A CONNECTED SUM FORMULA FOR THE SU(3) CASSON INVARIANT

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Abstract. We provide a formula for the SU(3) Casson invariant for 3-manifolds given as the connected sum of two integral homology 3-spheres.

1. Introduction

In [1], we introduced an invariant $\lambda_{SU(3)}$ of integral homology 3-spheres X defined by appropriately counting the conjugacy classes of representations $\varrho : \pi_1 X \to SU(3)$. Our main result here is the following theorem.

Theorem 1. *If X_1 and X_2 are integral homology 3-spheres, then*

$$\begin{aligned}\lambda_{SU(3)}(X_1 \# X_2) &= \lambda_{SU(3)}(X_1) + \lambda_{SU(3)}(X_2) \\ &\quad +4\,\lambda_{SU(2)}(X_1)\,\lambda_{SU(2)}(X_2),\end{aligned}$$

where $\lambda_{SU(2)}$ is Casson's original invariant, normalized as in [7].

Even though $\lambda_{SU(3)}$ is not additive under the connected sum operation, the theorem has the following consequence.

Corollary 2. *The difference $\lambda_{SU(3)} - 2\lambda^2_{SU(2)}$ defines an invariant of integral homology spheres which is additive under connected sum.*

The proof of Theorem 1 requires an understanding of how certain non-degenerate critical submanifolds of the (perturbed) Chern-Simons functional contribute to $\lambda_{SU(3)}$. The relevant results here are Propositions 8 and 11, which hold in rather general circumstances. Before delving into the details, we give a brief introduction to 3-manifold $SU(3)$ gauge theory and review the results of [1].

Suppose X is a closed, oriented, $\mathbb{Z}$-homology 3-sphere and set $P = X \times SU(3)$. Denote by θ the trivial (product) connection and by d the associated covariant derivative. Let

$$\mathcal{A} = \{d + A \mid A \in \Omega^1(X; su(3))\}$$

be the space of smooth connections in P. The gauge group $\mathcal{G}$ of smooth bundle automorphisms $g : P \to P$ acts on $\mathcal{A}$ by $g \cdot A = gAg^{-1} + gdg^{-1}$ with quotient $\mathcal{B} = \mathcal{A}/\mathcal{G}$, the space of gauge orbits of $SU(3)$ connections. For the most part we work with the Sobolev completions of $\mathcal{A}$ and $\mathcal{G}$ in the L^2_1 and L^2_2 norms, respectively, though occasionally we use the L^2 metric on $\mathcal{A}$.

Denote by $\Gamma_A = \{g \in \mathcal{G} \mid g \cdot A = A\}$ the stabilizer of A in $\mathcal{G}$. The curvature $F_A \in \Omega^2(X; su(3))$ is defined for $A \in \mathcal{A}$ by the formula

$$F_A = dA + A \wedge A$$

and the moduli space $\mathcal{M} \subset \mathcal{B}$ of flat connections by

$$\mathcal{M} = \{A \in \mathcal{A} \mid F_A = 0\}/\mathcal{G}.$$

For $[A] \in \mathcal{M}$, Γ_A is isomorphic to $\mathbb{Z}_3$, $U(1)$ or $SU(3)$ because X is a $\mathbb{Z}$-homology sphere. Set $\mathcal{M}^* = \{[A] \in \mathcal{M} \mid \Gamma_A = \mathbb{Z}_3\}$ and $\mathcal{M}^r = \{[A] \in \mathcal{M} \mid \Gamma_A = U(1)\}$. Here, as in [1], we call $A \in \mathcal{A}$ *reducible* if $\Gamma_A \cong U(1)$. Then $\mathcal{M}$ is the disjoint union $\mathcal{M}^* \cup \mathcal{M}^r \cup \{[\theta]\}$.

One can also view $\mathcal{M}$ as the quotient by $\mathcal{G}$ of the critical set of the Chern-Simons functional

$$\begin{aligned} CS: \quad & \mathcal{A} \longrightarrow \mathbb{R} \\ & A \mapsto \tfrac{1}{8\pi^2} \int_X tr(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A). \end{aligned}$$

First fix a Riemannian metric on X and let

$$* : \Omega^p(X; su(3)) \to \Omega^{3-p}(X; su(3))$$

be the resulting Hodge star operator. Then define an inner product on $\mathcal{A}$ by setting $\langle a, b\rangle_{L^2} = -\int_X tr(a \wedge *b)$. Now take the gradient of CS with respect to the L^2 metric on $\mathcal{A}$ to see that

$$\text{Grad}_A\, CS = -\tfrac{1}{4\pi^2} * F_A.$$

Consider the self-adjoint elliptic operator K_A which sends

$$(\xi, a) \in \Omega^0(X; su(3)) \oplus \Omega^1(X; su(3))$$

to $K_A(\xi, a) = (d_A^* a, d_A\xi - *d_A a)$. Assume A is flat. Then

$$\ker K_A = \mathcal{H}^0_A(X; su(3)) \oplus \mathcal{H}^1_A(X; su(3)),$$

the space of d_A-harmonic $su(3)$-valued (0+1)-forms. Choose a path $A_t \in \mathcal{A}$ with $A_0 = \theta$ and $A_1 = A$ and define $\mathrm{SF}(\theta, A)$ to be the spectral flow of the path of self-adjoint operators K_{A_t}. If A is reducible, then we can choose the path so that each A_t has $\Gamma_{A_t} = U(1)$ for $t \in (0,1]$. Adjusting by a path of gauge transformations, we can assume that, for $t \in [0,1]$, $A_t \in \mathcal{A}_{S(U(2)\times U(1))}$, the space of connections on $X \times S(U(2) \times U(1))$. Setting $\mathfrak{h} = s(u(2) \times u(1))$ to be the Lie subalgebra of $su(3)$, it follows that, for A reducible, the spectral flow decomposes as $\mathrm{SF}(\theta, A) = \mathrm{SF}_{\mathfrak{h}}(\theta, A) + \mathrm{SF}_{\mathfrak{h}^\perp}(\theta, A)$ according to the splitting $su(3) = \mathfrak{h} \oplus \mathfrak{h}^\perp$, where $\mathfrak{h}^\perp \cong \mathbb{C}^2$.

Now $\mathcal{M}$ is compact and has expected dimension zero (since K_A is self-adjoint), but it typically contains components of large dimension. So that we can work with a discrete space, we perturb the Chern-Simons functional using *admissible* functions. These are thoroughly described in Section 2 of [1]. Roughly, one alters $CS : \mathcal{A} \to \mathbb{R}$ by adding a gauge-invariant function $h : \mathcal{A} \to \mathbb{R}$ of the form $h = \tau \circ hol_\ell$, where $\tau : SU(3) \to \mathbb{R}$ is an invariant function (usually just the real or imaginary part of trace) and $hol_\ell : \mathcal{A} \to SU(3)$ is the holonomy around some loop $\ell \subset X$. In general circumstances, one must consider sums $h = \tau_1 \circ hol_{\ell_1} + \cdots + \tau_n \circ hol_{\ell_n}$ where $\ell_1, \ldots, \ell_n$ are loops in X and $\tau_1, \ldots, \tau_n$ are invariant functions (for analytical reasons, one averages these functions over tubular neighborhoods of the curves, see [1] for details). Denoting the space of admissible perturbation functions with respect to this choice of loops $\ell_1, \ldots, \ell_n$ by $\mathcal{F}$, by Definition 2.1 of [1], $\mathcal{F} \cong C^3(\mathbb{C}, \mathbb{R})^{\times n}$. Each $h \in \mathcal{F}$ induces a function, also denoted h, on $\mathcal{B}$.

A connection is called *h-perturbed flat* if it is a critical point of $CS + h$. Setting $\zeta_h(A) = *F_A - 4\pi^2 \operatorname{Grad}_A h$, the moduli space of h-perturbed flat connections is defined to be

$$\mathcal{M}_h = \zeta_h^{-1}(0)/\mathcal{G}.$$

We denote by $\mathcal{M}_h^*$ (and $\mathcal{M}_h^r$) the subset of gauge orbits of irreducible (reducible, respectively) perturbed flat connections.

Perturbing only changes the flatness equation in a small neighborhood of the supporting loops ℓ_i. For example, when $h = \tau \circ hol_\ell$, every perturbed flat connection A is actually flat outside a small tubular neighborhood of ℓ. In general if $h = \sum_{i=1}^n \tau_i \, hol_{\ell_i}$, then the same is true outside the union of small tubular neighborhoods of each ℓ_i. We showed in Section 3 of [1] that there exist loops $\ell_1, \ldots, \ell_n$ in X such that, for generic small $h \in \mathcal{F}$, $\mathcal{M}_h^*$ and $\mathcal{M}_h^r$ are compact 0-dimensional submanifolds of $\mathcal{B}^*$ and $\mathcal{B}^r$ consisting of gauge orbits that satisfy a cohomological regularity condition (see Definition 4 below and Theorem 3.13 of [1]). Moreover, if A is h-perturbed flat, then there is a flat connection $\widehat{A}$ near A (cf. Proposition 3.7, [1]).

