THE SU(3) CASSON INVARIANT FOR 3-MANIFOLDS SPLIT ALONG A 2-SPHERE OR A 2-TORUS

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Abstract. We describe the definition of the $SU(3)$ Casson invariant and outline an argument which determines the contribution of certain types of components of the flat moduli space. Two applications of these methods are detailed. The first is a connected sum formula for the $SU(3)$ Casson invariant [3]. The second presents a strategy for computing the $SU(3)$ Casson invariant for certain graph manifolds.

1. Introduction

The aim of this article is to give a non-technical survey of the results in [2, 3] concerning the $SU(3)$ Casson invariant $\lambda_{SU(3)}$ and to introduce a new technique for computing it.

We use ideas from equivariant Morse theory to motivate the definition of the invariant. The invariant involves counting critical points of the Chern-Simons function on gauge orbits of $SU(3)$ connections. If these critical points are not regular, then a perturbation of the Chern-Simons function is used to obtain a function with regular critical set. Although perturbations are an essential part of the definition of $\lambda_{SU(3)}$ in [2], their role is suppressed here.

The second part of the paper describes an approach to computing $\lambda_{SU(3)}$, based on an equivariant version of Bott-Morse theory, which allows computations under less strict regularity assumptions. We use this approach to derive the connected sum formula of [3] and to gain new information about $\lambda_{SU(3)}$ for graph manifolds obtained by gluing two $(2,q)$ torus knot complements together in a certain way. These examples include ± 1 surgery on the untwisted Whitehead double of a $(2,p)$ torus knot. For this family of graph manifolds, we prove that the correction term $\lambda''_{SU(3)}(X)$ vanishes and deduce that $\lambda_{SU(3)}(X) \in \mathbb{Z}$.[1] This is established by showing that only the zero dimensional components of

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[1] This result may be related to the vanishing of the $SU(2)$ Casson invariant for these graph manifolds. In the general case, it is not known if $\lambda_{SU(3)}(X) \in \mathbb{Q}$.

the *irreducible* flat $SU(3)$ moduli space contribute nontrivially to $\lambda_{SU(3)}(X)$. These connections are of a very specific form (see Section 8) and to complete the computation of $\lambda_{SU(3)}(X)$, one would need to enumerate them and to determine their $su(3)$ spectral flow mod 2. This problem is not discussed here and will be treated elsewhere.

Under the weaker regularity assumption, Theorems 6 and 8 describe how various components of the flat moduli space will contribute to $\lambda_{SU(3)}$ once a perturbation is turned on. This is analogous to the computation of the Euler characteristic $\chi(M)$ of a manifold M in terms of a Bott-Morse function. Recall that a function $f : M \to \mathbb{R}$ is called Bott-Morse if its critical point set is a union of smooth submanifolds of M and the Hessian of f is nondegenerate in the normal directions to those submanifolds. Each connected critical submanifold contributes plus or minus its Euler characteristic to $\chi(M)$ (and it is possible to determine this sign).

This paper is concerned with calculating $\lambda_{SU(3)}$ in situations where the flat moduli space satisfies certain regularity assumptions similar to the Bott-Morse condition. The definition of the invariant, and the nature of the regularity condition, are complicated by singularities in the space of connections modulo gauge. Before delving into gauge theory, we describe an equivariant Bott-Morse theory construction in finite dimensions which illustrates most of the ingredients in the gauge theory situation.

Our aim here is to give an accessible account of the invariants and the results derived from this Bott-Morse theoretic approach. Since full details appear elsewhere, most arguments are sketched.

2. Morse theory and the Euler characteristic

We begin by recalling the definition of the Euler characteristic because it provides a finite dimensional analog for our later constructions. For a compact manifold, the Euler characteristic can be viewed as a signed count of the zeros of a transverse vector field on the manifold. If the vector field is not transverse, then its zeros may not be isolated. In this case, the vector field can be made transverse with a small perturbation, and each component of the zero set of the original vector field gives rise to a finite number of transverse zeros after perturbation. The algebraic number of transverse zeros contributed by a particular component is independent of the (small) perturbation, so the Euler characteristic may be interpreted as a sum of contributions from the components of the zero set. The contribution from a component can often be determined without perturbing. For example, if one component is contained in the interior of a ball which does not intersect any other components, then the restriction of the vector field to the sphere bounding the ball has a well-defined Gauss map whose degree gives the contribution of the component.

Suppose M is a compact manifold and $f : M \to \mathbb{R}$ is a Morse function. For any Riemannian metric on M, the gradient vector field of f is transverse to the zero vector field. Let $\mathrm{Crit}(f) = \{p \in M \mid \nabla f(p) = 0\}$ be the set of critical points of f. For each critical point $p \in \mathrm{Crit}(f)$, define the Morse index $\mu(p; f)$ to be the dimension of the negative eigenspace of $\mathrm{Hess}\, f(p)$.

Define

$$\chi(M; f) = \sum_{p \in \mathrm{Crit}(f)} (-1)^{\mu(p;f)}. \tag{1}$$

Generally, a different Morse function will have a different critical point set. One can show, however, using an elementary cobordism argument, that the quantity $\chi(M; f)$ is independent of f. We sketch the argument below because it provides a model for the more subtle cobordism argument we need to establish that the $SU(3)$ Casson invariant is independent of perturbation.

Suppose f_0 and f_1 are Morse functions on M. Choose a generic path of functions f_t connecting f_0 and f_1. The parameterized critical point set

$$W = \bigcup_{t \in [0,1]} \mathrm{Crit}(f_t) \times \{t\},$$

is then an oriented one dimensional cobordism in $M \times [0, 1]$ with

$$\partial W = \mathrm{Crit}(f_1) \times \{1\} - \mathrm{Crit}(f_0) \times \{0\}.$$

For $i = 0, 1$ and $p \in \mathrm{Crit}(f_i)$, the orientation on $(p, i) \in \partial W$ agrees with $(-1)^{i+\mu(p;f_i)}$, which is the same as the sign with which $p \in \mathrm{Crit}(f_i)$ occurs in the sum

$$\sum_{p \in \mathrm{Crit}(f_0)} (-1)^{\mu(p;f_0)} - \sum_{p \in \mathrm{Crit}(f_1)} (-1)^{\mu(p;f_1)}.$$

It follows that $\chi(M; f_0) = \chi(M; f_1)$. In fact, this invariant of M equals the Euler characteristic $\chi(M)$.

3. The SU(2) Casson invariant via gauge theory

Now suppose X is an oriented *homology 3-sphere*. This means that X is a compact, oriented, three dimensional manifold with the same integral homology as S^3. Let $\mathcal{A}$ denote the space of $SU(2)$ connections on the trivial bundle $E = X \times \mathbb{C}^2$. We identify connections with $su(2)$-valued 1-forms using the trivialization of E. Let $\mathcal{G} = \{g : X \to SU(2)\}$ denote the corresponding gauge group (the group of bundle automorphisms). The gauge group $\mathcal{G}$ acts on $\mathcal{A}$

with quotient $\mathcal{B} = \mathcal{A}/\mathcal{G}$. We denote the gauge orbit of a connection $A \in \mathcal{A}$ by $[A]$. The set of irreducible connections is denoted $\mathcal{A}^*$ and its quotient is $\mathcal{B}^*$. Using appropriate Sobolev completions, $\mathcal{B}^*$ is a smooth, infinite dimensional Banach manifold. Each orbit of reducible connections is a singular point in $\mathcal{B}$ because its stabilizer subgroup is larger than that of an orbit of irreducible connections.

