$, it follows that over $\mathbb{CP}^1$,

$$\tilde{P}_t(\mathcal{C}_{ss}(L)) = P_t(\mathrm{BU}(1)) = \frac{1}{1-t^2}.$$

Thus if $|\lambda| = 3$,

$$\tilde{P}_t(\mathcal{C}_{\lambda,I}) = \prod_{i=1}^{3} \tilde{P}_t(\mathcal{C}_{ss}(L_i)) = \frac{1}{(1-t^2)^3}.$$

Let $d_i = \deg(L_i)$ and $\mu_i = \mu(L_i)$. Notice that the type $\lambda = ((0,0), (1,\mu_1), (1,\mu_2), (1,\mu_3))$ is determined by d_i and by the intersection matrix. More importantly, such a λ is the type of an unstable stratum if and only if

$$\mu_1 > \mu_2 > \mu_3. \tag{17}$$

The calculation runs through all possible degrees d_1, d_2, d_3 and all possible intersection matrices, including only those which satisfy (17) into the first sum of (16).

We further decompose the sum

$$\sum_{\substack{\lambda, I \\ |\lambda|=3}} t^{2d_{\lambda,I}} \tilde{P}_t(\mathcal{C}_{\lambda,I}) = S_{\mathrm{I}} + S_{\mathrm{II}} + S_{\mathrm{III}} + S_{\mathrm{IV}}$$

according to the cases: (I) $d_1 > d_2 > d_3$, (II) $d_1 = d_2 > d_3$, (III) $d_1 > d_2 = d_3$, and (IV) $d_1 = d_2 = d_3$. (Notice that if $d_1 < d_2$ or $d_2 < d_3$, then (17) cannot hold for this particular choice of weights, i.e. the four cases above are exhaustive.) Since $d_3 = -3 - d_1 - d_2$, Proposition 4.6 shows that a stratum of type λ, I has codimension given by

$$d_{\lambda,I} = 4d_1 + 2d_2 + 3 + \chi(\mathcal{Q})$$

and Proposition 4.7 makes it easy to compute $\chi(\mathcal{Q})$.

(I) If $d_1 > d_2 > d_3$ then (17) holds for all possible intersection matrices. It follows that $d_1 \geq 0$ and $-[d_1/2] - 1 \leq d_2 \leq d_1 - 1$. Splitting this into two pieces according to $d_1 = 2k$ is even and $d_1 = 2k+1$ is odd and then summing the geometric series, we see that

$$\begin{aligned}
S_{\mathrm{I}} &= \sum_{d_1,d_2} \tilde{P}_t(\mathcal{C}_{ss}(L)) t^{8d_1+4d_2+6+2\chi(\mathcal{Q})} \\
&= \frac{(1+t^2)^3(1+t^2+t^4)^3}{(1-t^2)^3} \sum_{d_1=0}^{\infty} \sum_{d_2=-[d_1/2]-1}^{d_1-1} t^{2(4d_1+2d_2+3)} \\
&= \frac{(1+t^2)^3(1+t^2+t^4)^3}{(1-t^2)^3} \sum_{k=0}^{\infty} \left(\sum_{d_2=-k-1}^{2k-1} t^{16k+4d_2+6} + \sum_{d_2=-k-1}^{2k} t^{16k+4d_2+14} \right) \\
&= \frac{(1+t^2)^3(1+t^2+t^4)^3}{(1-t^2)^3(t^4-1)} \sum_{k=0}^{\infty} (t^{24k+6} + t^{24k+18} - t^{12k+2} - t^{12k+10}) \\
&= \frac{(1+t^2)^2(1+t^2+t^4)^3}{(1-t^2)^4(1-t^{12})} (t^2 - t^6 + t^{10}).
\end{aligned}$$

For the remaining cases, it is helpful to notice that the weights over p_1 are *dominant* in the following way. Because the sum of the maximum difference between the weights over p_2 and p_3 is less than the minimum difference between the weights over p_1 (i.e. $2/9 + 3/31 < 1/3$), the intersection data over p_1 determines whether or not any given type satisfies (17).

(II) In this case, the type satisfies (17) if and only if $\mu_1 > \mu_2$, which is the case if and only if

$$I_{p_1} \in \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$$

Then we find that

$$\begin{aligned}
S_{\mathrm{II}} &= \frac{(1+t^2)^2(1+t^2+t^4)^3 t^2}{(1-t^2)^3} \sum_{d_1=0}^{\infty} t^{12d_1+6} \\
&= \frac{(1+t^2)^2(1+t^2+t^4)^3 t^8}{(1-t^2)^3(1-t^{12})}.
\end{aligned}$$

(III) Similarly, (17) is satisfied if and only if $\mu_2 > \mu_3$, which is the case if and only if

$$I_{p_1} \in \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$$

Again, we get that

$$S_{\rm III} = \frac{(1+t^2)^2(1+t^2+t^4)^3 t^8}{(1-t^2)^3(1-t^{12})}.$$

(IV) If $d_1 = d_2 = d_3 = -1$, then (17) is satisfied if and only if

$$I_{p_1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

This gives

$$S_{\rm IV} = \frac{(1+t^2)^2(t^4+t^2+1)^2}{(1-t^2)^3}.$$

To complete the calculation, we show that

$$\sum_{\substack{\lambda, I \\ |\lambda|=2}} t^{2d_{\lambda,I}} \tilde{P}_t(\mathcal{C}_{\lambda,I}) = 0.$$