Proposition 3. *For generic h sufficiently small, the quantity*

$$\lambda_{SU(3)}(X) := \sum_{[A]\in\mathcal{M}_h^*} (-1)^{\mathrm{SF}(\theta,A)} - \frac{1}{2}\sum_{[A]\in\mathcal{M}_h^r} (-1)^{\mathrm{SF}(\theta,A)}(\mathrm{SF}_{\mathfrak{h}^\perp}(\theta,A) - 4\,CS(\widehat{A}) + 2)$$

defines an invariant of integral homology 3-spheres X called the **Casson SU(3) invariant***.*

In reference to the second sum, only the difference $\mathrm{SF}_{\mathfrak{h}^\perp}(\theta,A)-4\,CS(\widehat{A})$ is well-defined on the gauge orbit $[A]$; each term individually depends on the choice of representative for $[A]$. It is proved in [1] that the above formula for $\lambda_{SU(3)}(X)$ is independent of the choice of h, Riemannian metric, and orientation of X.

2. The Gluing Construction and Point Components

Theorem 1 is proved by gluing together perturbed flat connections on X_1 and X_2. For $i=1,2$, set $P_i = X_i \times SU(3)$ and denote by θ_i the trivial connection in P_i. Choose h_i a generic sufficiently small admissible perturbation function so that $\mathcal{M}_{h_i}(X_i)$, the moduli space of perturbed flat connections in P_i, is *regular* according to the following definition.

We first introduce some notation. Given a smooth function $h : \mathcal{A} \to \mathbb{R}$, the Hessian of h at A is the map

$$\mathrm{Hess}_A\, h : \Omega^1(X; su(3)) \to \Omega^1(X; su(3))$$

defined in terms of the L^2 metric by

$$\langle \mathrm{Hess}_A\, f(a), b\rangle_{L^2} = \frac{\partial^2}{\partial s \partial t} h(A + sa + tb)\Big|_{s,t=0}.$$

Definition 4. *Suppose X is a $\mathbb{Z}$-homology 3-sphere, $P = X \times SU(3)$, $h : \mathcal{A} \to \mathbb{R}$ is an admissible perturbation function and A is an h-perturbed* *flat connection. Introduce the operator* $*d_{A,h} = *d_A - 4\pi^2 \operatorname{Hess}_A h$ *on* $\Omega^1(X; su(3))$ *and define the* **deformation complex** *to be*

$$\Omega^0(X; su(3)) \xrightarrow{d_A} \Omega^1(X; su(3)) \xrightarrow{*d_{A,h}} \Omega^1(X; su(3)) \xrightarrow{d_A^*} \Omega^0(X; su(3)).$$

Define groups $H^0_A(X; su(3)) = \ker d_A$ *(the Lie algebra of the stabilizer subgroup* Γ_A*) and* $H^1_{A,h}(X; su(3)) = \ker *d_{A,h} / \operatorname{im} d_A$*. A point* $[A] \in \mathcal{M}_h$ *is called* **regular** *if* $H^1_{A,h}(X, su(3)) = 0$*, and a subset* $S \subseteq \mathcal{M}_h$ *is regular if this condition holds for all* $[A] \in S$*.*

The procedure outlined in §7.2.1 of [3] constructs a nearly anti-self-dual connection on $X_1 \# X_2$ given anti-self-dual connections A_1 and A_2 on 4-manifolds X_1 and X_2. A key step is to approximate A_i by a connection that is flat in a small neighborhood of the basepoint $x_i \in X_i$. We use a similar (but simpler) procedure to construct perturbed flat connections on the connected sum of two 3-manifolds. We first review the construction for $X_1 \# X_2$, then construct the bundle $P_1 \# P_2$ and connection (see also [5]).

Given basepoints $x_i \in X_i$ and small, 3-balls B_i containing x_i, set $\dot{B}_i = B_i \setminus \{x_i\}$ and $\dot{X}_i = X_i \setminus \{x_i\}$. We take the metric to be flat on B_i. Choose an orientation reversing isometry $f : \dot{B}_1 \to \dot{B}_2$ of the deleted neighborhoods and define $X_1 \# X_2 = \dot{X}_1 \cup \dot{X}_2 / \sim$, where $x \sim f(x)$ for $x \in \dot{B}_1$.

Now suppose $h_1 = \sum_{j=1}^{n_1} \tau_{1,j}\, hol_{\ell_{1,j}}$ and $h_2 = \sum_{j=1}^{n_2} \tau_{2,j}\, hol_{\ell_{2,j}}$ are admissible perturbations on X_1 and X_2, respectively. We can choose x_i and B_i so that $\ell_{i,j}$ misses B_i for all $j = 1, \ldots n_i$ and each $i = 1, 2$. Thus, if A_i is an h_i-perturbed flat connection on X_i, its restriction to B_i is flat and parallel translation by A_i defines a trivialization of $P_i|_{B_i}$ in which the connection is also trivial.

Using these trivializations, we can extend any isomorphism $\sigma : (P_1)_{x_1} \to (P_2)_{x_2}$ to an isomorphism of $P_1|_{B_1} \to P_2|_{B_2}$. We then construct the bundle $P_1 \# P_2$ by gluing P_1 and P_2 by identifying $P_1|_{\dot{B}_1}$ and $P_2|_{\dot{B}_2}$. Since the restriction of A_i to B_i is trivial, we can also glue A_1 and A_2 to obtain the connection $A_1 \#_\sigma A_2$ on $P_1 \# P_2$. Of course, $P_1 \# P_2 \cong X \times SU(3)$ is independent of σ even though $A_1 \#_\sigma A_2$ is not, in general.

Since the loops $\ell_{i,j}$ do not intersect the balls B_i, setting $h_0 = h_1 + h_2$ defines an admissible perturbation on $X = X_1 \# X_2$. If A is an h_0-perturbed flat connection on X, then restricting A to each side of the connected sum, shows that A is gauge equivalent to one the form $A_1 \#_\sigma A_2$ for some A_1, A_2 and σ as above. Moreover, $A_1 \#_\sigma A_2$ and $A_1 \#_{\sigma'} A_2$ are gauge equivalent if and only if σ and σ' are in the same $\Gamma_{A_1} \times \Gamma_{A_2}$ orbit in $SU(3)$.

Observe that $\mathcal{M}_{h_0}(X)$ is not regular, even though both $\mathcal{M}_{h_1}(X_1)$ and $\mathcal{M}_{h_2}(X_2)$ are. In fact, the gauge orbit $[A_1 \#_\sigma A_2]$ is isolated in $\mathcal{M}_{h_0}(X)$ if and only if $A_i = \theta_i$ for $i = 1$ or 2. In that case, $[A_1 \#_\sigma A_2]$ is independent of σ and so we drop the subscript and simply write $[A_1 \# \theta_2]$ or $[\theta_1 \# A_2]$.

Since $\mathcal{M}_{h_0}(X)$ is not regular, one cannot compute $\lambda_{SU(3)}(X)$ from Proposition 3 without further perturbing the flatness equations. A method for doing this is presented in the next section, but first we explain the special role played by connections of the form $A_1 \# \theta_2$ and $\theta_1 \# A_2$. By a Mayer-Vietoris argument, the gauge orbits $[A_1 \# \theta_2]$ and $[\theta_1 \# A_2]$ in $\mathcal{M}_{h_0}(X)$ are regular whenever $[A_1] \in \mathcal{M}_{h_1}(X_1)$ and $[A_2] \in \mathcal{M}_{h_2}(X_2)$ are regular. If $C \subset \mathcal{M}_{h_0}(X)$ is a point component, then either $C = \{[A_1 \# \theta_2]\}$ or $C = \{[\theta_1 \# A_2]\}$.

It is well-known that for irreducible connections, the spectral flow is additive with respect to connected sum. Specifically, if $\theta = \theta_1 \# \theta_2$ and $A = A_1 \#_\sigma A_2$ where A_1 and A_2 are irreducible connections on X_1 and X_2, respectively, then

$$\mathrm{SF}_X(\theta, A) = \mathrm{SF}_{X_1}(\theta_1, A_1) + \mathrm{SF}_{X_2}(\theta_2, A_2). \tag{1}$$

(For proofs of this statement and the next in the $SU(2)$ setting, see Lemmas 2.2.1 and 2.2.2 in [5].) The next result treats the case where A_1 or A_2 is trivial and determines the contribution of point components to $\lambda_{SU(3)}(X_1 \# X_2)$.