The $SU(2)$ Casson invariant of X can be defined by a formula analogous to (1). The role of the Morse function is played by the Chern-Simons function $cs : \mathcal{A} \to \mathbb{R}$, defined by

$$cs(A) = \tfrac{1}{8\pi^2} \int_X tr(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A). \tag{2}$$

The critical point set of cs is exactly the set of flat connections, i.e.,

$$\mathrm{Crit}(cs) = \{A \in \mathcal{A} \mid F_A = 0\},$$

where $F_A = dA + A \wedge A$ is the curvature of A. The quotient of the set of flat connections is the moduli space

$$\mathcal{M} = \{A \in \mathcal{A} \mid F_A = 0\}/\mathcal{G} = \mathrm{Crit}(cs)/\mathcal{G},$$

which is compact and has expected dimension zero (though $\mathcal{M}$ is not generally a finite set). We set $\mathcal{M}^* = \mathcal{M} \cap \mathcal{B}^*$.

Floer [6] and Taubes [8] described a set of admissible perturbations, which are gauge invariant functions $h : \mathcal{A} \to \mathbb{R}$ such that:

(i) The perturbed moduli space $\mathcal{M}_h := \mathrm{Crit}(cs + h)/\mathcal{G}$ is compact.
(ii) For a generic small perturbation h, $\mathcal{M}_h^*$ is a smooth, compact, zero dimensional submanifold of $\mathcal{B}^*$.

The cobordism argument from Section 2 generalizes to show that an invariant of X can be defined by counting the critical orbits of cs in $\mathcal{B}^*$ with sign. The Hessian of cs has infinitely many positive and negative eigenvalues, so the usual definition of Morse index does not make sense, but one can define suitable signs by the following construction.

Fix a metric on X and associate to each $A \in \mathcal{A}$ the self-adjoint elliptic operator

$$D_A : \Omega^{0+1}(X; su(2)) \longrightarrow \Omega^{0+1}(X; su(2))$$

given by

$$D_A(\sigma, \tau) = (d_A^* \tau, d_A \sigma + * d_A \tau).$$

The spectral flow along a path A_t of connections is the signed number of eigenvalues of D_{A_t} which cross zero from negative to positive (crossings in the reverse direction count negatively, and our convention is to regard zero modes at $t = 0$ and $t = 1$ as positive). This quantity effectively gives a *relative Morse index* between two critical points. We choose the trivial connection θ as a basepoint, i.e., replace $(-1)^{\mu(p)}$ in formula (1) by $(-1)^{SF(\theta,A)}$. The spectral flow changes by an even integer under gauge transformation of A, so the sign is well-defined on gauge orbits $[A]$.

Theorem 1 (Taubes)**.** *For generic small perturbations h, the quantity*

$$\sum_{[A]\in\mathcal{M}_h^*}(-1)^{SF(\theta,A)} \tag{3}$$

is independent of the metric and perturbation. It equals minus the Casson invariant $-\lambda_{SU(2)}(X)$, normalized as in [9]*. This invariant is defined by counting conjugacy classes of nontrivial representations $\rho:\pi_1X\to SU(2)$ with sign* [1]*.*

Unless X is a homology sphere, $\mathcal{M}^*$ is not generally compact and so the quantity in (3) will typically depend on the choice of perturbation h (see Walker's generalization of $\lambda_{SU(2)}$ to rational homology spheres [9], for example). A similar problem occurs for homology spheres in $SU(n)$ gauge theory for $n>2$. In [2], an invariant of homology spheres is defined using $SU(3)$ gauge theory by adding a correction term to the $SU(3)$ analogue of the sum (3). This correction term involves reducible (perturbed) flat connections. It is needed to compensate for the births and deaths of irreducible gauge orbits of critical points of $cs+h$ from the stratum of reducible connections as h varies. We will come back to this problem in Section 5, but first we illustrate the birth and death phenomena with a finite dimensional model.

4. Equivariant Morse theory and the relative Euler characteristic

Suppose M is a compact Riemannian manifold and G is a compact Lie group acting smoothly on M. Let L be the subset of M consisting of points with nontrivial stabilizer. Setting $M^*=M-L$ (suggestively), we note that G acts freely on M^*. We use

$$\Gamma_p=\{g\in G\mid g(p)=p\}$$

to denote the stabilizer subgroup of G at $p\in M$. It is sufficient for our purposes to assume that Γ_p is either trivial or isomorphic to $U(1)$ for each $p\in M$. This implies L is a submanifold of M with smooth quotient L/G and that M/G is a manifold with singularities along L/G.

Suppose $f:M\to\mathbb{R}$ is a C^3 G-invariant function. For $p\in \mathrm{Crit}(f)\cap L$, we refine the Morse index $\mu(p)$ as follows. The tangent space T_pM decomposes into

$$T_pM=T_pL\oplus N_pL, \tag{4}$$

where N_pL is the normal bundle fiber at p. The stabilizer subgroup $\Gamma_p \cong U(1)$ acts nontrivially on N_pL with weight one, and since f is G-invariant, $\langle \nabla f(p), \vec{v} \rangle = 0$ for all $\vec{v} \in N_pL$. In addition, invariance of f implies that $\operatorname{Hess} f(p)$ respects the decomposition in (4). Thus,

$$\mu(p) = \mu_t(p) + \mu_n(p),$$

where μ_t and μ_n denote the dimensions of the negative eigenspaces of $\operatorname{Hess} f(p)$ on the two summands in the decomposition (4).

Definition 2. A C^3 G-invariant function $f : M \to \mathbb{R}$ is called *equivariantly Morse* if the following properties hold:

(i) The gradient vector field of the restriction $f|_{L/G}$ is transverse to zero, so that the set of critical points of $f|_{L/G}$ is a regular, zero dimensional manifold.
(ii) For all $p \in \operatorname{Crit}(f|_L)$, the Hessian $\operatorname{Hess} f(p)|_{N_p(L)}$ restricted to the normal directions is nondegenerate.
(iii) The set of critical points of the induced function $f : M^*/G \to \mathbb{R}$ is compact, regular, and zero dimensional.

Note that compactness in the third condition is not automatic, since M^*/G is not compact. One can show that it follows from the first two conditions, however.

The proof that Morse functions are generic in the nonequivariant case has been generalized by Wasserman [10] to prove the following proposition.

Proposition 3. *For generic C^3 equivariant functions $f : M \to \mathbb{R}$, f is equivariantly Morse.*

Given an equivariantly Morse function $f : M \to \mathbb{R}$, define

$$\chi_G(M) = \sum_{[p]\in \operatorname{Crit}(f)\cap M^*/G} (-1)^{\mu(p)} - \tfrac{1}{2} \sum_{[p]\in \operatorname{Crit}(f)\cap L/G} (-1)^{\mu_t(p)} \mu_n(p). \tag{5}$$

We will argue that this quantity is independent of f.