This follows from the following

Claim: If W is a rank 2 parabolic sub- or quotient bundle of E, then $\mathcal{C}_{ss}(W) = \emptyset$.

proof of claim: For such bundles W, there are 3^3 different ways that W could inherit weights from E. We prove this in the case that W inherits the weights $(0, \frac{1}{3}), (\frac{2}{9}, \frac{1}{3})$ and $(\frac{9}{31}, \frac{10}{31})$, the other cases being identical. Here, by tensoring with an appropriate line bundle, we can assume that $\deg(W) \in \{0, 1\}$. Suppose first that $\deg(W) = 0$. Then $\mu(W) = \frac{419}{558}$. Suppose that $L \subset W$ is a holomorphic subbundle, then L is destabilizing if and only if either (i) $\deg(L) \geq 1$ or (ii) $\deg(L) = 0$ and L_{p_1} intersects the flag at p_1 nontrivially. Using formulas (17) and (18) of [4], it now follows that $\tilde{P}_t(\mathcal{C}_{ss}) = 0$.

As for the case $\deg(W) = 1$, then $\mu(W) = 1\frac{140}{558}$ and one can show that any holomorphic subbundle $L \subset W$ is destabilizing if and only if $\deg(L) \geq 1$. Using formulas (17) and (18) of [4], and it again follows that $\tilde{P}_t(\mathcal{C}_{ss}) = 0$. This completes the proof of the claim. ♠

It now follows that

$$S_{\rm I} + S_{\rm II} + S_{\rm III} + S_{\rm IV} = \frac{(1+t^2)(1+t^2+t^4)^2}{(1-t^2)^5}$$

and an application of (5) gives that $P_t(\mathfrak{N}) = 0$, i.e. $\mathfrak{N} = \emptyset$ and this completes the first example.

For the second example, we suppose the weights are $(0, \frac{1}{3}, \frac{2}{3}), (\frac{2}{9}, \frac{1}{3}, \frac{4}{9})$ and $(\frac{3}{31}, \frac{10}{31}, \frac{18}{31})$. This is similar to the first example and so we give only the results of the calculation, leaving the details to the interested reader. Just as before,

$$P_t(\mathrm{B}\mathcal{P}) = \frac{(1+t^2)(1+t^2+t^4)^2}{(1-t^2)^5}.$$

Also, direct computation reveals

$$\sum_{\substack{\lambda,I\\|\lambda|=3}} t^{2d_{\lambda,I}}\tilde{P}_t(\mathcal{C}_{\lambda,I}) = \frac{3t^2+5t^4+20t^6+5t^8+3t^{10}}{(1-t^2)^4(1-t^4)},$$

$$\sum_{\substack{\lambda,I\\|\lambda|=2}} t^{2d_{\lambda,I}}\tilde{P}_t(\mathcal{C}_{\lambda,I}) = \frac{3t^2+10t^4+3t^6}{(1-t^2)^2(1-t^4)}.$$

Applying formula (5) gives

$$\tilde{P}_t(\mathcal{C}_{ss}) = \frac{1+t^2}{1-t^2},$$

while applying Proposition 4.8 shows that $P_t(\mathfrak{N}) = 1+t^2$. Thus, for these weights, $\mathfrak{N} \cong \mathbb{CP}^1$.

5 Results for Brieskorn Spheres

5.1 The SU(3) Representation Space of Brieskorn Spheres

In this section, we apply the results of the previous sections to the the SU(3) representation spaces of Brieskorn spheres. First, we use Theorem 3.1 to identify these representation spaces with certain components of the representations of an associated triangle group. Then we use Theorem 4.1 of [4] to identify the latter representation spaces with moduli spaces of certain parabolic bundles. It is important to ensure that the condition $\mathcal{C}_s = \mathcal{C}_{ss}$ is satisfied by the bundles which arise. This can be verified in two ways, either by working directly with bundles or by using the correspondence between semistable bundles and representations and arguing that there are no reducible representations. For example, consider the triangle groups $T(2,p,q)$, (p,q odd and relatively prime). Since there are no nontrivial reducible SU(3) representations, the corresponding parabolic structures on rank 3 bundles satisfy $\mathcal{C}_s = \mathcal{C}_{ss}$.

The other way to check that $\mathcal{C}_s = \mathcal{C}_{ss}$ uses "numerology." For example, suppose that E is a rank n parabolic bundle and $\mathbf{pd}(E) = 0$. Suppose further that no subcollection of weights has integer sum. Then this forbids $\mathbf{pd}(E') \in \mathbb{Z}$ for any proper subbundle, and it follows that $\mathcal{C}_s = \mathcal{C}_{ss}$. In the case of rank 3 bundles, it suffices to check that no choice of weights, one for each parabolic point, has integer sum.

Example: The representation space of $\Sigma(2,3,7)$.

First we prove a result similar to but stronger than Theorem 3.1 for this special case.