Lemma 5. *Set $\theta = \theta_1 \# \theta_2$ and suppose that A_i is a nontrivial, h_i-perturbed flat $SU(3)$ connection on X_i for $i = 1, 2$. In parts (ii) and (iii), assume further that A_i is reducible and that $\widehat{A}_i$ is the reducible flat connection on X_i close to A_i for $i = 1, 2$. Then*

(i) $\mathrm{SF}_X(\theta, A_1 \# \theta_2) = \mathrm{SF}_{X_1}(\theta_1, A_1)$ *and* $\mathrm{SF}_X(\theta, \theta_1 \# A_2) = \mathrm{SF}_{X_2}(\theta_2, A_2)$.
(ii) $\mathrm{SF}_{X,\mathfrak{h}^\perp}(\theta, A_1 \# \theta_2) = \mathrm{SF}_{X_1,\mathfrak{h}^\perp}(\theta_1, A_1)$ *and* $\mathrm{SF}_{X,\mathfrak{h}^\perp}(\theta, \theta_1 \# A_2) = \mathrm{SF}_{X_2,\mathfrak{h}^\perp}(\theta_2, A_2)$.
(iii) $CS_X(\widehat{A}_1 \# \theta_2) = CS_{X_1}(\widehat{A}_1)$ *and* $CS_X(\theta_1 \# \widehat{A}_2) = CS_{X_2}(\widehat{A}_2)$.

Using Lemma 5 and summing over the set

$$\mathcal{M}^0_{h_0}(X) = \{[A] \in \mathcal{M}_{h_0}(X) \mid A = A_1 \# \theta_2 \text{ or } A = \theta_1 \# A_2\}$$

of point components of $\mathcal{M}_{h_0}(X)$, we see that

$$\begin{aligned}
&\sum_{[A]\in\mathcal{M}^{0,*}_{h_0}(X)} (-1)^{\mathrm{SF}(\theta,A)} - \tfrac{1}{2} \sum_{[A]\in\mathcal{M}^{0,r}_{h_0}(X)} (-1)^{\mathrm{SF}(\theta,A)} (\mathrm{SF}_{\mathfrak{h}^\perp}(\theta, A) - 4\, CS(\widehat{A}) + 2) \\
&\quad = \sum_{[A_1]\in\mathcal{M}^{*}_{h_1}(X_1)} (-1)^{\mathrm{SF}(\theta_1,A_1)} + \sum_{[A_2]\in\mathcal{M}^{*}_{h_2}(X_2)} (-1)^{\mathrm{SF}(\theta_2,A_2)} \\
&\qquad - \tfrac{1}{2} \sum_{[A_1]\in\mathcal{M}^{r}_{h_1}(X_1)} (-1)^{\mathrm{SF}(\theta_1,A_1)} (\mathrm{SF}_{\mathfrak{h}^\perp}(\theta_1, A_1) - 4\, CS(\widehat{A}_1) + 2) \\
&\qquad - \tfrac{1}{2} \sum_{[A_2]\in\mathcal{M}^{r}_{h_2}(X_2)} (-1)^{\mathrm{SF}(\theta_2,A_2)} (\mathrm{SF}_{\mathfrak{h}^\perp}(\theta_2, A_2) - 4\, CS(\widehat{A}_2) + 2) \\
&\quad = \lambda_{SU(3)}(X_1) + \lambda_{SU(3)}(X_2).
\end{aligned}$$

Thus, the point components in $\mathcal{M}_{h_0}(X)$ give rise to the first two terms on the right hand side of formula (1).

3. Higher Dimensional Components

In this section, we study connected components C of $\mathcal{M}_{h_0}(X)$ with $\dim C > 0$ and analyze their contribution to $\lambda_{SU(3)}(X_1 \# X_2)$. Here and elsewhere in this section, $h_0 = h_1 + h_2$ is the perturbation from the previous section obtained by perturbing over X_1 and X_2 separately. Suppose C is such a component and suppose $[A_1 \#_\sigma A_2] \in C$. Then, since $\mathcal{M}_{h_1}(X_1)$ and $\mathcal{M}_{h_2}(X_2)$ are both regular, we obtain an explicit description of C as the double coset space of $SU(3)$ by Γ_{A_1} and Γ_{A_2}.

We also introduce the *based* gauge group $\mathcal{G}_0 = \{g \in \mathcal{G} \mid g_{x_0} = 1\}$, where $x_0 \in X$ is a fixed basepoint. Set $\widetilde{\mathcal{B}} = \mathcal{B}/\mathcal{G}_0$, the space of based gauge orbits of connections, and $\widetilde{\mathcal{M}}_h = \zeta_h^{-1}(0)/\mathcal{G}_0$, the based perturbed flat moduli space. Using the gluing construction, it is not difficult to see that $\widetilde{\mathcal{M}}_{h_0}(X_1 \# X_2) = \widetilde{\mathcal{M}}_{h_1}(X_1) \times \widetilde{\mathcal{M}}_{h_2}(X_2)$.

The projection $\pi : \widetilde{\mathcal{B}} \to \mathcal{B}$ has fiber modeled on $SU(3)/\Gamma_A$ over $[A]$. The two fiber types relevant here are $PU(3) = SU(3)/\mathbb{Z}_3$ and the homogeneous 7-manifold N obtained as the space of left cosets of the $U(1)$

subgroup

$$\left\{ \left(\begin{array}{ccc} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u^{-2} \end{array} \right) \middle| \, u \in U(1) \right\} \tag{2}$$

of $SU(3)$. From now on, since we will be dealing almost exclusively with connections on $X = X_1 \# X_2$, we write $\mathcal{M}_h$ for $\mathcal{M}_h(X)$. The following proposition summarizes what we now know about the components $C \subset \mathcal{M}_{h_0}$ with $\dim C > 0$.

Proposition 6. *Suppose $C = \{[A_1 \#_\sigma A_2] \mid \sigma \in \Gamma_{A_1} \backslash SU(3)/\Gamma_{A_2}\}$ is a connected component of $\mathcal{M}_{h_0}$, where both A_1 and A_2 are nontrivial (so C is not a point component).*

(i) *If A_1 or A_2 is irreducible, then C is a smooth submanifold of $\mathcal{B}^*$ with $C \cong PU(3)$ if A_1 and A_2 are both irreducible, and $C \cong N$ if A_1 or A_2 is reducible.*

(ii) *If both A_1 and A_2 are reducible, then $\widetilde{C} \cong N \times N$ is a smooth submanifold of $\widetilde{\mathcal{B}}$, where $\widetilde{C}$ is the preimage of C under the projection $\pi : \widetilde{\mathcal{B}} \to \mathcal{B}$.*

In (i), the component C is nondegenerate, that is, the Hessian of $CS + h_0$ is nondegenerate in the normal directions to C. In (ii), the same is true of $\widetilde{C}$.

Obviously $h_0 \in \mathcal{F}$, and for generic h near h_0, the moduli space $\mathcal{M}_h$ will be regular and every $[A] \in \mathcal{M}_h$ will be close to some $[A_0] \in \mathcal{M}_{h_0}$. Moreover, for components C of type (i), the restriction $h|_C$ will generically be a Morse function. To see this, consider the bundle E over $\mathcal{F} \times C$ obtained from $TC \to C$ by pullback under $\mathcal{F} \times C \to C$. Define a section $s : \mathcal{F} \times C \to E$ by setting $s(h, [A]) = \mathrm{Grad}_{[A]}(h|_C)$. The abundance condition implies s is a submersion, and thus we have an open set V in $\mathcal{F}$ containing h_0 and a subset $V' \subset V$ of second category such that $h \in V'$ implies $h|_C$ is Morse.

Using such h, we can evaluate the contribution to $\lambda_{SU(3)}(X_1 \# X_2)$ of the critical points in $\mathcal{M}_h$ arising from each component $C \subset \mathcal{M}_{h_0}$. For components of type (i), we apply the following lemma. Although the result is well-known, we include a proof because we could not find one in the literature. This proof will later be generalized to establish Lemma 10, an equivariant version of this result which is new, as far as we know.

Lemma 7. *Suppose $C \subset \mathcal{M}^*_{h_0}$ is a nondegenerate critical submanifold and f is an admissible function with $f|_C$ Morse. Set $h_t = h_0 + tf$ for t small. Then there is an open set $U \subset \mathcal{B}^*$ containing C and an $\epsilon > 0$ such that, for every $0 < t < \epsilon$, $\mathcal{O}_t := \mathcal{M}_{h_t} \cap U$ is a regular subset of $\mathcal{M}_{h_t}$ with a natural bijection $\varphi_t : \mathrm{Crit}(f|_C) \to \mathcal{O}_t$. Given a smooth family of connections A_t with $[A_0] \in \mathrm{Crit}(f|_C)$ and $[A_t] = \varphi_t([A_0])$ for $0 < t < \epsilon$, then*

$$\mathrm{SF}(\theta, A_t) = \mathrm{SF}(\theta, A_0) + \mathrm{ind}_{[A_0]}(f), \tag{3}$$

where $\mathrm{ind}_{[A_0]}(f)$ is the Morse index of the critical point $[A_0]$ with respect to the function $f|_C$.