For a generic path of functions connecting two equivariant Morse functions, three distinct types of topological changes, or *bifurcations*, in the critical set can occur. These are illustrated in Figure 1, where the dotted curve represents the parameterized critical set in L/G and the solid curve represents the parameterized critical set in M^*/G.

The points labeled A and B represent the standard births/deaths of cancelling pairs of critical points in the zero dimensional critical sets in L/G and M^*/G.

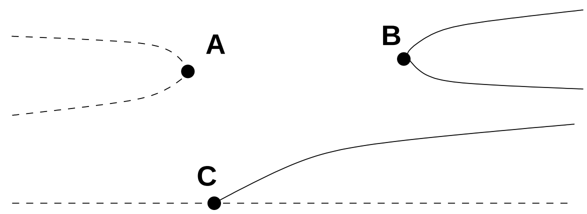

FIGURE 1. The three types of bifurcations of the critical set.

The point labeled C in the figure represents the third type of bifurcation, in which a free critical orbit in M^* pops out of a critical orbit in L.

This third type of bifurcation can be visualized by taking $M = S^2$ and $G = U(1)$ acting by equatorial rotations. In this case, L consists of two points, the North and South poles, denoted N and S. Representing functions on the sphere as height functions, Figure 2 shows a deformation of the standard height function during which a critical circle C pops out of S. Note that, in this birth, not only does the topology of $\mathrm{Crit}(f)$ change, but the *normal Morse index* $\mu_n(S)$ also changes. In fact, one can show that this third type of bifurcation occurs precisely when there is a change in normal Morse index at a critical orbit in L, and a careful check of orientations shows that the formula (5) remains invariant under all three types of bifurcations. Thus $\chi_G(M)$ is independent of the choice of equivariantly Morse function f.

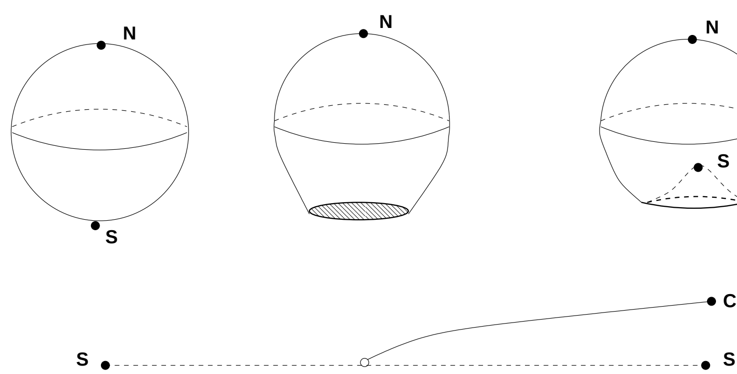

FIGURE 2. A critical circle C popping out of the South pole.

The following result is proved by choosing an equivariantly Morse function whose values at all the critical points in L are lower than its values at any other critical points.

Lemma 4. *The quantity in formula (5) equals the relative Euler characteristic* $\chi(M/G, L/G)$.

Remark 5. *This invariant may be viewed as a differential topological invariant of the quotient space M/G, which is smooth except along L/G. In this sense, we have come as close as possible to a differential topological description of an Euler characteristic of M/G.*

5. The SU(3) Casson invariant

In this section, we outline the definition of the $SU(3)$ Casson invariant, using ideas from the previous section for motivation.

Suppose X is a homology sphere and denote by $\mathcal{A}$ the space of $SU(3)$ connections in the trivial bundle $E = X \times \mathbb{C}^3$. The gauge group $\mathcal{G} = \{g : X \to SU(3)\}$ acts on $\mathcal{A}$ with quotient $\mathcal{B} = \mathcal{A}/\mathcal{G}$. Let $\mathcal{A}^*$ be the subspace of irreducible connections, with quotient $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$. Choosing appropriate Sobolev completions, it follows that $\mathcal{B}^*$ is a smooth, infinite dimensional Banach manifold.

Connections in the complement of $\mathcal{A}^*$ are called reducible and are characterized by the fact that their stabilizer in $\mathcal{G}$ is larger than the center

$$Z(SU(3)) = \left\{ \left(\begin{array}{ccc} e^{2\pi i k/3} & 0 & 0 \\ 0 & e^{2\pi i k/3} & 0 \\ 0 & 0 & e^{2\pi i k/3} \end{array} \right) \middle| \, k = 0, 1, 2 \right\} \cong \mathbb{Z}_3.$$

Throughout this article, given a group G and a subset $S \subset G$, we use $Z(S) = \{g \in G \mid gs = sg \text{ for all } s \in S\}$ to denote the centralizer of S.

The $SU(3)$ Casson invariant is defined by adapting the finite dimensional model from the previous section to the Chern-Simons function of equation (2). As in the $SU(2)$ case, the critical point set $\text{Crit}(cs)$ consists of flat connections and the flat moduli space is $\mathcal{M} = \text{Crit}(cs)/\mathcal{G}$. Note that for a nontrivial flat $SU(3)$ connection A on a homology sphere, the stabilizer subgroup Γ_A is isomorphic to either $U(1)$ or $\mathbb{Z}_3$, depending on whether A is reducible or not.

We can dispense with the infinite dimensional group action, and work instead with an $SU(3)$ action, as follows. Fix a basepoint $x_0 \in X$ and consider the normal subgroup of based gauge transformations $\mathcal{G}_0 = \{g \in \mathcal{G} \mid g(x_0) = 1\}$. The quotient $\widetilde{\mathcal{B}} = \mathcal{A}/\mathcal{G}_0$ is a smooth manifold and $\mathcal{B} = \widetilde{\mathcal{B}}/SU(3)$. The singularities in $\mathcal{B}$ occur at orbits in $\widetilde{\mathcal{B}}$ of connections whose stabilizer (with respect to the $SU(3)$ action) is larger than $\mathbb{Z}_3$.

While cs is not $\mathcal{G}$-invariant, $cs(g \cdot A) = cs(A) + \deg(g)$, and so this function descends to give a smooth function $cs : \widetilde{\mathcal{B}} \to \mathbb{R}/\mathbb{Z}$ which is equivariant with respect to the $SU(3)$ action.

The role of $L \subset M$ is played by the space of $\mathcal{G}_0$-orbits of reducible connections in $\widetilde{\mathcal{B}}$. It is useful to work with gauge representatives of these orbits that are in a standard form. Let $\mathcal{A}^r \subset \mathcal{A}$ be the subspace of connections preserving the decomposition of $E = X \times \mathbb{C}^3$ into the sum $(X \times \mathbb{C}^2) \oplus (X \times \mathbb{C})$. This splitting of $\mathbb{C}^3$ determines a decomposition of the Lie algebra $su(3) = \mathfrak{h} \oplus \mathfrak{h}^\perp$, where $\mathfrak{h} = su(2) \oplus \mathbb{R}$ and $\mathfrak{h}^\perp = \mathbb{C}^2$. The group $SU(2)$ acts via the adjoint action on the first summand (and trivially on the $\mathbb{R}$) and via the canonical representation on the second.