Claim: If p relatively prime to 6, then $R^*(\Sigma(2,3,p),\mathrm{SU}(3)) \cong R^*(T(2,3,p),\mathrm{SU}(3))$.

proof of claim: Theorem 3.1 shows that we can understand the irreducible SU(3) representations of $\Sigma(2,3,p)$ by studying those irreducible SU(3) representations ρ of the triangle group

$$T(2,9,p) = \langle x_1, x_2, x_3 \mid x_1^2 = x_2^9 = x_3^p = 1 = x_1x_2x_3\rangle$$

with $\rho(x_2)^3$ a central element. But ρ irreducible implies that $\rho(x_1)$ is conjugate to the matrix $diag(-1,-1,1)$, and, by Proposition 3.2, it follows that each of $\rho(x_2)$ and $\rho(x_3)$ must have three distinct eigenvalues. It is now a simple exercise to write out a list of all 9^{th} roots of unity with three distinct eigenvalues which are also cube roots of a central element. There is only one such matrix (up to conjugacy), namely $\exp(diag(0,\frac{1}{3},\frac{2}{3}))$. This shows that ρ is an irreducible representation of $T(2,3,p)$. The claim now follows since any irreducible SU(3) representation of $T(2,3,p)$ has $\rho(x_2)$ conjugate to $\exp(diag(0,\frac{1}{3},\frac{2}{3}))$. ♠

Now assume $p=7$ and let $T=T(2,3,7)$. Then by Theorem 3.2, it follows that every nonempty component $R_{\vec{\alpha}} \subset R(T,\mathrm{SU}(3))$ has dimension zero. Moreover, writing a list of all $2^{nd}, 3^{rd}$ and 7^{th} roots of unity in SU(3) we see that $\alpha_1=\exp(diag(0,\frac{1}{2},\frac{1}{2}))$ and $\alpha_2=\exp(diag(0,\frac{1}{3},\frac{2}{3}))$. Using the natural inclusion $\mathrm{SO}(3)\subset\mathrm{SU}(3)$, the results of [8] show that there are two irreducible representations ρ_1,ρ_2 with $\rho_1(x_3)$ and $\rho_2(x_3)$ conjugate to $\exp(diag(0,\frac{2}{7},\frac{5}{7}))$ and $\exp(diag(0,\frac{3}{7},\frac{4}{7}))$. The new bit of information is that there are precisely two additional irreducible SU(3) representations, neither of which is the complexification of SO(3) representations. For example, setting $\alpha_3=\exp(diag(\frac{1}{7},\frac{2}{7},\frac{4}{7}))$ and interpreting $R^*_{\vec{\alpha}}$ as a moduli space of parabolic bundles, an application of the inductive procedure shows that $R_{\vec{\alpha}}$ is a point. In a similar way, we find that (i) $R^*_{\vec{\alpha}}$ is a point for $\alpha_3=\exp(diag(\frac{3}{7},\frac{5}{7},\frac{6}{7}))$, and (ii) $R^*_{\vec{\alpha}}$ is empty for all other possible choices for α_3. This shows that $R^*(\Sigma(2,3,7)$ consists of 4 points.

Of course, such simple results (e.g. distinguishing these two new SU(3) representations for $\Sigma(2,3,7)$) requires repeated application of the inductive procedure, which is itself a somewhat long and cumbersome computation. Since it is not entirely reasonable to expect to have the time (never mind the patience) to do this by hand, I have written a batch of MAPLE programs for this purpose. Specifying the Brieskorn sphere $\Sigma(p,q,r)$ as input, one obtains output consisting of the vectors $\vec{\alpha}$ with $R_{\vec{\alpha}}$ nonempty. In addition, the total number of points and 2-sphere components in $R^*(\Sigma,\mathrm{SU}(3))$ is given. In the case $\Sigma=\Sigma(2,p,q)$, $R^*(\Sigma,\mathrm{SU}(3))$ is a discrete collection of points and the table at the end of this section summarizes some of the output of this MAPLE program by listing $\chi(R^*)$. In addition, since all of these representations have *even* spectral flow, this table also gives the leading term in a gauge-theoretic definition of the generalized Casson invariant for the group SU(3) (see the next section for more details).

We remark, however, that no computer has enough patience necessary to produce *all* the output listed! The problem is of course that the computer only provides finitely many computations, whereas we have listed results for infinite families of Brieskorn spheres, e.g. the manifolds $\Sigma(2,3,6k\pm1)$. We now indicate, as briefly as possible, the argument used to make this last deduction in the specific case of $\Sigma(2,3,6k\pm1)$; the other cases being similar.

Consider the group $\Gamma=\langle x_1,x_2,x_3 \mid x_1^2=x_2^3=x_1x_2x_3=1\rangle$. For a representation $\rho:\Gamma\to\mathrm{SU}(3)$ to be irreducible, $\rho(x_1)$ and $\rho(x_2)$ must be conjugate to the matrices $X_1=\exp(diag(0,\frac{1}{2},\frac{1}{2}))$ and $X_2=\exp(diag(0,\frac{1}{3},\frac{2}{3}))$, respectively.

There are surjections from Γ onto the triangle groups $T(2,3,6k\pm1)$ gotten by imposing the relation $x_3^{6k\pm1}=1$. Thus, there are injections

$$R^*(T(2,3,6k\pm1),\mathrm{SU}(3)) \longrightarrow R^*(\Gamma,\mathrm{SU}(3)).$$

It turns out that we can parameterize $R^*(\Gamma,\mathrm{SU}(3))$ as a subset of $\mathrm{SU}(3)/conj$, namely the subset of *allowable* values for X_3, a diagonal matrix conjugate to $\rho(x_3)$. As we shall see, this is is the shaded region in Figure 1.