Proof. We begin by introducing some notation and recalling some basic material from [6] and [1]. Let $\mathcal{J}$ be the trivial bundle over $\mathcal{A} \times \mathcal{F}$ with fiber $\Omega^{0+1}(X; su(3))$. Impose the L^2 pre-Hilbert space structure on the fibers and consider the smooth subbundle $\mathcal{L} \subset \mathcal{J}\ |_{\mathcal{A}^* \times \mathcal{F}}$ whose fiber above (A, h) is

$$\mathcal{L}_{A,h} = \{(\xi, a) \in \mathcal{J}_{A,h} \mid \xi = 0, d_A^* a = 0\}.$$

The bundle $\mathcal{L}$ over $\mathcal{A}^* \times \mathcal{F}$ is $\mathcal{G}$-equivariant and hence descends to give a bundle, also denoted by $\mathcal{L}$, over $\mathcal{B}^* \times \mathcal{F}$, which we regard as the tangent bundle to $\mathcal{B}^*$ with the L^2 metric as opposed to a Sobolev metric.

Recall the operator K_A on $\Omega^{0+1}(X; su(3))$ defined by

$$K_A(\xi, a) = (d_A^* a, d_A \xi - *d_A a).$$

It can be extended to give an operator $K : \mathcal{J} \to \mathcal{J}$ by setting

$$K_{A,h}(\xi, a) = (d_A^* a, d_A \xi - *d_{A,h} a) = \left(d_A^* a, d_A \xi - *d_A a + 4\pi^2 \,\mathrm{Hess}_A\, h(a)\right).$$

Then $K_{A,h}$ is a closed, essentially self-adjoint Fredholm operator with dense domain, depending smoothly on A and h. It has discrete spectrum with no accumulation points, and each eigenvalue has finite multiplicity. If A is h-perturbed flat, then $K_{A,h}$ respects the splitting $\mathcal{J} = \mathcal{L}' \oplus \mathcal{L}$ where $\mathcal{L}' = \Omega^0 \oplus \mathrm{Im}(d_A : \Omega^0 \to \Omega^1)$.

Remark. Note that $K_{A,h}$ as defined here differs from the operator used in [1]. However, the formula for $\lambda_{SU(3)}(X)$ is the same, because changing the sign of $*d_A$ in K is equivalent to changing the orientation of the 3-manifold, and it is proved in [1] that $\lambda_{SU(3)}(-X) = \lambda_{SU(3)}(X)$.

We now introduce a closely related operator on $\mathcal{L}$. Let $\pi_{A,h} : \mathcal{J}_{A,h} \to \mathcal{L}_{A,h}$ be the L^2-orthogonal projection and let $\widehat{K}_{A,h}$ be the operator on

$\mathcal{L}_{A,h}$ obtained by restricting $\pi_{A,h} \circ K_{A,h}$. For paths in $\mathcal{F} \times \mathcal{B}^*$, the spectral flow of $K_{A,h}$ and $\widehat{K}_{A,h}$ are identical.

Let

$$\lambda_0 = \min\{|\lambda| \mid \lambda \neq 0, \lambda \in \operatorname{Spec}(\widehat{K}_{A_0,h_0}) \text{ for } [A_0] \in C\}. \tag{4}$$

Choose open neighborhoods $U \subset \mathcal{B}^*$ of C and $V \subset \mathcal{F}$ of h_0 small enough so that $([A], h) \in U \times V$ implies $\lambda_0/2 \notin \operatorname{Spec}(\widehat{K}_{A,h})$. Over $U \times V$ it is possible to decompose $\mathcal{L}$ into $\mathcal{L}_0 \oplus \mathcal{L}_1$ where

$$\mathcal{L}_0 = \bigoplus_{|\lambda| < \lambda_0/2} E_\lambda \qquad \text{and} \qquad \mathcal{L}_1 = \bigoplus_{|\lambda| > \lambda_0/2} E_\lambda. \tag{5}$$

Here $\lambda \in \operatorname{Spec}(K_{A,h})$ is an eigenvalue and E_λ is its eigenspace.

Let $p_i : \mathcal{L} \to \mathcal{L}_i$ be the projection and choose $\epsilon > 0$ so that $h_t \in V$ for $t \in (-\epsilon, \epsilon)$. For $i = 0, 1$, define

$$\psi_i : U \times (-\epsilon, \epsilon) \to \mathcal{L}_i$$

by setting $\psi_i([A], t) = p_i(\zeta_{h_t}(A))$. (Recall that $\zeta_h(A) = *F_A - 4\pi^2 \operatorname{Grad}_A h$.) A standard argument shows that ψ_1 is a submersion along $C \times \{0\}$ and so, by the Inverse Function Theorem, for U and ϵ small enough, $\psi_1^{-1}(0)$ is a submanifold of $U \times (-\epsilon, \epsilon)$ parameterized by a C^3 function $\Phi : C \times (-\epsilon, \epsilon) \to U \times (-\epsilon, \epsilon)$ of the form $\Phi([A], t) = (\phi_t([A]), t)$, where $\phi_t : C \to U$ is smooth.

Consider part of the parameterized moduli space

$$W = \bigcup_{t \in (-\epsilon, \epsilon)} \mathcal{M}_{h_t} \times \{t\}$$

defined by

$$W_\epsilon = \{([A], t) \mid [A] \in U, \, -\epsilon < t < \epsilon, \, \zeta_{h_t}(A) = 0\}.$$

Then W_ϵ is the image under Φ of the zero set of the map Q from $C \times (-\epsilon, \epsilon)$ to $\mathcal{L}_0$ defined by $Q = \psi_0 \circ \Phi$. This zero set is not cut out transversely since $\mathcal{M}_{h_0}$ is not regular along C. We expand $Q(x, t)$ about $t = 0$ for $x \in C$. For clarity we are using x instead of $[A]$ to denote gauge orbits. Since $x \in C$, $\zeta_{h_0}(x) = 0$ and we have

$$\zeta_{h_t}(\phi_t(x)) = t \operatorname{Hess}_x(CS + h_0) \left(\left. \frac{d\phi_t(x)}{dt} \right|_{t=0} \right) - 4\pi^2 t \operatorname{Grad}_x f + O(t^2).$$

It then follows that

$$\begin{aligned} Q(x,t) &= p_0\left(\zeta_{h_t}(\phi_t(x))\right) \\ &= p_0\left[t\,\mathrm{Hess}_x(CS+h_0)\left(\left.\frac{d\phi_t(x)}{dt}\right|_{t=0}\right) - 4\pi^2 t\,\mathrm{Grad}_x\, f\right] + O(t^2) \\ &= -4\pi^2\, t\, p_0(\mathrm{Grad}_x\, f) + O(t^2). \end{aligned}$$

This last step follows since p_0 is the projection onto the kernel of the Hessian of $CS+h_0$. Thus the function Q/t extends to a C^2 function $\widehat{Q}: C \times (-\epsilon,\epsilon) \longrightarrow \mathcal{L}_0$ defined by

$$\widehat{Q}(x,t) = \begin{cases} Q(x,t)/t & \text{if } t \neq 0 \\ -4\pi^2 p_0(\mathrm{Grad}_x\, f) & \text{otherwise.} \end{cases}$$

Obviously, for $t \neq 0$, the zero set of $\widehat{Q}$ coincides with that of Q. Moreover, the restriction of $\widehat{Q}$ to $C\times\{0\}$ is transverse to the zero section of $\mathcal{L}_0$, since by hypothesis $f|_C$ is a Morse function. Therefore, for ϵ small enough, $\widehat{Q}^{-1}(0)$ is a smooth, 1-dimensional submanifold of $C \times (-\epsilon,\epsilon)$ which intersects $C \times \{0\}$ transversely and

$$\widehat{Q}^{-1}(0) \cap (C \times \{0\}) = \mathrm{Crit}(f|_C).$$

Following this product cobordism gives a natural bijection

$$\varphi_t : \mathrm{Crit}(f|_C) \to \mathcal{O}_t.$$

To prove (3), let $[A_0] \in \mathrm{Crit}(f|_C)$ and denote by A_t a differentiable family of connections representing the path of orbits $\varphi_t([A_0])$. Consider the differentiable family of closed, essentially self-adjoint Fredholm operators $K(t) := \widehat{K}_{A_t,h_t}$. (Here we could equally well work with the path K_{A_t,h_t} of operators on $\mathcal{J}$ since we are only concerned with the behavior of the small eigenvalues.)

The eigenvalues of $K(t)$ of modulus less than λ_0 vary continuously differentiably in t, and their derivatives at $t=0$ are given by the eigenvalues of $\left.\frac{\partial K(t)}{\partial t}\right|_{t=0}$ restricted and projected to $\ker K(0) = \ker \widehat{K}_{A_0,h_0}$ (see Theorem II.5.4 and Section III.6.5 of [4]). However, one can see directly that the restriction of $p_0\left(\left.\frac{\partial K(t)}{\partial t}\right|_{t=0}\right)$ to $\mathcal{L}_0$ agrees with $\mathrm{Hess}_{[A]}(f|_C)$ and this completes the proof. □

The next result applies to components of type (i) and determines their contribution to $\lambda_{SU(3)}(X)$. It is an immediate consequence of Lemma 7.