For connections $A \in \mathcal{A}^r$, the tangent space $T_A\mathcal{A} = \Omega^1(X; su(3))$ decomposes into the sum of $T_A\mathcal{A}^r = \Omega^1(X; \mathfrak{h})$ and $N_A\mathcal{A}^r = \Omega^1(X; \mathfrak{h}^\perp)$. Since the connection (acting on forms by the adjoint representation) respects this decomposition, the twisted signature operator D_A splits into two operators $D_A^{\mathfrak{h}}$ and $D_A^{\mathfrak{h}^\perp}$, each acting on the spaces of twisted $(0+1)$-forms with the specified coefficients. In fact, since $\mathfrak{h} \cong su(2) \oplus \mathbb{R}$, $D_A^{\mathfrak{h}}$ splits further into an operator on $su(2)$-valued forms and an *untwisted* operator on $\mathbb{R}$-valued forms. Furthermore, since $\mathcal{A}^r$ is connected, any $A \in \mathcal{A}^r$ can be connected to θ by a path in $\mathcal{A}^r$ and $SF(\theta, A)$ can thereby be split into "tangential" and "normal" parts.

Based on the finite dimensional case, it is natural to examine the quantity $\lambda_1(X) - \lambda_2(X)$ where

$$\lambda_1(X) = \sum_{[A] \in \mathcal{M}_h^*} (-1)^{SF_{su(3)}(\theta, A)} \tag{6}$$

and

$$\lambda_2(X) = \frac{1}{2} \sum_{[A] \in \mathcal{M}_h^r} (-1)^{SF_{\mathfrak{h}}(\theta, A)} SF_{\mathfrak{h}^\perp}(\theta, A). \tag{7}$$

Unfortunately, the formula (7) is not well-defined, for the following reason. For $g \in \mathcal{G}$, the spectral flows $SF(\theta, A)$ and $SF(\theta, gA)$ differ by an even integer. Thus formula (6) and the leading sign in formula (7) do not depend on the gauge representative A, but the normal spectral flow $SF_{\mathfrak{h}^\perp}(\theta, A)$ does. One way to eliminate this problem is to replace $SF_{\mathfrak{h}^\perp}(\theta, A)$ by $SF_{\mathfrak{h}^\perp}(\theta, A) - 2cs(A)$, but the new quantity depends on the perturbation (varying the perturbation will cause the critical point to move, thus changing the value of cs at the critical point).

This difficulty is overcome by using the following formula:

$$\lambda_2(X) = \frac{1}{2} \sum_{[A] \in \mathcal{M}_h^r} (-1)^{SF_{\mathfrak{h}}(\theta, A)} (SF_{\mathfrak{h}^\perp}(\theta, A) - 4cs(\hat{A}) + 2), \tag{8}$$

where A is a representative of the gauge orbit $[A]$ of reducible perturbed flat connections, and $\hat{A}$ is an *unperturbed* flat connection close to A. Since cs is constant on components of flat connections, this is well-defined (for small h, so that A is close to a unique component of flat connections). The addition of the constant after the Chern-Simons value has the effect of modifying the $SU(3)$ invariant by addition of a multiple of $\lambda_{SU(2)}$, and we choose this constant so that $\lambda_{SU(3)}(-X) = \lambda_{SU(3)}(X)$.

Theorem 6 (Theorem 1, [2]). *For a generic set of small perturbations h, the quantity $\lambda_{SU(3)} = \lambda_1 - \lambda_2$, where λ_2 is defined in equation (8), is an invariant of X, independent of perturbation and gauge representatives.*

6. Contributions from components of the flat moduli space

To evaluate $\lambda_{SU(3)}(X)$ using the formula in the previous section requires that the function $cs : \widetilde{\mathcal{B}} \to \mathbb{R}/\mathbb{Z}$ satisfies a nondegeneracy condition which is an immediate generalization of the equivariantly Morse condition in Definition 2. In this section we determine the contribution to $\lambda_{SU(3)}$ of several types of components of the (unperturbed) flat moduli space under a less restrictive nondegeneracy assumption.

The simplest case is that of a point component $C = \{[A]\}$ in $\mathcal{M}$. If A is irreducible and $H^1_A(X; su(3)) = 0$, then $[A]$ contributes ± 1 to $\lambda_{SU(3)}(X)$, where the sign is given by the parity of $SF_{su(3)}(\theta, A)$. If A is reducible, then provided $H^1_A(X; su(3)) = 0$, it contributes the rho invariant $\varrho_{A'}(X)$ to $\lambda_{SU(3)}(X)$. Here, $H^1_A(X; su(3))$ means cohomology with coefficients in the $su(3)$ bundle twisted by A and A' is the flat $SU(2)$ reduction of A. The term $\varrho_{A'}(X)$ is the Atiyah-Patodi-Singer rho invariant of X associated to the holonomy map $hol_{A'} : \pi_1 X \to SU(2)$ using the canonical action[2] of $SU(2)$ on $\mathbb{C}^2$. The condition that $H^1_A(X; su(3)) = 0$ is the analog of the conditions (i) and (ii) in Definition 2.

Consider a path component C of the flat moduli space $\mathcal{M}$ such that $C \subset \mathcal{B}^*$ and C is a smooth submanifold. Then C is a compact, critical submanifold of the circle-valued Chern-Simons function on $\mathcal{B}^*$. We call such a component a *nondegenerate critical submanifold* (in the sense of Bott-Morse theory) if the Hessian of the Chern-Simons function is a nondegenerate bilinear form on each fiber of the normal bundle to C.

Theorem 7 (Proposition 7, [3]). *Suppose that $C \subset \mathcal{M}^*$ is a compact component. Suppose further C is a nondegenerate critical submanifold for the Chern-Simons function. Then C contributes $(-1)^{SF(\theta,A)}\chi(C)$ to $\lambda_{SU(3)}(X)$, where A is any gauge representative of an orbit in C.*

[2] $\varrho_{A'}(X)$ is not the same as the rho invariant obtained using the adjoint action of $SU(2)$ on its Lie algebra, which is also studied in gauge theory.

As is well-known, the same result holds verbatim in the $SU(2)$ context, and we use it to outline a gauge theoretic proof of the additivity of the $SU(2)$ Casson invariant.

Corollary 8 (Casson)**.** $\lambda_{SU(2)}(X_1 \# X_2) = \lambda_{SU(2)}(X_1) + \lambda_{SU(2)}(X_2)$.

Sketch of proof. By perturbing, we can assume that the moduli space of X_1 and X_2 are compact, regular, and zero dimensional. Set $X = X_1 \# X_2$ and let θ_i denote the trivial (product) connection on $X_i \times \mathbb{C}^2$ for $i = 1, 2$. For a connection A on X, we denote by A_i the restriction of A to X_i for $i = 1, 2$.

There are precisely two component types in the perturbed flat moduli space of X:

(1) Point components of the form $[\theta_1 \# A_2]$ or $[A_1 \# \theta_2]$.
(2) $SO(3)$ components of the form $[A_1 \#_\phi A_2]$, where A_1 and A_2 are irreducible and $\phi \in SO(3)$ is the gluing parameter.