One word about choice of conventions. We parameterize $\mathrm{SU}(3)/conj$ by identifying the two regions

$$\triangle_l = \{(x,y,z) \mid 0 \le x \le y \le z \le 1, x+y+z=1\}$$

$$\triangle_r = \{(x,y,z) \mid 0 \le x \le y \le z \le 1, x+y+z=2\}$$

in $\mathbb{R}^3$. I.e. $\mathrm{SU}(3)/conj = \triangle_l \cup_\sim \triangle_r$ where $\triangle_l \ni (0,x,y) \sim (x,y,1) \in \triangle_r$.

Now for convenience set $R = R^*(\Gamma, \mathrm{SU}(3))$. For $\alpha \in \mathrm{SU}(3)/conj$, let $R_\alpha \subset R$ denote those irreducible representations ρ with $\rho(x_3) \in \alpha$. As before, we can use parabolic bundles to study R_α. In particular, it follows from an easy dimension count that each nonempty R_α is a point. As α varies, R_α changes from being the empty set to being a point along *hyperplanes* in $\mathrm{SU}(3)/conj$ (this is a specific example of the general phenomenon studied in [5]).

To be more specific, write $\alpha = \exp(diag(\alpha_1, \alpha_2, \alpha_3))$. Then if a change occurs, it must occur along one of the hyperplanes illustrated in Figure 1 and given algebraically by the equations $\alpha_i \in \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$. (Not all of these hyperplanes are drawn since some would violate the conditions $0 \le \alpha_1 \le \alpha_2 \le \alpha_3 < 1$ and $\sum_i \alpha_i \in \{1,2\}$, e.g. $\alpha_1 = \frac{5}{6}$ and $\alpha_3 = \frac{1}{6}$.) In particular, the topological type of R_α is constant within *chambers*, i.e. the connected components of the complement of these hyperplanes. There are only finitely many chambers, and doing one (computer-aided) computation for each chamber, one finds that the α with R_α nonempty is precisely the shaded region in Figure 1. No change occurs along the hyperplanes interior to the shaded region; this is strictly a genus zero phenomenon and is a consequence of certain moduli spaces of rank 2 parabolic bundles being empty.

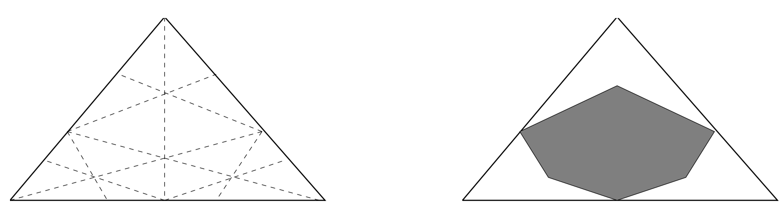

Figure 1: The α with $R_\alpha \neq \emptyset$

A little thought convinces one that $R^*(T(2,3,6k\pm1), \mathrm{SU}(3))$ can be identified with certain lattice points in R, namely those whose coordinates are rational numbers with denominator $6k\pm1$. So, not surprisingly, the number of irreducible representations of $\Sigma(2,3,6k\pm1)$ are given by counting lattice points inside R. Using symmetry, we can instead count (with correct multiplicity) the number of lattice points in either $\triangle_l \cap R$ or $\triangle_r \cap R$, the correct multiplicity being 2 for interior points and 1 for points along the common boundary of $\triangle_l \cap R$ and $\triangle_r \cap R$. The formula results from the following elementary considerations: first, enlarge the region by multiplying by $6k \pm 1$ and count *integer* lattice points in this larger region (with the same convention for multiplicities). Next, project down to a region in the plane $z = 0$ (since $x, y \in \mathbb{Z}$ and $x+y+z = 6k\pm1$ implies $z \in \mathbb{Z}$). This is the convex quadrilateral illustrated in Figure 2. It is an easy exercise to show that the number of integer lattice points, counted with the appropriate multiplicities, is given by $3k^2 \pm k$.

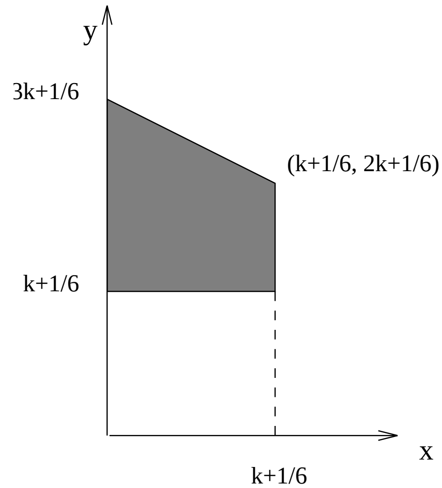

Figure 2: The number of integer lattice points in this region is $3k^2 - k$.