Proposition 8. *Suppose $C \subset \mathcal{M}^*_{h_0}$ is a nondegenerate critical submanifold and $[A] \in C$. Then the contribution of C to $\lambda_{SU(3)}(X)$ is $(-1)^{\mathrm{SF}(\theta,A)}\chi(C)$.*

Next we develop similar results for components C of type (ii). In this case, since C is not smooth, we work equivariantly on $\widetilde{C}$, which has a natural $SU(3) \cong \mathcal{G}/\mathcal{G}_0$ action. First, we introduce a relevant definition.

Definition 9. *Suppose G is a compact Lie group acting smoothly on a compact manifold Y. Then a smooth G-invariant function $f : Y \to \mathbb{R}$ is called* **equivariantly Morse** *if its critical point set $\mathrm{Crit}(f)$ is a union of orbits isolated in Y/G and along any such orbit the Hessian of f is nondegenerate in the normal directions.*

Note that an equivariantly Morse function is not necessarily Morse, though it is always Bott-Morse.

Let $\widetilde{C}^*$ and $\widetilde{C}^r$ be the preimages of C^* and C^r under the projection $\pi : \widetilde{\mathcal{B}} \to \mathcal{B}$. They determine a stratification $\widetilde{C} = \widetilde{C}^* \cup \widetilde{C}^r$ given by orbit type. We denote by $[\![A]\!] \in \widetilde{\mathcal{B}}$ the $\mathcal{G}_0$ orbit of $A \in \mathcal{A}$. Observe that $\Gamma_A \cong \mathbb{Z}_3$ for $[\![A]\!] \in \widetilde{C}^*$ and $\Gamma_A \cong U(1)$ for $[\![A]\!] \in \widetilde{C}^r$. This latter isomorphism endows $\nu(\widetilde{C}^r)$, the normal bundle of $\widetilde{C}^r$ in $\widetilde{C}$, with a natural $U(1)$ action. Every $h \in \mathcal{F}$ defines an invariant function on $\widetilde{C}$ by restriction. If h is equivariantly Morse and $\tau \subset \widetilde{C}$ is an open, $SU(3)$ invariant tubular neighborhood of $\widetilde{C}^r$, then the induced functions $(\widetilde{C}^* \setminus \tau)/SU(3) \to \mathbb{R}$ and $\widetilde{C}^r/SU(3) \to \mathbb{R}$ obtained by restricting and passing to the quotient are both Morse functions with only finitely many critical points.

We now prove that generic $h \in \mathcal{F}$ induce equivariantly Morse functions on $\widetilde{C}$. This is achieved in two steps. First, let ξ be the bundle over $\mathcal{F} \times \widetilde{C}^r$ obtained by pulling back the bundle $T\widetilde{C}^r \oplus \mathrm{Sym}(\nu)$ under $\mathcal{F} \times \widetilde{C}^r \to \widetilde{C}^r$, where $\mathrm{Sym}(\nu)$ is the bundle of $U(1)$ equivariant symmetric bilinear forms on $\nu(\widetilde{C}^r)$. Define a section $s : \mathcal{F} \times \widetilde{C}^r \longrightarrow \xi$ by setting

$$s(h, [\![A]\!]) = \left(\mathrm{Grad}_{[\![A]\!]}(h|_{\widetilde{C}^r}),\, (\mathrm{Hess}_{[\![A]\!]}\, h)\big|_{\nu(\widetilde{C}^r)}\right).$$

The abundance condition implies that s is a submersion along $\{h_0\} \times \widetilde{C}^r$ (see Proposition 3.4 of [1]).

Hence there is an open set $V \subset \mathcal{F}$ containing h_0 and a subset $V_1 \subset V$ of second category such that $h \in V_1$ implies $h|_{\widetilde{C}^r}$ satisfies Definition 9. It follows that there is an $SU(3)$ invariant neighborhood τ of $\widetilde{C}^r$ in $\widetilde{C}$ and an open neighborhood V_2 of h such that $h' \in V_2$ implies $\mathrm{Crit}(h'|_\tau) \subset \widetilde{C}^r$. Consider the compact subset $C^0 \subset C^*$ obtained by taking the quotient of $\widetilde{C} \setminus \tau'$ under $SU(3)$, where $\tau' \subset \tau$ is some smaller invariant tubular neighborhood. Repeating the argument given just before Lemma 7 with C replaced by C^0 shows that there is a second category subset of $V_3 \subset V_2$ such that $h' \in V_3$ implies that $h'|_{C^0}$ satisfies Definition 9 as well. This shows that $h|_{\widetilde{C}}$ is equivariantly Morse for generic $h \in \mathcal{F}$ near h_0.

Lemma 10. *Suppose $\widetilde{C} \subset \widetilde{\mathcal{M}}_{h_0}$ is a nondegenerate critical submanifold and f is an admissible function such that $f|_{\widetilde{C}}$ is equivariantly Morse. Set $h_t = h_0 + tf$ and let $C \subset \mathcal{M}_{h_0}$ be the image of $\widetilde{C}$ under $\widetilde{\mathcal{M}}_{h_0} \to \mathcal{M}_{h_0}$. Then there is an open set $U \subset \mathcal{B}$ containing C and an $\epsilon > 0$ such that, for every $0 < t < \epsilon$, $\mathcal{O}_t := \mathcal{M}_{h_t} \cap U$ is a regular subset of $\mathcal{M}_{h_t}$. Let $\widetilde{\mathcal{O}}_t$ be the preimage of $\mathcal{O}_t$ under $\pi : \widetilde{\mathcal{B}} \to \mathcal{B}$. There is bijection $\varphi_t : \mathrm{Crit}(f|_C) \to \mathcal{O}_t$ which lifts to an $SU(3)$ equivariant diffeomorphism $\tilde{\varphi}_t : \mathrm{Crit}(f|_{\widetilde{C}}) \to \widetilde{\mathcal{O}}_t$. Given a smooth family A_t with $[\![A_0]\!] \in \mathrm{Crit}(f|_{\widetilde{C}})$ and $[\![A_t]\!] = \tilde{\varphi}_t([\![A_0]\!])$, then for $0 < t < \epsilon$,*

$$\mathrm{SF}(\theta, A_t) = \mathrm{SF}(\theta, A_0) + \mathrm{ind}_{[\![A_0]\!]}(f), \tag{6}$$

where $\mathrm{ind}_{[\![A_0]\!]}(f)$ is the Morse index of the critical point $[\![A_0]\!]$ of $f|_{\widetilde{C}}$. If, in addition, A_0 and A_t are reducible, then (6) holds for the $\mathfrak{h}$ and $\mathfrak{h}^\perp$ components separately:

$$\begin{aligned} \mathrm{SF}_{\mathfrak{h}}(\theta, A_t) &= \mathrm{SF}_{\mathfrak{h}}(\theta, A_0) + \mathrm{ind}^t_{[\![A_0]\!]}(f), \\ \mathrm{SF}_{\mathfrak{h}^\perp}(\theta, A_t) &= \mathrm{SF}_{\mathfrak{h}^\perp}(\theta, A_0) + \mathrm{ind}^n_{[\![A_0]\!]}(f), \end{aligned} \tag{7}$$

where $\mathrm{ind}^t_{[\![A_0]\!]}(f)$ and $\mathrm{ind}^n_{[\![A_0]\!]}(f)$ are the indices of $\mathrm{Hess}_{[\![A_0]\!]}(f|_{\widetilde{C}})$ in the directions tangent and normal to $\widetilde{C}^r$ in $\widetilde{C}$, respectively.

Proof. Since the argument is nearly identical to the proof of Proposition 7, we only explain the modifications one needs to make. The tangent space to the gauge group $\mathcal{G}$ at the identity is given by the space of 0-forms completed in the L^2_2 norm. Therefore, the tangent space to the subgroup $\mathcal{G}_0 \subset \mathcal{G}$ of based gauge transformations is the subspace

$$\Omega^0_0 = \{\xi \in L^2_2(\Omega^0(M; su(3))) \mid \xi(x_0) = 0\}$$

consisting of 0-forms vanishing at the basepoint. Consider the bundle $\widetilde{\mathcal{L}}$ whose fiber above (A, h) is

$$\widetilde{\mathcal{L}}_{A,h} = \{(\xi, a) \in \mathcal{J}_{A,h} \mid \xi = 0, a \perp d_A(\Omega^0_0)\},$$

and denote again by $\widetilde{\mathcal{L}}$ the induced bundle on the quotient $\widetilde{\mathcal{B}}\times\mathcal{F}$. Notice that the fiber $\widetilde{\mathcal{L}}_{A,h}$ contains $\ker d_A^*$ as a subspace of codimension $8-\dim\Gamma_A$. We regard $\widetilde{\mathcal{L}}_{A,h}$ as the tangent space of $\widetilde{\mathcal{B}}$ at $[\![A]\!]$.