Since $\chi(SO(3)) = 0$, Theorem 7 implies that components of the second type do not contribute to $\lambda_{SU(2)}(X)$. For the point components, we use additivity of the spectral flow

$$SF_X(A_1 \# A_2, B_1 \# B_2) = SF_{X_1}(A_1, B_1) + SF_{X_2}(A_2, B_2)$$

to prove Corollary 8. □

The next result is an equivariant generalization of Theorem 7 and covers many components of $\mathcal{M}$ with mixed isotropy.

Theorem 9. *Suppose $\widetilde{C} \subset \widetilde{\mathcal{B}}$ is a connected critical submanifold of the Chern-Simons function which is nondegenerate in the Bott-Morse sense. Set $C = \widetilde{C}/SU(3)$, $C^* = C \cap \mathcal{B}^*$, and $C^r = C \cap \mathcal{B}^r$, and choose $[A] \in C^*$ and $[B] \in C^r$. Then C contributes*

$$(-1)^{SF_{su(3)}(\theta,A)} \chi(C, C^r) + (-1)^{SF_{\mathfrak{h}}(\theta,B)} \chi(C^r)(SF_{\mathfrak{h}^\perp}(\theta, B) - 4cs(B) + 2)$$

to $\lambda_{SU(3)}(X)$.

Sketch of proof. Choose a perturbation h such that $h|_{\widetilde{C}}$ is equivariantly Morse (this is possible by [2], Proposition 3.4). Consider the 1-parameter family $f_t = cs + th$. Several factors complicate the argument slightly. First, the parameterized critical set is not generally a manifold. Second, one does not have the freedom in general to choose h so that for $t > 0$ the critical set $\text{Crit}(f_t)$ near $\widetilde{C}$ remains a subset of $\widetilde{C}$. Nevertheless, one can show using the implicit function theorem that, for small $\epsilon > 0$, the parameterized critical set

$$\bigcup_{0 \le t < \epsilon} \text{Crit}(f_t) \times \{t\} \subset \widetilde{\mathcal{B}} \times [0, \epsilon)$$

is homeomorphic to the union of $\widetilde{C} \times \{0\}$ and $\mathrm{Crit}(h|_{\widetilde{C}}) \times [0, \epsilon)$, identified along $t = 0$ (and the homeomorphism is a diffeomorphism on either part of the union). $\square$

7. The Connected Sum Formula

In this section, we outline the proof of the following theorem.

Theorem 10 (Theorem 1 of [3])**.** *Suppose X_1 and X_2 are integral homology 3-spheres. Then*

$$\lambda_{SU(3)}(X_1 \# X_2) = \lambda_{SU(3)}(X_1) + \lambda_{SU(3)}(X_2) + 4\,\lambda_{SU(2)}(X_1)\,\lambda_{SU(2)}(X_2), \tag{9}$$

where $\lambda_{SU(2)}$ is Casson's original invariant, normalized as in [9].

Remark. Even though $\lambda_{SU(3)}$ is not additive under connected sum, the theorem implies that the difference $\lambda_{SU(3)} - 2\lambda^2_{SU(2)}$ is additive under connection sum.

Sketch of proof. Assume that the moduli spaces of flat $SU(3)$ connections on X_1 and X_2 are regular (see Definition 3.8 of [2]). In particular, $\mathcal{M}(X_1)$ and $\mathcal{M}(X_2)$ are compact zero dimensional manifolds.

The moduli space $\mathcal{M}(X_1 \# X_2)$ is not regular because it contains higher dimensional components. We prove (9) by interpreting point components in the usual way and applying Theorems 7 and 9 to components of positive dimension.

Given flat connections A_1 on X_1 and A_2 on X_2, a well-known procedure (see section 7.2.1 of [5]) constructs a flat connection A on $X = X_1 \# X_2$. The gauge orbit $[A]$ is not uniquely determined by $[A_1]$ and $[A_2]$. The reason is that one can gauge transform A_1 while keeping A_2 fixed and the newly constructed connection will not be gauge equivalent to the old one.

This is perhaps easiest to understand in terms of $SU(3)$ representation varieties. For a homology sphere X, set

$$\widetilde{R}(X, SU(3)) = \mathrm{Hom}(\pi_1(X), SU(3)).$$

For $g \in SU(3)$ and $\rho \in \widetilde{R}(X, SU(3))$, we define $g \cdot \rho$ to be the representation sending $x \in \pi_1 X$ to $g\,\rho(x)\,g^{-1}$. This defines an action of $SU(3)$ on $\widetilde{R}(X, SU(3))$ and we denote the quotient by

$$R(X, SU(3)) = \widetilde{R}(X, SU(3))/SU(3).$$

By a well-known theorem in differential geometry, $\widetilde{\mathcal{M}}(X) \cong \widetilde{R}(X, SU(3))$ and $\mathcal{M}(X) \cong R(X, SU(3))$. We use the notation $[\rho]$ to denote the conjugacy class of a representation $\rho : \pi_1 X \to SU(3)$. Thus $[\rho] \in R(X, SU(3))$.

In case $X = X_1 \# X_2$ decomposes as a connected sum, the fundamental group is given by the free product

$$\pi_1 X = \pi_1 X_1 * \pi_1 X_2.$$

Thus, $SU(3)$ representations ρ_1 and ρ_2 of $\pi_1 X_1$ and $\pi_1 X_2$ define, in an obvious way, a representation $\rho = \rho_1 * \rho_2 : \pi_1 X \to SU(3)$ and ρ is uniquely determined by ρ_1 and ρ_2. This gives a canonical isomorphism

$$\widetilde{R}(X, SU(3)) \cong \widetilde{R}(X_1, SU(3)) \times \widetilde{R}(X_2, SU(3)),$$

and conjugation acts diagonally. Notice that the conjugacy class $[\rho]$ is *not* determined by $[\rho_1]$ and $[\rho_2]$ because one can conjugate ρ_2 relative to ρ_1 and produce new, inequivalent representations.

Fix representations ρ_1 and ρ_2 of X_1 and X_2 and consider the family of representations ρ^g defined for $g \in SU(3)$ by

$$\rho^g = \rho_1 * (g \cdot \rho_2).$$

By taking $[\rho^g]$, this defines a parameterized family in the representation space $R(X, SU(3))$. As we shall see, the correct parameter space for this family is a double coset space.

For $i = 1, 2$ define the isotropy subgroup of ρ_i to be

$$\Gamma_i = \{\gamma \in SU(3) \mid \gamma \cdot \rho_i = \rho_i\}.$$

If $\gamma \in \Gamma_2$ and $g \in SU(3)$ is arbitrary, then

$$\rho^{g\gamma} = \rho_1 * (g\gamma) \cdot \rho_2 = \rho_1 * g \cdot \rho_2 = \rho^g.$$

On the other hand, if $\gamma \in \Gamma_1$

$$\rho^{\gamma g} = \rho_1 * (\gamma g) \cdot \rho_2 = \gamma \cdot ((\gamma^{-1} \cdot \rho_1) * (g \cdot \rho_2)),$$

and thus $[\rho^{\gamma g}] = [\rho^g]$ if $\gamma \in \Gamma_1$. Thus the subset $C = \{[\rho^g] \mid g \in SU(3)\} \subset R(X, SU(3))$ is homeomorphic to

$$\Gamma_1 \backslash SU(3) / \Gamma_2.$$

If ρ_1 is the trivial representation, then $\Gamma_1 = SU(3)$ and $\Gamma_1 \backslash SU(3)/\Gamma_2$ is a single point. The same is true if ρ_2 is trivial. Pairing each nontrivial representation of X_1 with the trivial representation on X_2 and vice versa, and using additivity of the spectral flow and the Chern-Simons invariants, we obtain the first two terms on the right of equation (9).