SU(3) Representations of Brieskorn spheres

Brieskorn sphere Σ	$\chi(R^*)$
$\Sigma(2,3,6k\pm1)$	$3k^2 \pm k$
$\Sigma(2,5,10k\pm1)$	$33k^2 \pm 9k$
$\Sigma(2,5,10k\pm3)$	$33k^2 \pm 19k + 2$
$\Sigma(2,7,14k\pm1)$	$138k^2 \pm 26k$
$\Sigma(2,7,14k\pm3)$	$138k^2 \pm 62k + 4$
$\Sigma(2,7,14k\pm5)$	$138k^2 \pm 102k + 16$
$\Sigma(2,9,18k\pm1)$	$390k^2 \pm 58k$
$\Sigma(2,9,18k\pm5)$	$390k^2 \pm 210k + 24$
$\Sigma(2,9,18k\pm7)$	$390k^2 \pm 298k + 52$

5.2 Spectral Flow and Generalized Casson Invariants

Taubes gave a gauge theoretic description of Casson's invariant in [18]. For example, it is proved that if Σ is a $\mathbb{Z}$ homology sphere such that $\mathrm{H}^1(\Sigma, \mathfrak{ad}_\rho) = 0$ for all $\rho \in R^*(\Sigma, \mathrm{SU}(2))$, then

$$\lambda(\Sigma) = \frac{1}{2} \sum_{\rho \in R^*} (-1)^{SF(\Theta,\rho)}. \tag{18}$$

Here, $SF(\Theta, \rho)$ is the spectral flow of the self-duality operator from the product connection Θ to the flat connection induced by ρ. The same formula holds in the general case of an arbitrary $\mathbb{Z}$ homology sphere, provided one first perturbs the flatness conditions to obtain a perturbed representation space R^* which is finite [18].

More recently, Mrowka and Walker have generalized this approach to provide a gauge theoretic description of Walker's invariant of $\mathbb{Q}$ homology spheres. Their invariant has the form of two sums, the first given by equation (18) and the second includes contributions from each reducible representation in the form of a Maslov index, which can be expressed in terms of the spectral flow and the Chern-Simons invariant.

Their approach seems very promising for providing a rigorous definition and allowing for explicit computations of the generalized Casson invariants λ_G. In particular, one expects that the leading order term in $\lambda_{\mathrm{SU}(n)}(\Sigma)$ to be given by the sum

$$\sum_{\rho \in R^*} (-1)^{SF(\Theta,\rho)}, \tag{19}$$

where $R^* = R^*(\Sigma, \mathrm{SU}(n))$ is suitably perturbed so that it is finite. There are admittedly subtle and difficult questions regarding the invariance of the generalized Casson invariants under perturbations, however, we can obviate these deliberations when working with SU(3) representations of $\Sigma = \Sigma(2,p,q)$ because Proposition 2.4 implies that the reducibles are isolated and Proposition 3.2 implies that $R^*(\Sigma, \mathrm{SU}(3))$ is finite. In this section, we give a computation of (19) in this special case. The results of the previous section identify $R^*(\Sigma, \mathrm{SU}(3))$ explicitly, which, together with a formula for the SU(3) spectral flow of any $\rho \in R^*(\Sigma, \mathrm{SU}(3))$, completes the computation.

Suppose that $\Sigma = \Sigma(a_1, \ldots, a_n))$ is a Seifert fibered homology sphere and that E is a complex vector bundle over Σ with structure group SU(N). Then E is trivial, and a given trivialization allows us to identify the space of connections $\mathcal{A}$ with $\Omega^1 \otimes \mathfrak{su}(N)$. Pick a metric on Σ. For $\alpha \in \mathcal{A}$, consider the elliptic operator

$$D_\alpha : (\Omega^0 \oplus \Omega^1) \otimes \mathfrak{su}(N) \to (\Omega^0 \oplus \Omega^1) \otimes \mathfrak{su}(N),$$

defined by $D_\alpha(\phi, \tau) = (d_\alpha^* \tau, *d_\alpha \tau + d_\alpha \phi)$ where d_α is the covariant derivative of α, d_α^* is its adjoint, and $*$ is the Hodge star operator. Given another connection $\beta \in \mathcal{A}$, choose $0 < \delta < \inf |\mu|$ where μ is any nonzero eigenvalue of either D_α or D_β. Choose a path a_t in $\mathcal{A}$ with $a_0 = \alpha$ and $a_1 = \beta$. Then $SF(\alpha, \beta)$ is defined to be the spectral flow of D_{a_t}, i.e. the number $m_+ - m_-$, where $m_\pm$ is the order of the set $M_\pm$ defined by

$$M_+ = \{\mu_t \text{ an eigenvalue of } D_{a_t} \text{ with } \mu_0 < -\delta \text{ and } \mu_1 > \delta\},$$

$$M_- = \{\mu_t \text{ an eigenvalue of } D_{a_t} \text{ with } \mu_0 > -\delta \text{ and } \mu_1 < \delta\}.$$

This number is not independent of the various choices (e.g. trivialization, path), however its value in $\mathbb{Z}_{4N}$ is well defined. This follows from the Index Theorem and the simple computation that $c_2(\mathfrak{ad} E \otimes \mathbb{C}) = 2N c_2(E)$ (see §2 of [13]). Note also that by our convention, $SF(\beta, \alpha) = -\dim \mathrm{SU(N)} - SF(\alpha, \beta)$.