By restricting and projecting $K_{A,h}$, we obtain an operator $\widetilde{K}$ on $\widetilde{\mathcal{L}}$. This operator agrees with $K_{A,h}$ on $\ker d_A^*$ and vanishes on the orthogonal complement to $\ker d_A^*$ in $\widetilde{\mathcal{L}}_{A,h}$, which is just the tangent space to the orbit of the residual $SU(n)$ action. Choose open subsets $\widetilde{U}\subset\widetilde{\mathcal{B}}$ containing $\widetilde{C}$ and $V\subset\mathcal{F}$ containing h_0 and define λ_0 as in (4), with $\widehat{K}$ replaced by $\widetilde{K}$. Over $\widetilde{U}\times V$, decompose $\widetilde{\mathcal{L}}=\widetilde{\mathcal{L}}_0\oplus\widetilde{\mathcal{L}}_1$ into the two eigenbundles as in (5). Let $\widetilde{Q}:\widetilde{C}\times(-\epsilon,\epsilon)\to\widetilde{\mathcal{L}}_0$ be the analog of the map Q from before.

The only substantial difference is that now $f|_{\widetilde{C}}$ is not Morse but rather equivariantly Morse. This implies that f induces Morse functions on C^* and C^r with only finitely many critical points. The argument from Proposition 7 which produced the map $\widehat{Q}$ on $C\times(-\epsilon,\epsilon)$ can also be applied here and results in equivariant maps $\widetilde{C}^*\times(-\epsilon,\epsilon)\to\widetilde{\mathcal{L}}_0$ and $\widetilde{C}^r\times(-\epsilon,\epsilon)\to\widetilde{\mathcal{L}}_0$ whose zero sets together coincide with that of $\widetilde{Q}$. Reducing modulo $SU(n)$, we obtain 1-dimensional (product) cobordisms in $\mathcal{B}^*$ and $\mathcal{B}^r$ which we follow to define the map φ_t. The preimages of the cobordisms under $\pi:\widetilde{\mathcal{B}}\to\mathcal{B}$ are equivariant product cobordisms in $\widetilde{\mathcal{B}}$. Nondegeneracy of $\operatorname{Hess} f$ in the normal direction to $\widetilde{C}^r$ guarantees that there are no irreducible orbits in $\widetilde{Q}^{-1}(0)$ nearby, and the claims about the spectral flow follow as in the previous case. □

The following proposition applies to components of type (ii) and determines their contribution to $\lambda_{SU(3)}(X_1\#X_2)$.

Proposition 11. *Suppose h_0 is a small perturbation and $C\subset\mathcal{M}_{h_0}$ is a connected component satisfying the following conditions:*

- (i) *For each $[A]\in C$, the isotropy group Γ_A is isomorphic to either $\mathbb{Z}_3$ or $U(1)$.*
- (ii) *The lift $\widetilde{C}$ of C under the projection $\pi:\widetilde{\mathcal{B}}\to\mathcal{B}$ is a nondegenerate critical submanifold.*
- (iii) *Both C^* and C^r are connected.*

Choose connections A_0,B_0 with $[A_0]\in C^*$ and $[B_0]\in C^r$. Then the contribution of C to $\lambda_{SU(3)}(X)$ is

$$\begin{aligned}(-1)^{\mathrm{SF}(\theta,A_0)}&\chi(C,C^r)\\ &-\tfrac{1}{2}(-1)^{\mathrm{SF}(\theta,B_0)}\chi(C^r)\left(\mathrm{SF}_{\mathfrak{h}^\perp}(\theta,B_0)-4\,CS(\widehat{B}_0)+2\right)\end{aligned}\tag{8}$$

where $\widehat{B}_0$ is a flat, reducible connection close to B_0.

Remark. We do not assume X is a connected sum in either Proposition 8 or 11 as there may be other interesting applications of these results, e.g., to components of the flat moduli space of positive dimension. Condition (iii) holds for components C arising from connected sums but is not an essential hypothesis. For example, if C^r is not connected, then decompose it into its connected components

$$C^r = \bigcup_{i=1}^m C_i^r$$

and choose $B_i \in \mathcal{A}$ with $[B_i] \in C_i^r$ for $i = 1, \ldots, m$. Then the correct statement is obtained by replacing (8) by

$$(-1)^{\mathrm{SF}(\theta,A_0)}\chi(C, C^r) \\ -\tfrac{1}{2}\sum_{i=1}^m (-1)^{\mathrm{SF}(\theta,B_i)}\chi(C_i^r)\left(\mathrm{SF}_{\mathfrak{h}^\perp}(\theta, B_i) - 4\,CS(\widehat{B}_i) + 2\right).$$

Proof. We first show that (8) is independent of the choices of A_0, B_0 and $\widehat{B}_0$. The argument of Theorem 5.1 of [1] shows that (8) depends only on the gauge orbits $[A_0], [B_0] \in C$ and not on their gauge representatives. That argument also shows that (8) is independent of the choice of $\widehat{B}_0$. So, it suffices to show that (8) is independent of the choice of $[A_0] \in C^*$ and $[B_0] \in C^r$.

The Lie group $SU(3)$ acts smoothly on $\widetilde{C}$, and hence Corollary VI.2.5 of [2] implies $\widetilde{C}^r$ is a smooth submanifold of $\widetilde{C}$. Since $PU(3) = SU(3)/\mathbb{Z}_3$ acts freely on $\widetilde{C}^*$, the quotient C^* is also smooth. Thus the dimension of the kernel of $\mathrm{Hess}_A(CS + h_0)$ is constant as a function of $[A] \in C^*$ (the tangent space of C^* at $[A]$ can be identified with the space of zero modes of the Hessian). The same is true of the signature operator

$$K_A : \Omega^{0+1}(X, su(3)) \longrightarrow \Omega^{0+1}(X, su(3)),$$

since it is just the Hessian enlarged by putting $d_A : \Omega^0(X, su(3)) \to \Omega^1(X, su(3))$ and its adjoint $d_A^* : \Omega^1(X, su(3)) \to \Omega^0(X, su(3))$ in opposite off-diagonal blocks.

Given $[A_0], [A_0'] \in C^*$, there is by (iii) a path in C^* from $[A_0]$ to $[A_0']$ which we lift to a path A_t of irreducible connections from A_0 to $A_1 = g \cdot A_0'$, where $g \in \mathcal{G}$. Since none of the eigenvalues of K_A cross zero along A_t, it follows that $\mathrm{SF}(\theta, A_0) = \mathrm{SF}(\theta, A_1)$. This proves (8) is independent of the choice of $[A_0] \in C^*$.

To prove (8) is independent of the choice of $[B_0] \in C^r$, choose a lift $[\![B_0]\!] \in \widetilde{C}^r$ of $[B_0]$ and decompose the tangent space of $\widetilde{C}$ at $[\![B_0]\!]$ into the subspaces of vectors tangent to $\widetilde{C}^r$ and vectors normal to $\widetilde{C}^r$ in $\widetilde{C}$. Now C^r connected implies $\widetilde{C}^r$ is connected, and hence the dimension of the kernel of $\mathrm{Hess}_B(CS + h_0)$ is constant as a function of $[\![B]\!] \in \widetilde{C}^r$. The same is true for the restriction

$$\mathrm{Hess}_B(CS + h_0)|_{\Omega^1(X;\mathfrak{h}^\perp)}$$

because its kernel can be identified with the normal bundle of $\widetilde{C}^r$ in $\widetilde{C}$. Similar statements hold for the signature operator K_B and its restriction $K_B|_{\Omega^{0+1}(X;\mathfrak{h}^\perp)}$ (notice that $H^0_B(X; su(3)) = \mathbb{R}$ and $H^0_B(X; \mathfrak{h}^\perp) = 0$ for $[\![B]\!] \in \widetilde{C}^r$).

Given $[B_0], [B_0'] \in C^r$, there is a path in C^r from $[B_0]$ to $[B_0']$ which we lift to a path B_t of reducible connections from B_0 to $B_1 = g \cdot B_0'$. Since none of the eigenvalues of K_B or its restriction $K_B|_{\Omega^{0+1}(X;\mathfrak{h}^\perp)}$ cross zero along B_t, it follows that $\mathrm{SF}(\theta, B_0) = \mathrm{SF}(\theta, B_1)$ and $\mathrm{SF}_{\mathfrak{h}^\perp}(\theta, B_0) = \mathrm{SF}_{\mathfrak{h}^\perp}(\theta, B_1)$. This proves that (8) is independent of the choice of $[B] \in C^r$.