The remaining cases to consider are:

(1) Both ρ_1 and ρ_2 are irreducible.
(2) ρ_1 is irreducible and ρ_2 is reducible, or vice versa.
(3) Both ρ_1 and ρ_2 are reducible and nontrivial.

Note that cases 1 and 2 give rise to components completely contained in the subvariety $R^*(X, SU(3))$ of irreducible representations. Case 3 gives rise to components of mixed isotropy; that is, the constructed representation may or may not be reducible, depending on the gluing parameter g.

First, assume both ρ_1 and ρ_2 are irreducible. Then $\Gamma_1 = \Gamma_2 = Z(SU(3)) \cong \mathbb{Z}_3$, and hence $\Gamma_1 \backslash SU(3)/\Gamma_2 \cong PU(3)$. Since $\chi(PU(3)) = 0$, Theorem 7 implies that these components do not contribute to $\lambda_{SU(3)}(X)$.

Next, assume without loss of generality that ρ_1 is irreducible and ρ_2 is reducible but nontrivial. Then $\Gamma_1 \cong \mathbb{Z}_3$ and $\Gamma_2 = H$, where H is the $U(1)$ subgroup

$$H = \left\{ \left(\begin{array}{ccc} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u^{-2} \end{array} \right) \middle| \, u \in U(1) \right\}. \tag{10}$$

Thus $\Gamma_1 \backslash SU(3)/\Gamma_2$ is a homogeneous 7-manifold, hence its Euler characteristic vanishes. Another application of Theorem 7 shows that such components do not contribute to $\lambda_{SU(3)}(X)$.

The last case is the most interesting because one gets nontrivial contributions to the $SU(3)$ Casson invariant. Indeed, the nonadditivity of $\lambda_{SU(3)}$ under connected sum is a direct result of the nonvanishing of the relative Euler characteristic in this case.

Observe that even though both ρ_1 and ρ_2 are reducible, the induced representation $\rho = \rho_1 * \rho_2$ may be irreducible. To visualize this, let $L_i \subset \mathbb{C}^3$ be the one dimensional subspace invariant under ρ_i. Then $\rho = \rho_1 * \rho_2$ is reducible if and only if $L_1 = L_2$.

Alternatively, if ρ_1 and ρ_2 are reducible with $L = L_1 = L_2$, then the representation $\rho^\gamma = \rho_1 * (\gamma \cdot \rho_2)$ is reducible if and only if γ preserves L. For the standard reduction, i.e., if $\operatorname{im}(\rho_i) \subset S(U(2) \times U(1))$ and $L = \operatorname{span}\{(0,0,1)\}$, this occurs precisely when $\gamma \in S(U(2) \times U(1))$. In this case, $\Gamma_1 = \Gamma_2 = H$ is the subgroup (10). (This agrees with the proof of Corollary 8 because the quotient of $S(U(2) \times U(1))$ by Γ_1 is indeed isomorphic to $SO(3)$.)

Let $C = \Gamma_1 \backslash SU(3)/\Gamma_2$ be the associated component of the representation variety $R(X, SU(3))$. As we have already observed, the subset C^r of reducible representations is parameterized by $SO(3)$ and $\chi(C^r) = 0$. One can show (see the proof of Proposition 10 of [3]) that $\chi(C) = \chi(C, C^r) = 4$, thus an application of Theorem 9, together with additivity of spectral flow, completes the proof of the theorem. $\square$

8. Graph Manifolds of two $(2,q)$ torus knot complements

In this section, we consider the family of graph manifolds obtained by gluing the complements of two torus knots together as follows.

Given knots K_1 and K_2 in S^3 with complements X_1 and X_2, orient K_1 and K_2 and denote by μ_i and λ_i the standard oriented meridian/longitude pair of K_i. Construct a homology sphere X by gluing X_1 to X_2 by identifying μ_1 and λ_2 and λ_1 and μ_2. Thus $X = X_1 \cup_T X_2$, where T is the 2-torus and the gluing is specified by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in terms of the bases $\{\mu_1, \lambda_1\}$ and $\{\mu_2, \lambda_2\}$ for $H_1(\partial X_1)$ and $H_1(\partial X_2)$. A Mayer-Vietoris argument proves that X is a homology sphere.

We will consider homology spheres of this form in the special case K_1 and K_2 are torus knots of type $(2,q)$. Recall that a torus knot is determined by a pair (p,q) of relatively prime integers. It is the knot $K : [0, 2\pi] \to \mathbb{R}^3 \subset S^3$ given by

$$K(\theta) = ((2 + \cos q\theta)\cos p\theta, (2 + \cos q\theta)\sin p\theta, -\sin q\theta).$$

We first consider representations of the knot complement.

Lemma 11. *Suppose X is the complement of the $(2,p)$ torus knot K and let μ denote its meridian. If $\rho : \pi_1 X \to SU(3)$ is an irreducible representation, then $\rho(\mu)$ has three distinct eigenvalues.*

Proof. We prove this by contradiction, using the well-known fact that μ normally generates $\pi_1 X$. This fact shows immediately that $\rho(\mu)$ has at least two distinct eigenvalues, so suppose $\rho(\mu)$ has an eigenvalue with eigenspace $U \subset \mathbb{C}^3$ and $\dim(U) = 2$.

Writing $\pi_1 X = \langle x, y \mid x^2 = y^p \rangle$, notice that x^2 is central in $\pi_1 X$. By irreducibility of ρ, it follows that $\rho(x^2)$ is central in $SU(3)$. (Hence $\rho(x^6) = 1$.) Thus $\rho(x)$ also has an eigenspace of dimension two, which we denote by V. For dimensional reasons, $U \cap V \neq 0$. Choosing a nonzero vector $v \in U \cap V$, we see that v is a common eigenvector for $\rho(x)$ and $\rho(\mu)$.

We claim that v is also an eigenvector of $\rho(y)$. Since $\pi_1 X$ is generated by x and y, this claim contradicts irreducibility of ρ, as the linear span of v would then be an invariant subspace.

To prove the claim, recall that $\mu = xy^{(1-p)/2}$, and hence $x^5\mu = x^6 y^{(1-p)/2}$. Since v is an eigenvector of both $\rho(x)$ and $\rho(\mu)$, it is also an eigenvector of

$$\rho(x^5\mu) = \rho(x^6 y^{(1-p)/2}) = \rho(x^6)\rho(y^{(1-p)/2}) = \rho(y^{(1-p)/2}).$$

It follows that v is an eigenvector of $\rho(y)^{1-p} = \rho(y)\rho(y)^{-p}$. The proof of the claim is completed by noting that $\rho(y)^{-p}$ is central (and hence is a scalar matrix). □

The previous lemma is no longer true if one drops the assumption that ρ is irreducible. It obviously fails in the abelian case, but one can also find nonabelian counterexamples.

