Representations of Orbifold Groups and Parabolic Bundles

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Contents

1 Introduction

Let X be a compact holomorphic orbifold of dimension 2. Such orbifolds are topologically classified by their genus and a finite collection of integers giving the cone angles at the cone points in X. By a smoothing process which replaces singular neighborhoods of the cone points with holomorphic disks, we obtain a Riemann surface X_s with a collection of distinguished points (called parabolic points). Let $E \to X$ be a holomorphic orbifold bundle. By the push forward construction (a smoothing process on the level of bundles), we obtain a holomorphic bundle $\mathcal{E} \to X_s$ with parabolic structure, i.e. a weighted (partial) flag in the fiber $\mathcal{E}_p$ over each parabolic point p. In this thesis, we establish that the bundle $\mathcal{E}$ is parabolic stable if and only if there is a unitary connection A on E with constant central curvature (Theorem 5.1). In particular, E is projectively flat. Thus, we get a description of the space of projective unitary representations of the orbifold group as stable parabolic bundles and use it to compute the cohomology of the SU(2)-representation space of any Seifert-fibered homology sphere.

In order to put this result into context, let us consider for a moment the case of holomorphic bundles (without parabolic structure) over a Riemann surface X. The big picture includes the three moduli:

- $\mathcal{SS}$ = the moduli of semistable holomorphic structures on E,
- $\mathcal{R}$ = certain[1] PU(n)-representations of $\pi_1 X$,
- $\mathcal{M}$ = the moduli of Yang-Mills[2] connections on E.

Each of these spaces is a quotient space; in order to avoid singularities and non-Hausdorff behavior, we consider the subspaces:

- $\mathcal{S} \subseteq \mathcal{SS}$ of stable holomorphic structures,
- $\mathcal{R}^* \subseteq \mathcal{R}$ irreducible representations,
- $\mathcal{M}^* \subseteq \mathcal{M}$ of Yang-Mills minima.

In [15], Narasimhan and Seshadri prove that $\mathcal{SS} \approx \mathcal{R}$, with $\mathcal{S} \approx \mathcal{R}^*$. In [4], Donaldson gives a gauge theoretic proof of the result of Narasimhan and Seshadri by showing $\mathcal{M}^* \approx \mathcal{S}$. Atiyah and Bott, in [1], give an inductive procedure based on the stratification of $\mathcal{C}$, the space of all holomorphic structures, to compute $\mathrm{H}^*(\mathcal{S})$ in the case where $\mathcal{SS} = \mathcal{S}$.

The three moduli have counterparts in world of orbifolds and parabolic bundles. Namely, given a holomorphic orbifold bundle $E \to X$ with push forward $\mathcal{E} \to X_s$, we have

- $\mathcal{SS}$ = the moduli of semistable holomorphic parabolic structures on $\mathcal{E}$,
- $\mathcal{R}$ = certain PU(n)-representations of $\pi_1^{orb} X$,
- $\mathcal{M}$ = the moduli of Yang-Mills orbifold connections on E.

with analogous subspaces $\mathcal{S} \subseteq \mathcal{SS}$, $\mathcal{R}^* \subseteq \mathcal{R}$, and $\mathcal{M}^* \subseteq \mathcal{M}$. Mehta and Seshadri [14] prove $\mathcal{SS} \approx \mathcal{R}$ (with $\mathcal{S} \approx \mathcal{R}^*$) for genus $g \geq 2$ and one parabolic point. Moreover, the Atiyah-Bott program is extended in [17] to parabolic bundles. In this thesis we

[1]or equivalently, representations of π_1 of the once-punctured surface with prescribed holonomy.
[2]These are connections which are critical points for the Yang-Mills functional.

give another proof of the result of Mehta and Seshadri, i.e. we show that $\mathcal{S} \approx \mathcal{M}^*$ for arbitrary genus. The approach used is essentially Donaldson's [4], adapted to orbifolds. Consequently, we have $\mathcal{S} \approx \mathcal{R}^*$ for an appropriately defined[3] representation space. This, along with the Atiyah-Bott program for parabolic bundles, allows for the cohomology of the representation space of certain Fuchsian groups (orbifold fundamental groups).

Lately, the work of Casson and Floer has stimulated interest in the theory of SU(2)-representation space of $\pi_1(\Sigma^3)$, where Σ^3 is a homology 3-sphere. Let $\mathcal{R}(\Sigma)$ denote the representations modulo conjugation. If, in addition, Σ is Seifert-fibered, then there is a canonical orbifold X so that $\mathcal{R}(\Sigma) \approx \mathcal{R}(X)$. Thus, the above program gives a method for computing the cohomology of $\mathcal{R}(\Sigma)$. In carrying this out, we find that $\mathrm{H}^i(\mathcal{R}(\Sigma)) = 0$ for i odd. This is not suprising in light of the conjecture of Fintushel and Stern[6], proved by Kirk and Klassen [11] (see also [3] and [7]). In both [3] and [7], it is proved that $\mathcal{R}(\Sigma)$ is a rational variety and therefore simply connected. We have tried to find a simple topological proof of the fact $\pi_1\mathcal{R}(\Sigma) = 0$, but the usual techniques (i.e. Newstead's [16]) fail.