To compute the contribution of C to $\lambda_{SU(3)}(X)$, we choose an admissible function f so that $f|_{\widetilde{C}}$ is equivariantly Morse and consider the parameterized moduli space

$$W = \bigcup_{0 \leq t \leq t_0} \mathcal{M}_{h_t} \times \{t\}$$

for the 1-parameter family of perturbations $h_t = h_0 + tf$. For t_0 small, W is a union of connected components corresponding to the connected components of h_0. Let U be the component of W containing $C \times \{0\}$, and let U_t denote the "t-slice" $U \cap (\mathcal{M}_{h_t} \times \{t\})$.

Then, by definition, the contribution of C to $\lambda_{SU(3)}(X)$ is the sum

$$\sum_{[A] \in U^*_{t_0}} (-1)^{\mathrm{SF}(\theta, A)} - \tfrac{1}{2} \sum_{[B] \in U^r_{t_0}} (-1)^{\mathrm{SF}(\theta, B)} (\mathrm{SF}_{\mathfrak{h}^\perp}(\theta, B) - 4\, CS(\widehat{B}) + 2) \tag{9}$$

where t_0 is a small positive number and $U_t = U_t^* \cup U_t^r$ is the decomposition into irreducible and reducible gauge orbits.

From equation (6) of Lemma 10, it follows that

$$\begin{aligned}\sum_{[A]\in U^*_{t_0}} (-1)^{\mathrm{SF}(\theta,A)} &= \sum_{[A]\in \mathrm{Crit}(f|_{C^*})} (-1)^{\mathrm{SF}(\theta,A)}(-1)^{\mathrm{ind}_{[\![A]\!]}(f)} \\ &= (-1)^{SF(\theta,A_0)} \sum_{[A]\in \mathrm{Crit}(f|_{C^*})} (-1)^{\mathrm{ind}_{[A]}(f)}.\end{aligned} \tag{10}$$

This uses the previously established fact that $(-1)^{\mathrm{SF}(\theta,A)} = (-1)^{\mathrm{SF}(\theta,A_0)}$ for all $[A] \in C^*$, together with the observation that the Morse index of f at $[\![A]\!] \in \widetilde{C}^*$ equals that of the induced function f on C^* at $[A]$.

Similarly, from equation (7) of Lemma 10, it follows that

$$\begin{aligned}&\sum_{[B]\in U^r_{t_0}} (-1)^{\mathrm{SF}(\theta,B)}(\mathrm{SF}_{\mathfrak{h}^\perp}(\theta,B) - 4\,CS(\widehat{B}) + 2) \\ &\quad= \sum_{[B]\in \mathrm{Crit}(f|_{C^r})} (-1)^{\mathrm{SF}(\theta,B)}(-1)^{\mathrm{ind}_{[\![B]\!]}(f)} \Big(\mathrm{ind}^n_{[\![B]\!]}(f) \\ &\qquad\qquad + \mathrm{SF}_{\mathfrak{h}^\perp}(\theta,B) - 4\,CS(\widehat{B}) + 2\Big) \\ &\quad= \sum_{[B]\in \mathrm{Crit}(f|_{C^r})} (-1)^{\mathrm{SF}(\theta,B_0)}(-1)^{\mathrm{ind}^t_{[\![B]\!]}(f)} \Big(\mathrm{ind}^n_{[\![B]\!]}(f) \\ &\qquad\qquad + \mathrm{SF}_{\mathfrak{h}^\perp}(\theta,B_0) - 4\,CS(\widehat{B}_0) + 2\Big) \\ &\quad= (-1)^{\mathrm{SF}(\theta,B_0)}(\mathrm{SF}_{\mathfrak{h}^\perp}(\theta,B_0) - 4\,CS(\widehat{B}_0) + 2) \sum_{[B]\in \mathrm{Crit}(f|_{C^r})} (-1)^{\mathrm{ind}^t_{[\![B]\!]}(f)} \\ &\qquad - (-1)^{\mathrm{SF}(\theta,A_0)} \sum_{[B]\in \mathrm{Crit}(f|_{C^r})} (-1)^{\mathrm{ind}^t_{[\![B]\!]}(f)} (\mathrm{ind}^n_{[\![B]\!]}(f)).\end{aligned} \tag{11}$$

The second step follows since $(-1)^{SF(\theta,B)}$ and $\mathrm{SF}_{\mathfrak{h}^\perp}(\theta,B) - 4\,CS(\widehat{B})$ are independent of $[B] \in C^r$ and since $\mathrm{ind}^n_{[\![B]\!]}(f)$ is even. The last step is justified by the following lemma.

Lemma 12. *For all* $[A] \in C^*$ *and all* $[B] \in C^r$, $(-1)^{\mathrm{SF}(\theta,B)} = (-1)^{\mathrm{SF}(\theta,A)+1}$.

Proof. To prove the lemma, suppose β_t is a 1-parameter family in $\mathcal{A}$ with $[\beta_0] \in C^r$ and $[\beta_t] \in C^*$ for $t > 0$. Then

$$\dim H^0_{\beta_1}(X; su(3)) = \dim H^0_{\beta_0}(X; su(3)) - 1, \text{ and}$$
$$\dim H^1_{\beta_1,h_0}(X; su(3)) = \dim H^1_{\beta_0,h_0}(X; su(3)) - 1.$$

Indeed, as t increases from $t = 0$, a pair of eigenvalues of $K_{\beta_t,h}$ of equal magnitude and opposite sign leave zero. This proves that $\mathrm{SF}(\theta, \beta_0) = \mathrm{SF}(\theta, \beta_1) - 1$. It also proves the claim since, as we have already seen, $(-1)^{\mathrm{SF}(\theta,B)}$ is independent of $[B] \in C^r$ and $(-1)^{\mathrm{SF}(\theta,A)}$ is independent of $[A] \in C^*$. ☐

We now complete the proof of Proposition 11. Substituting equations (10) and (11) into (9), we see that the contribution of C to $\lambda_{SU(3)}(X)$ is given by

$$(-1)^{SF(\theta,A_0)}\Big[\sum_{[A]\in \mathrm{Crit}(f|_{C^*})} (-1)^{\mathrm{ind}_{[A]}(f)} + \tfrac{1}{2} \sum_{[B]\in \mathrm{Crit}(f|_{C^r})} (-1)^{\mathrm{ind}^t_{[\![B]\!]}(f)} (\mathrm{ind}^n_{[\![B]\!]}(f))\Big] \tag{12}$$
$$-\tfrac{1}{2}(SF_{\mathfrak{h}^\perp}(\theta, B_0) - 4\, CS(\widehat{B}_0) + 2) \cdot (-1)^{SF(\theta,B_0)} \sum_{[B]\in \mathrm{Crit}(f|_{C^r})} (-1)^{\mathrm{ind}^t_{[\![B]\!]}(f)}$$

Notice that quantity in brackets on the first line of (12) is independent of the equivariantly Morse function f on $\widetilde{C}$. (This follows from an argument similar to but simpler than that given in [1] to show that $\lambda_{SU(3)}$ is independent of perturbation.) Hence we can compute it using any equivariantly Morse function we want. Choosing a function whose Hessian in the normal directions to $\widetilde{C}^r$ is positive definite and whose critical values along $\widetilde{C}^*$ are all larger than the values along $\widetilde{C}^r$, we see that the quantity in brackets on the first line of (12) equals the relative Euler characteristic $\chi(C, C^r)$. A standard argument shows that

$$\sum_{[B]\in \mathrm{Crit}(f|_{C^r})} (-1)^{\mathrm{ind}^t_{[\![B]\!]}(f)} = \chi(C^r).$$

This proves (12) equals (8) and we are done. ☐

We can now complete the proof of Theorem 1. As explained earlier, the point components in $\mathcal{M}_{h_0}$ give rise to the first two terms on the right of formula (1). Further, if C is a connected component of $\mathcal{M}_{h_0}$ of type (i), then it contributes algebraically zero to $\lambda_{SU(3)}(X)$. This follows from Proposition 8 since $\chi(C) = 0$ for such C. (See Proposition 6. In the case $C \cong N$, this follows simply because N is an orientable manifold of odd dimension.)

It remains to determine the contribution to $\lambda_{SU(3)}(X_1 \# X_2)$ of components C of type (ii). Our first step will be to calculate the relative Euler characteristic $\chi(C, C^r)$. By the exactness property of singular homology,

$$\chi(C, C^r) = \chi(C) - \chi(C^r) = \chi(C),$$

where the last step follows from the fact that $C^r \cong SO(3)$, which is well-known in $SU(2)$ gauge theory. (See p.134, [5].) Our computation of $\chi(C)$ utilizes the following description of C as the quotient of a certain $U(1)$ action on N.