We now turn our attention to $SU(3)$ representations of the homology sphere $X = X_1 \cup_T X_2$ obtained by gluing a $(2, p_1)$ torus knot complement X_1 to a $(2, p_2)$ complement X_2 with the specific boundary identification described at the beginning of the section.

Lemma 12. *Suppose X_i is the complement of a $(2, p_i)$ torus knot K_i for $i = 1, 2$ and μ_i, λ_i are the standard meridian and longitude for K_i. Let $X = X_1 \cup_T X_2$ be the homology sphere obtained by identifying μ_1 with λ_2 and λ_1 with μ_2. If $\rho : \pi_1 X \to SU(3)$ is a representation whose restriction to $\pi_1 X_1$ is abelian then ρ is trivial. (Similarly for $\pi_1 X_2$.)*

Proof. Given $\rho : \pi_1 X \to SU(3)$, we define representations $\rho_i : \pi_1 X_i \to SU(3)$ for $i = 1, 2$ by precomposing with the natural map $\pi_1 X_i \to \pi_1 X$ induced by inclusion. Notice that ρ is determined by the pair (ρ_1, ρ_2).

If ρ_1 is abelian, then because λ_1 is null-homologous in X_1, it follows that $\rho_1(\lambda_1)$ is the identity. In $\pi_1 X$, we have $\mu_2 = \lambda_1$, hence $\rho_2(\mu_2)$ is also the identity. But μ_2 normally generates $\pi_1 X_2$, so this implies that ρ_2 is trivial. This then implies that ρ_1 is trivial, hence ρ is trivial. □

Suppose $\rho : \pi_1 X \to SU(3)$, and let ρ_i denote the restriction of ρ to $\pi_1 X_i$ as in the proof of the lemma. Note that ρ is completely determined by the pair (ρ_1, ρ_2). Note also that if ρ is reducible, then so are ρ_1 and ρ_2. Finally, notice that the converse is false.

Our strategy for computing $\lambda_{SU(3)}(X)$ is to parameterize the various components of $R(X, SU(3))$ and apply Theorem 7. Specifically, for fixed ρ_1 and ρ_2, we use the stabilizer group of $\rho|_{\pi_1 T}$ to parameterize the set

$$C = \{[\rho] \mid \rho|_{\pi_1 X_i} \text{ is conjugate to } \rho_i \text{ for } i = 1, 2\} \subset R(X, SU(3)).$$

The next result implies that the stabilizer subgroup of $\rho(\pi_1 T)$ is the maximal torus of $SU(3)$.

Theorem 13. *Suppose K_1 and K_2 are $(2, p_1)$ and $(2, p_2)$ torus knots, respectively, and X_1 and X_2 are their complements. Let $X = X_1 \cup_T X_2$ be the* *homology sphere obtained by gluing X_1 to X_2 by identifying meridians and longitudes. Denote by μ_1, λ_1 and μ_2, λ_2 the standard meridian and longitude pair for the knots K_1 and K_2, respectively. If $\rho : \pi_1 X \to SU(3)$ is nontrivial, then both $\rho(\mu_1)$ and $\rho(\mu_2)$ have three distinct eigenvalues.*

Proof. As before, associate to ρ the representations ρ_1 of $\pi_1 X_1$ and ρ_2 of $\pi_1 X_2$. If both ρ_1 and ρ_2 are irreducible, then the conclusion follows from Lemma 11. The remaining cases are:

(i) Both ρ_1 and ρ_2 are reducible.
(ii) ρ_1 is reducible and ρ_2 is irreducible or vice versa.

By conjugating, if necessary, we can assume that $\rho(\mu_1)$ and $\rho(\mu_2)$ are diagonal. In each case, we shall assume $\rho(\mu_1)$ has only two distinct eigenvalues and arrive at a contradiction.

Writing $\pi_1 X_1 = \langle x_1, y_1 \mid x_1^2 = y_1^{p_1} \rangle$ and $\pi_1 X_2 = \langle x_2, y_2 \mid x_2^2 = y_2^{p_2} \rangle$. In terms of these generators, the meridian and longitude are given by the elements

$$\begin{array}{ll} \mu_1 = x_1 y_1^{(1-p_1)/2}, & \mu_2 = x_2 y_2^{(1-p_2)/2}, \\ \lambda_1 = x_1^2 \mu_1^{-2p_1}, & \lambda_2 = x_2^2 \mu_2^{-2p_2}. \end{array}$$

Regarding μ_i and λ_i as words in x_i and y_i, we obtain the following presentation of the fundamental group of $X = X_1 \cup_T X_2$:

$$\pi_1 X = \langle x_1, y_1, x_2, y_2 \mid x_1^2 = y_1^{p_1}, x_2^2 = y_2^{p_2}, \mu_1 = \lambda_2, \mu_2 = \lambda_1 \rangle.$$

In case (i), Lemma 12 implies both ρ_1 and ρ_2 are nonabelian (otherwise ρ is trivial). Now, λ_2 is homologically trivial in X_2, and since ρ_2 is reducible, it follows that $\rho_2(\lambda_2)$ has 1 as one of its eigenvalues. Since $\mu_1 = \lambda_2$ in $\pi_1 X$, the same is true of $\rho_1(\mu_1)$. Since the other two eigenvalues are equal, $\rho_1(\mu_1)$ must have eigenvalues $1, -1$ and -1. It follows that $\rho_1(\mu_1)^2$ is the identity matrix.

Applying ρ_1 to the relation $\lambda_1 = x_1^2 \mu_1^{-2p_1}$, we see that $\rho_1(\lambda_1) = \rho_1(x_1^2)$. Since x_1^2 is central in $\pi_1 X_1$, $\rho_1(x_1^2) \in Z(\mathrm{im}(\rho_1))$ has at most two distinct eigenvalues. But we have already noted that one of its eigenvalues equals 1, hence the other two are ± 1. Since $\rho_1(x_1^2) = \rho_1(\lambda_1) = \rho_2(\mu_2)$ and ρ_2 is nonabelian, $\rho_1(x_1^2)$ is nontrivial and thus has eigenvalues $-1, -1$, and 1. Now $\rho(x_1)$ is a square root of $\rho(x_1^2)$, thus it has eigenvalues $\{\pm i, \pm i, -1\}$ or $\{i, -i, 1\}$. In the first case, $\rho(x_1) \in Z(\mathrm{im}(\rho_1))$ since two of its eigenvalues are equal. Thus $\rho(x_1)$ commutes with $\rho(y_1)$ implying ρ_1 is abelian and contradicting Lemma 12. So we can assume $\rho_1(x_1)$ has eigenvalues $i, -i$ and 1.