The spectral flow can be viewed as the index of the self-duality operator $d_A^* \oplus d_A^-$ where A denotes the connection on $\Sigma \times [0,1]$ gotten from the path a_t. This gives the formula (see Theorem 7.1 of [13])

$$SF(\alpha, \beta) = \mathrm{Index}(d_A^* \oplus d_A^-),$$

where the index is taken with respect to the Atiyah-Patodi-Singer boundary conditions [2]. In [8], Fintushel and Stern compute the SU(2) spectral flow of any representation of Σ and prove that it is even. A generalization of their argument computes the SU(N) spectral flow of any irreducible representation and shows it to be even. In the proof of the following proposition, we adopt the notation of [8] and refer to it for those statements which are routine.

Proposition 5.1 *Suppose $\Sigma(a_1,\ldots,a_n)$ is a Seifert fibred homology sphere and that $\varphi \in R^*(\Sigma,\mathrm{SU(N)})$. Then $SF(\Theta,\varphi)$ is even.*

proof: Consider the mapping cylinder W of the Seifert fibration $\Sigma \to S^2$. Then it is well known that W is an orbifold with singularities having neighborhoods which are cones on the lens spaces $L(a_i,b_i)$. Let L denote the disjoint union of these lens spaces. The orbifold fundamental group of W is $\pi_1\Sigma/\langle h=1\rangle$. Thus, since ρ is irreducible, $\rho(h)$ is central and it follows that the adjoint representation of ρ extends over W. Hence we can define an orbifold bundle $\mathfrak{ad}_\varphi$ over W. Let $W_0 = W \setminus C(L)$, where $C(L)$ is the union of open cones about the lens spaces. Then $\pi_1 W_0$ is just a free group on $n-1$ generators. It is not hard to verify that $\mathrm{H}^1(\Sigma,\mathfrak{ad}_\varphi) = \mathrm{H}^1(W_0,\mathfrak{ad}_\varphi)$ and $\mathrm{H}^2(W_0,\mathfrak{ad}_\varphi) \to \mathrm{H}^2(\Sigma,\mathfrak{ad}_\varphi)$ is injective (cf. Lemma 2.6 of [8]). By [2], it follows that

$$\mathrm{Index}(d^*_{A_\varphi} \oplus d^-_{A_\varphi})(W_0) = \mathrm{Ind}_\delta(d^*_{A_\varphi} \oplus d^-_{A_\varphi})(W_0) - h_\infty(F)_{W_0},$$

where $h_\infty(F)_{W_0}$ is the dimension of the subspace of limiting values of extended $L^4_{0;\delta}$ sections f of $(\Omega^0 \oplus \Omega^2_-) \otimes \mathfrak{su}(N)$ satisfying $(d^*_{A_\varphi} \oplus d^-_{A_\varphi})^*(f) = 0$. But the proof of Proposition 3.3 of [8] shows that $\mathrm{Ind}_\delta(d^*_{A_\varphi} \oplus d^-_{A_\varphi})(W_0) = 0$. Furthermore we have

Claim: $h_\infty(F)_{W_0} = 0$.

proof of claim: As explained in [8], such f lie in $\mathrm{H}^0(W_0,\mathfrak{ad}_\varphi) \oplus \mathrm{H}^2_-(W_0,\mathfrak{ad}_\varphi)$, thus it is enough to show $h^0(W_0) = 0 = h^2_-(W_0)$ (here, $\mathfrak{ad}_\varphi$ coefficients understood). For each generator x_i of $\pi_1\Sigma$, let Γ_i denote the isotropy group of $\varphi(x_i)$. By Proposition 3.2, it follows that $h^1(W_0) = \sum_1^n d_i - 2\dim\mathrm{SU(N)}$, where recall $d_i = \dim\mathrm{SU(N)} - \dim\Gamma_i$ is the dimension of the conjugacy class of the image of $\varphi(x_i)$. Also, since φ is irreducible, $h^0(W_0) = 0$ and additionally, $h^4(W_0) = 0$. By duality, $h^3(W_0) = h^1(W_0, \Sigma\cup L)$ and the exact sequence for the pair $(W_0,\Sigma\cup L)$ implies

$$\mathrm{H}^0(\Sigma\cup L;\mathfrak{ad}_\varphi) \to \mathrm{H}^1(W_0,\Sigma\cup L;\mathfrak{ad}_\varphi)$$

is an isomorphism. Since $0 = h^0(\Sigma)$ it follows that $h^1(W_0,\Sigma\cup L) = \sum_1^n h^0(L(a_i,b_i))$. Now $\pi_1 L(a_i,b_i) = \mathbb{Z}_{a_i}$ with generator x_i. Suppose $\varphi(x_i) = \exp(diag(\theta_1(i),\ldots,\theta_N(i)))$, where $0 \le \theta_1(i) \le \cdots \le \theta_N(i) < 1$. Since $x_i^{a_i} = 1$, it follows that $\theta_j(i) = 2\pi l_j(i)/a_i$ for some integer $l_j(i)$. Because $L(a_i,b_i)$ has cyclic fundamental group, the restriction of $\mathfrak{ad}_\varphi$ to $L(a_i,b_i)$, decomposes as

$$\mathfrak{ad}_\varphi|_{L(a_i,b_i)} = \bigoplus_{j<k} \mathbb{L}_{jk}(i) \oplus \mathbb{R}^{N-1}.$$