Recall that $N = SU(3)/U(1)$ is our model for fibers of $\widetilde{\mathcal{B}} \to \mathcal{B}$ above reducible orbits. In terms of a reducible $SU(3)$ representation ϱ of $\pi_1(X)$, N is just the adjoint orbit of ϱ, namely points in N correspond to $SU(3)$ representations conjugate to ϱ. Because these representations are all reducible, associated to each point in N there is a canonical 1-dimensional subspace of $\mathbb{C}^3$ given by the invariant linear subspace of the corresponding representation. This defines a map $N \to \mathbb{CP}^2$ which is, in fact a fibration. The fiber above $[0,0,1] \in \mathbb{CP}^2$ consists of $SU(3)$ representations ϑ conjugate to ϱ with $\operatorname{im}(\vartheta) \subset SU(2) \times 1$. The two irreducible $SU(2)$ representations ϱ' and ϑ' associated to ϱ and ϑ are conjugate, and hence the fiber of $N \to \mathbb{CP}^2$ is $SO(3)$, the adjoint $SU(2)$ orbit of ϱ'.

In general, define

$$\Gamma_\varrho = \{g \in SU(3) \mid g\varrho g^{-1} = \varrho\}$$

and recall that ϱ is reducible and nontrivial if and only if $\Gamma_\varrho \cong U(1)$. Suppose ϱ_1 and ϱ_2 are nontrivial reducible $SU(3)$ representations of X_1 and X_2, respectively. Then C consists of the conjugacy classes of representations ϱ of $X_1 \# X_2$ such that the restriction of ϱ to $\pi_1(X_i)$ is conjugate to ϱ_i for $i = 1, 2$. Proposition 6 shows that

$$\widetilde{C} = SU(3)/\Gamma_{\varrho_1} \times SU(3)/\Gamma_{\varrho_2} \cong N \times N,$$

and by fixing the first factor, it follows that C is the quotient of the second factor by the induced action of $\Gamma_{\varrho_1} \cong U(1)$. If ϱ_1 is chosen with image contained in $SU(2) \times 1$, then Γ_{ϱ_1} is simply the $U(1)$ subgroup described in (2). The subgroup of this group consisting of cube roots of 1 acts trivially.

The $U(1)$ action descends to the base of the fibration $\pi : N \to \mathbb{CP}^2$, where it acts by $[x, y, z] \mapsto [ux, uy, u^{-2}z]$ and has fixed point set $\{[0, 0, 1]\} \cup \{[x, y, 0]\} = \{pt\} \cup \mathbb{CP}^1$. Notice that $\pi^{-1}([0, 0, 1]) = C^r$ and set $B_1 = \mathbb{CP}^2 \setminus \{[0, 0, 1]\}$ and $B_2 = \mathbb{CP}^2 \setminus \mathbb{CP}^1$. Define $C_i = \pi^{-1}(B_i)/U(1)$ and observe that $C = C_1 \cup C_2$ and $C^r \subset C_2$. The Mayer-Vietoris sequence gives that

$$\chi(C) = \chi(C_1) + \chi(C_2) - \chi(C_1 \cap C_2).$$

However, $U(1)/\mathbb{Z}_3$ acts freely on $B_2 \setminus \{[0, 0, 1]\} \cong \mathbb{C}^2 \setminus \{0\}$ and trivially on the fiber above $[0, 0, 1]$, and hence C_2 is an $SO(3)$ bundle over $B_2/U(1)$. Thus, $\chi(C_2) = 0$. Similarly, $\chi(C_1 \cap C_2) = 0$.

Now B_1 certainly retracts to $\mathbb{CP}^1$, and we claim that C_1 also retracts to $C_0 = \pi^{-1}(\mathbb{CP}^1)/U(1)$. This follows by considering the $U(1)$ action on the fibers above $[x, y, 0] \in \mathbb{CP}^1$. For example, take $p = [1, 0, 0]$, the north pole. If $\varrho_2 \in \pi^{-1}(p)$, then $\operatorname{im}(\varrho_2) \subset H$ where

$$H = \left\{ \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -\bar{b} & \bar{a} \end{array} \right) \middle| \, a\bar{a} + b\bar{b} = 1. \right\}.$$

In this case, $U(1)$ acts on $\pi^{-1}(p) = SO(3)$ in the standard way by rotation of the off-diagonal entries and has quotient S^2. Hence $\pi^{-1}(p)/U(1) \cong S^2$ and nearby, the $SO(3)$ fibers in C^* retract to the S^2 fibers in $\pi^{-1}(\mathbb{CP}^1)/U(1)$ via the cone structure.

The same is true for $[x, y, 0] \in \mathbb{CP}^1$. For suppose x and y are complex numbers satisfying $x\bar{x} + y\bar{y} = 1$ and suppose that

$$\alpha = \left(\begin{array}{ccc} x & -\bar{y} & 0 \\ y & \bar{x} & 0 \\ 0 & 0 & 1 \end{array} \right).$$

Then $\alpha(p) = [x, y, 0]$ and $\operatorname{im}(\varrho_2) \subset \alpha H \alpha^{-1}$ whenever ϱ_2 is a reducible $SU(3)$ representation with invariant linear subspace $[x, y, 0]$. Since the $U(1)$ action commutes with multiplication by α, it acts on $\alpha H \alpha^{-1}$ in the same way as it did on H. Thus $\pi^{-1}([x, y, 0])/U(1) \cong S^2$ and there is a fibration $C_0 \to \mathbb{CP}^1$ with fiber S^2. Hence $\chi(C_0) = 4$ and we conclude that $\chi(C, C^r) = 4$.

Since $\chi(C^r) = 0$ for components of type (ii), all the terms involving B in Proposition 11 vanish and it follows that each such component contributes $(-1)^{\mathrm{SF}(\theta,A)}\chi(C,C^r)$ to $\lambda_{SU(3)}(X_1\#X_2)$, where $A=A_1\#_\sigma A_2$ is chosen so that $[A]\in C^*$. In order to compute $\mathrm{SF}(\theta,A) \mod 2$, it is convenient to set $B=A_1\#_\tau A_2$ with $[B]\in C^r$. Since B is reducible, $\dim H^0_B(X;su(3))=1$ and we compute that $\dim H^1_B(X;su(3))=7$ (this uses the splitting $su(3)=su(2)\oplus\mathbb{C}^2\oplus\mathbb{R}$ together with the facts: $\dim H^1_B(X;su(2))=\dim C^r=3$, $H^1_B(X;\mathbb{R})=0$, and $H^1_B(X;\mathbb{C}^2)=4$). On the other hand, if $[A]\in C^*$, then $H^0_A(X;su(3))=0$ and $\dim H^1_A(X;su(3))=\dim C^*=6$. By Lemma 12,

$$\mathrm{SF}(\theta,A)\equiv \mathrm{SF}(\theta,B)-1 \mod 2. \tag{13}$$

Using the splitting $su(3)=su(2)\oplus\mathbb{C}^2\oplus\mathbb{R}$, and applying equation (1) to the $su(2)$ component of $\mathrm{SF}(\theta,B)$, we see that

$$\begin{aligned}\mathrm{SF}(\theta,B)&=\mathrm{SF}(\theta,A_1\#_\tau A_2)\\&=\mathrm{SF}_{su(2)}(\theta,A_1\#_\tau A_2)+\mathrm{SF}_{\mathbb{C}^2}(\theta,A_1\#_\tau A_2)\\&+\mathrm{SF}_{\mathbb{R}}(\theta,A_1\#_\tau A_2)\\&\equiv \mathrm{SF}_{su(2)}(\theta_1,A_1)+\mathrm{SF}_{su(2)}(\theta_2,A_2)-1 \mod 2\\&\equiv \mathrm{SF}(\theta_1,A_1)+\mathrm{SF}_{su(2)}(\theta_2,A_2)-1 \mod 2.\end{aligned} \tag{14}$$

The third and fourth steps follow because all the $\mathbb{C}^2$ spectral flows are even and all the $\mathbb{R}$ spectral flows equal -1 (the coefficients are untwisted, X is a homology sphere, and we use the $(-\epsilon,\epsilon)$ convention for computing spectral flows). Combining equations (13) and (14), we conclude that

$$\mathrm{SF}(\theta,A)\equiv\mathrm{SF}(\theta,B)-1\equiv\mathrm{SF}(\theta_1,A_1)+\mathrm{SF}(\theta_2,A_2) \mod 2.$$

This, together with the above computation of $\chi(C,C^r)$, implies

$$\begin{aligned}\sum(-1)^{\mathrm{SF}(\theta,A)}\chi(C,C^r) &= 4\sum_{[A_1]\in\mathcal{M}^r_{h_1}(X_1)}(-1)^{\mathrm{SF}(\theta_1,A_1)}\sum_{[A_2]\in\mathcal{M}^r_{h_2}(X_2)}(-1)^{\mathrm{SF}(\theta_2,A_2)}\\&= 4\,\lambda_{SU(2)}(X_1)\,\lambda_{SU(2)}(X_2),\end{aligned}$$

where the first sum is over all components $C\subset\mathcal{M}_{h_0}$ of type (ii) and $[A]\in C^*$. Recall that $h_0=h_1+h_2$ and $\mathcal{M}_{h_i}(X_i)$ is regular for $i=1,2$. This completes the proof of Theorem 1.

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