Conjugate ρ so that ρ_1 has image in $S(U(2) \times U(1))$ and $\rho(\mu_1)$ and $\rho(\mu_2)$ are diagonal. Using the fact that μ_1 normally generates $\pi_1 X_1$ and that $\rho_1(\mu_1)$ has eigenvalues ± 1, it follows that

$$\mathrm{im}(\rho_1) \subset \left\{ \begin{pmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \right\}.$$

But $\rho_1(y_1^{p_1}) = \rho_1(x_1^2)$ has $(3,3)$ entry equal to 1. Since p_1 is odd, this implies $\rho_1(y_1)$ also has $(3,3)$ entry equal to 1. Thus $\rho_1(x_1)$ and $\rho_1(y_1)$, and consequently all matrices in $\mathrm{im}(\rho_1)$, are of the form

$$\begin{pmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consequently

$$\rho(\mu_1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which implies ρ_1 is abelian, a contradiction.

The proof in case (ii) is similar. Assume without loss of generality that ρ_1 is reducible and ρ_2 is irreducible. As before, we arrange by conjugation that $\rho_1(\mu_1)$ and $\rho_2(\mu_2)$ are diagonal. We know from Lemma 11 that $\rho_2(\mu_2)$ has three distinct eigenvalues, so the only way the theorem can fail is if $\rho_1(\mu_1)$ has eigenvalues t, t, t^{-2} for some $t \in U(1)$. We can conjugate ρ further so that $\mathrm{im}(\rho_1) \subset S(U(2) \times U(1))$. Since λ_1 is homologically trivial, one of its eigenvalues equals 1 and it follows that

$$\rho_1(\lambda_1) = \begin{pmatrix} s & 0 & 0 \\ 0 & s^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some $s \in U(1)$. Using the relations $\mu_2 = \lambda_1$, $\lambda_2 = \mu_1$, and $\lambda_2 = x_2^2(\mu_2)^{-2p_2}$, it follows that

$$\rho(\mu_1) = \rho(x_2^2)\rho(\lambda_1)^{-2p_2}. \tag{11}$$

However $\rho(x_2^2) \in Z(SU(3))$ since x_2^2 is central in $\pi_1 X_2$ and ρ_2 is irreducible.

Now $\rho(\mu_1)$ is one of the following three matrices:

$$\begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{pmatrix}, \quad \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-2} & 0 \\ 0 & 0 & t \end{pmatrix}, \quad \begin{pmatrix} t^{-2} & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}.$$

In the first case, notice that $\rho_1(\mu_1) \in Z(\mathrm{im}(\rho_1))$, hence ρ_1 is abelian (since μ_1 normally generates $\pi_1 X_1$). In the second case, equation (11) leads to the matrix equation:

$$\begin{pmatrix} t & 0 & 0 \\ 0 & t^{-2} & 0 \\ 0 & 0 & t \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} s^{-2p_2} & 0 & 0 \\ 0 & s^{2p_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The only solutions in $SU(3)$ occur when $t = u$, in which case $\rho_1(\mu_1)$ is also central and ρ_1 is abelian. In the third case one can construct a similar argument with the same conclusion. Therefore, the assumption that $\rho(\mu_1)$ has eigenvalues t, t, t^{-2} cannot hold, and $\rho(\mu_1)$ has three distinct eigenvalues. □

This theorem shows that if ρ_1 and ρ_2 are fixed so that $\rho_i(\mu_i)$ and $\rho_i(\lambda_i)$ are diagonal, then the component of $R(X, SU(3))$ consisting of conjugacy classes of representations ρ with $\rho|_{\pi_1 X_i}$ conjugate to ρ_i is parameterized by a quotient of the form

$$\Gamma_1 \backslash T / \Gamma_2$$

where $T = S^1 \times S^1$ is the maximal torus of $SU(3)$. As in the previous section, Γ_i denotes the isotropy subgroup of ρ_i. The various different cases are:

(i) If ρ is reducible, then so are ρ_1 and ρ_2 and up to conjugation, we have $\Gamma_1 = \Gamma_2 = H$, the $U(1)$ subgroup (10). The corresponding component $C = \Gamma_1 \backslash T / \Gamma_2 \cong U(1)$ has $\chi(C) = 0$. Notice that $C = C^r$, i.e., $C^* = \emptyset$. Theorem 9 then shows that these components do not contribute to $\lambda_{SU(3)}(X)$.

(ii) If ρ_1 and ρ_2 are both irreducible, then $\Gamma_1 = \Gamma_2 = Z(SU(3)) \cong \mathbb{Z}_3$ and $C = \Gamma_1 \backslash T / \Gamma_2 \cong T$ again has $\chi(C) = 0$. Theorem 7 now shows that these components do not contribute to $\lambda_{SU(3)}(X)$.

(iii) If ρ_1 is irreducible and ρ_2 is reducible (or vice versa), then we conjugate until $\mathrm{im}(\rho_2) \subset S(U(2) \times U(1))$ and so $\Gamma_1 \cong \mathbb{Z}_3$ and $\Gamma_2 = H$. Hence $C = \Gamma_1 \backslash T / \Gamma_2 \cong U(1)$ and $\chi(C) = 0$. Again by Theorem 7, we conclude that these components do not contribute to $\lambda_{SU(3)}(X)$.

(iv) If ρ_1 and ρ_2 are reducible and ρ is irreducible, then Γ_1 and Γ_2 are isomorphic to $U(1)$ subgroups of $SU(3)$ but $\Gamma_1 \neq \Gamma_2$. (In other words, if L_i is the 1-dimensional subspace of $\mathbb{C}^3$ invariant under ρ_i, then $L_1 \neq L_2$). In this case, $C = \Gamma_1 \backslash T / \Gamma_2 \cong \{*\}$ is a point which contributes ± 1 to $\lambda_{SU(3)}(X)$ depending on the parity of the $su(3)$ spectral flow.

Implicit in the above statements is the fact that the components described there are nondegenerate. That is, for any orbit $[A] \in \widetilde{\mathcal{B}}$ in a component of the flat moduli space, the kernel of Hess(cs) on $T_{[A]}\widetilde{\mathcal{B}}$ is exactly the tangent space of the component (the double coset space). This can be verified by using Fox differential calculus to find the first cohomology with twisted coefficients for the two knot complements and identify their images in the cohomology of the splitting torus. Then a Mayer-Vietoris argument completes the proof of this fact.

This proves the following theorem.

Theorem 14. *If X is the homology sphere obtained by gluing X_1 to X_2 as above, where X_1 is a $(2, p_1)$ torus knot complement and X_2 is a $(2, p_2)$ torus knot complement, then the correction term $\lambda''_{SU(3)}(X)$ vanishes and $\lambda_{SU(3)}(X)$ is the algebraic count of irreducible representations of $\pi_1 X$ of type (iv) above. In particular, $\lambda_{SU(3)}(X) \in \mathbb{Z}$.*

Based on our description of the component types (i) – (iv) of $R(X, SU(3))$, a reasonable guess is that $\lambda_{SU(3)}(X) = 4\lambda_{SU(2)}(X_1)\lambda_{SU(2)}(X_2)$ for graph manifolds $X = X_1 \cup_T X_2$ as in Theorem 14. To prove this, we need to develop a method for computing the $su(3)$ spectral flow for manifolds split along a 2-torus. We leave this problem to a future article.

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