Here, $\mathbb{L}_{jk}(i)$ is a complex line bundle over $L(a_i,b_i)$ and the $\pi_1 L(a_i,b_i) = \mathbb{Z}_{a_i}$ action is given by rotation in an angle of $\theta_k(i)-\theta_j(i)$ on $\mathbb{L}_{jk}$ and is trivial on $\mathbb{R}^{N-1}$. Every pair $j<k$ with $\theta_j(i) = \theta_k(i)$ contributes 2 to $h^0(L(a_i,b_i))$ and it is easy to verify that $h^0(L(a_i,b_i) = \dim\Gamma_i$. Thus $h^3(W_0) = \sum_1^n \dim\Gamma_i$ and since $\sum_0^4 (-1)^i h^i(W_0) = \dim\mathrm{SU(N)}\chi(W_0) = \dim\mathrm{SU(N)}(2-n)$, it follows that

$$h^2(W_0) = \dim\mathrm{SU(N)}\chi(W_0) + h^1(W_0) + h^3(W_0) = 0,$$

which proves the claim.

Using 4.2 of [2], we get

$$\begin{aligned}\mathrm{Index}(d_A^*\oplus d_A^-)(\varphi,\Theta) &= \int_{\Sigma\times\mathbb{R}}\hat{A}(\Sigma\times\mathbb{R})\mathrm{ch}(V_-)\mathrm{ch}(\mathfrak{ad}P)\\ &\quad -\frac{1}{2}(h_\Theta+\eta_\Theta(0))+\frac{1}{2}(-h_\varphi+\eta_\varphi(0)).\end{aligned} \tag{20}$$

and

$$\begin{aligned}\mathrm{Index}(d_{A_\varphi}^*\oplus d_{A_\varphi}^-)(W_0) &= \int_{W_0}\hat{A}(W_0)\mathrm{ch}(V_-)\mathrm{ch}(\mathfrak{ad}E)-\frac{1}{2}(h_\varphi+\eta_\varphi(0))(\Sigma)\\ &\quad +\frac{1}{2}\sum_{i=1}^n(-h_\varphi+\eta_\varphi(0))(L(a_i,b_i)).\end{aligned} \tag{21}$$

Here, V_- is the bundle of negative spinors and $\hat{A}$ and $\mathrm{ch}(V_-)$ are computed using the Riemannian connection. Also $h_\alpha(X)=h^0(X;\mathfrak{ad}_\alpha)+h^1(X;\mathfrak{ad}_\alpha)$ and η_α is the η-invariant of the signature operator twisted by α restricted to even forms.

We can now build a connection $\tilde{A}$ over $\tilde{W}=W\cup(\Sigma\times\mathbb{R}^+)$ by using φ on W, the connection A on $\Sigma\times[0,1]$ gotten from the path a_t with $a_0=A_\varphi$ and $a_1=\Theta$, and the trivial connection on $\Sigma\times[1,\infty)$. Since $\mathrm{Index}(d_{A_\varphi}^*\oplus d_{A_\varphi}^-)(W_0)=0$,

$$\begin{aligned}SF(\varphi,\Theta) &= \mathrm{Index}(d_A^*\oplus d_A^-)(\varphi,\Theta)\\ &= \mathrm{Index}(d_{A_\varphi}^*\oplus d_{A_\varphi}^-)(W_0)+\mathrm{Index}(d_A^*\oplus d_A^-)(\varphi,\Theta)\\ &= \mathrm{Index}(d_{\tilde{A}}^*\oplus d_{\tilde{A}}^-)(\tilde{W}_0)\end{aligned}$$

By adding (20) and (21), it follows that

$$\begin{aligned}SF(\varphi,\Theta) &= \int_{\tilde{W}_0}\hat{A}(\tilde{W}_0)\mathrm{ch}(V_-)\mathrm{ch}(\mathfrak{ad}E)-h_\varphi(\Sigma)-\frac{1}{2}(h_\Theta+\eta_\Theta(0))(\Sigma)\\ &\quad +\frac{1}{2}\sum_{i=1}^n(-h_\varphi+\eta_\varphi(0))(L(a_i,b_i)).\end{aligned} \tag{22}$$

To proceed, evaluate the integral term as in 4.19 of [2] using $\mathcal{L}$ to denote the Hirzebruch L-polynomial and $\mathcal{E}$ the Euler form.

$$\begin{aligned}\int_{\tilde{W}_0}\hat{A}(\tilde{W}_0)\mathrm{ch}(V_-)\mathrm{ch}(\mathfrak{ad}E) &= \int_{\tilde{W}_0}\left(2p_1(\mathfrak{ad}E)+\frac{1}{2}\dim\mathrm{SU(N)}(\mathcal{L}-\mathcal{E})\right)\\ &= 2p_1(A)+\frac{1}{2}\dim\mathrm{SU(N)}\left(\sigma(W_0)-\chi(W_0)+\eta_\theta(0)(\partial W_0)\right).\end{aligned}$$

But $\sigma(W_0)-\chi(W_0)=n-1$, so we get that

$$\begin{aligned}SF(\varphi,\Theta) &= 2p_1(A)+\frac{1}{2}\dim\mathrm{SU(N)}(n-1)-h_\varphi(\Sigma)-\frac{1}{2}(h_\Theta+\rho_\Theta)(\Sigma)\\ &\quad +\frac{1}{2}\sum_{i=1}^n(-h_\varphi+\rho_\varphi)(L(a_i,b_i)),\end{aligned} \tag{23}$$

where $\rho_\alpha = \eta_\alpha - \dim(\alpha)\eta_\theta$. Now by irreducibility, $h^0(\Sigma, \mathfrak{ad}_\varphi) = 0$, thus

$$h_\varphi(\Sigma) = h^1(\Sigma, \mathfrak{ad}_\varphi) = \sum_{i=1}^{n} d_i - 2\dim \mathrm{SU(N)}$$

by Proposition 3.2. Obviously, $\rho_\Theta(\Sigma) = 0$ and $h_\Theta(\Sigma) = \dim \mathrm{SU(N)}$. Furthermore, over $L(a_i, b_i)$, the bundle $\mathfrak{ad}_\varphi$ lifts to S^3 and thus $h^1(L, \mathfrak{ad}_\varphi) = 0$, so that

$$h_\varphi = h^0(L, \mathfrak{ad}_\varphi) = \sum_{i=1}^{n} \dim \Gamma_i.$$

Using the fact that $d_i + \dim \Gamma_i = \dim \mathrm{SU(N)}$ to rewrite (23), we get

$$SF(\varphi, \Theta) = 2p_1(A) + \dim \mathrm{SU(N)} - \frac{1}{2}\sum_{i=1}^{n} d_i + \frac{1}{2}\sum_{i=1}^{n} \rho_\varphi(L(a_i, b_i)).$$

Since $SF(\Theta, \varphi) = -\dim \mathrm{SU(N)} - SF(\varphi, \Theta)$, once we show that

$$2p_1(A) - \frac{1}{2}\sum_{1}^{n} d_i + \frac{1}{2}\sum_{i=1}^{n} \rho_\varphi(L(a_i, b_i))$$

is even, it will follow that $SF(\Theta, \varphi)$ is too.

The remaining terms are the Chern-Simons invariant $p_1(A)$ and the ρ-invariants $\rho_\varphi(L(a_i, b_i))$ which can be calculated as follows. Denote by $\mathbb{L}_{jk}$ the complex line bundle over ∂W_0 which is trivial over Σ and is $\mathbb{L}_{jk}(i)$ over $L(a_i, b_i)$. An elementary obstruction theory argument shows that this extends to a line bundle over W_0. By assembling these line bundles together with a trivial $\mathbb{R}^{N-1}$ bundle we get a reducible bundle E' over W_0 which is equal to E over ∂W_0. Let A' denote a reducible connection on E'. Since E and E' match on the boundary, $p_1(A) = p_1(A')(\mathrm{mod}\mathbb{Z})$ (this is merely saying that the Pontrjagin class of a bundle over a closed manifold is an integer). Writing $A' = \bigoplus_{j<k} A'_{jk}$ according to the decomposition of E', we have that

$$2p_1(A) + \frac{1}{2}\sum_{i=1}^{n} \rho_\varphi(L(a_i, b_i)) = \sum_{j<k} \left(2p_1(A'_{jk}) + \frac{1}{2}\sum_{i=1}^{n} \rho_{\varphi_{jk}}(L(a_i, b_i))\right),$$

where $\rho_{\varphi_{jk}}(L(a_i, b_i))$ is the contribution of the $\mathbb{L}_{jk}$ part to $\rho_\varphi(L(a_i, b_i))$. Set $M_{jk} = 2p_1(A'_{jk}) + \frac{1}{2}\sum_{i=1}^{n} \rho_{\varphi_{jk}}(L(a_i, b_i))$, for convenience and let $m_{jk} = \#\{i \mid l_j(i) \neq l_k(i)\}$. We now claim that the parity of M_{jk} is the same as that of m_{jk}. This follows from the fact that the invariant $R(e_{jk}) = M_{jk} + m_{jk} - 3$ of [9] is *odd*. Here, e_{jk} denotes the Euler number of the orbifold bundle $\mathbb{L}_{jk}$, it is given by the formula $e_{jk} = a\sum_{i=1}^{n}(l_k(i) - l_j(i))/a_i$. But writing $\sum_1^n d_i = \sum_{j<k} D_{jk}$, where $D_{jk} = 2\#\{i \mid l_j(i) \neq l_k(i)\}$, it is clear that $m_{jk} - \frac{1}{2}D_{jk} = 0$. Hence $SF(\Theta, \varphi)$ is even. ♠

Remark: With just a little more work, we could get $SF(\Theta, \varphi)\mathrm{mod}4N$. In fact,

$$\frac{1}{2}\sum_{i=1}^{n} \rho_{\varphi_{jk}}(L(a_i, b_i)) = \sum_{i=1}^{n} \frac{2}{a_i} \sum_{h=1}^{a_i - 1} \cot\left(\frac{\pi a h}{a_i^2}\right) \cot\left(\frac{\pi k}{a_i}\right) \sin^2\left(\frac{\pi e_{jk} h}{a_i}\right),$$

leaving only the term $p_1(A)$, which is the Chern-Simons invariant. The techniques of [3] and [11], properly generalized, should give a precise computation of this. We hazard the guess that

$$p_1(A) = p_1(A') = \sum_{j<k} \frac{e_{jk}^2}{a}.$$

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