Having completed this work, we learned of the work of Furuta and Steer [7] giving the same results by similar methods. In this thesis, we extend the results to compute the cohomology of representation spaces of Seifert fibrations which are torsion free (arbitrary genus). In particular, we have complete results for genus 1 and partial results for genus ≥ 2. This includes simple connectivity of all but one component of $\mathcal{R}(\Sigma)$. This one component is diffeomorphic to the SU(2)-representation space of a surface of genus g and is singular for g > 2. Andrew Nicas pointed out to me that one can use Kirwan's explicit formulas (§4 and 5 of [12]) to find the intersection Betti numbers of this component (up to 2-torsion). In a future article, we hope to address the problem of higher rank bundles (i.e. $U(n)$ and $SU(n)$ representations). The presence of reducibles, reflected by the fact that $\mathcal{SS} \neq \mathcal{S}$, is the main obstacle to this program. Kirwan's theory appears to be the best hope for dealing with these issues.

We introduce the notion of orbifolds and orbifold bundles in §2. The category of parabolic bundles is introduced in §3, where we also define stability (Definition 3.9) and obtain a result (Proposition 3.8) which we will need in §5. In §4, we establish an equivalence between the categories of holomorphic orbifold bundles and parabolic bundles (Propositions 4.1 & 4.4). We also prove the technical result (Proposition 4.5) which is needed for Theorem 5.1, our main result. §5 contains the proof of this result and establishes the relationship between representations and semistable parabolic bundles. In §6, we give, as an application, the computation of the cohomology of $\mathcal{S}$ in the rank 2 case and describe its relationship to $\mathcal{R}(\Sigma)$ for Seifert-fibered spaces Σ. We close this section with some explicit calculations where $\mathcal{S}$ (a component of $\mathcal{R}$) is of dimensions four and six.

[3] i.e. representations of π_1^{orb} of the once-punctured orbifold with prescribed holonomy.

2 Orbifolds

In this section, we briefly define holomorphic orbifolds, classifying (topologically) those of dimension 2. We also describe the orbifold fundamental group π_1^{orb}. We give a presentation for this group in case the orbifold has dimension 2. We then turn attention to orbifold bundles and develop the complex differential geometry which we shall use throughout our thesis. We end with a description of the second fundamental form for a short exact sequence of holomorphic orbifold bundles.

We now define orbifolds (also called V-manifolds), using the notion of a *local uniformizing system*, which we abbreviate l.u.s. Before we get into the formalities, intuitively, an orbifold is locally modelled on an open set in $\mathbf{C}^n$ modulo a finite group. Of course, saying what happens on the overlaps is the tricky part.

Definition 2.1 *A connected metric space X is a holomorphic orbifold if*
(a) *For a base of open sets $U \subset X$, we have a local uniformizing system, i.e. triples $\{\tilde{U}, \Gamma, \phi\}$ where*

1. *$\tilde{U}$ is a connected open subset of $\mathbf{C}^n$,*
2. *Γ is a finite set of biholomorphic bijections of $\tilde{U}$,*
3. *$\phi : \tilde{U} \to U$ is Γ-invariant and induces a homeomorphism $\tilde{U}/\Gamma \overset{\phi}{\approx} U$.*

(b) *If $U \subseteq U'$, then we have an injection, which is a pair $\{\lambda, \psi\}$ so that*

1. *$\lambda : \Gamma \to \Gamma'$ is a monomorphism,*
2. *$\psi : \tilde{U} \hookrightarrow \tilde{U}'$ is a holomorphic embedding such that the diagram*

$$\begin{array}{ccc} \tilde{U} & \overset{\psi}{\to} & \tilde{U}' \\ \downarrow\gamma' & & \downarrow\gamma \\ \tilde{U} & \overset{\psi}{\to} & \tilde{U}' \\ \downarrow\phi' & & \downarrow\phi \\ U & \subseteq & U' \end{array}$$

commutes for all $\gamma \in \Gamma$, where $\gamma' = \lambda(\gamma)$.

We call the collection of l.u.s. and corresponding injections a *defining family* $\mathcal{F}$, and, as usual, consider two families $\mathcal{F}$ and $\mathcal{F}'$ equivalent if $\mathcal{F} \cup \mathcal{F}'$ is a defining family (i.e. satisfies **b**). We shall be mainly concerned with holomorphic orbifolds of dimension 2, which are (topologically) classified by a finite list (see [19] for a list of all 2-dimensional orbifolds). This follows because any finite subgroup of U(1) is a cyclic group $\mathbf{Z}_a$. So, any singular point $c \in X$ has an l.u.s. of the form $D^2/\mathbf{Z}_a$ where $\mathbf{Z}_a$ is the standard action on D^2 (i.e. multiplication by an a^{th} root of unity). In this case, the cone point c has cone angle $2\pi/a$. So compact holomorphic orbifolds X are topologically classified by their genus g and a finite collection of integers $(a_1, \ldots, a_n)$ giving the cone angles at the cone points $(c_1, \ldots, c_n)$. We use $X(g; a_1, \ldots, a_n)$ to

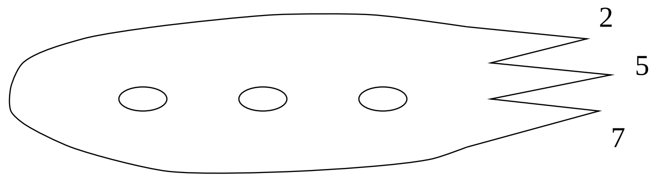

Figure 1: $X(3; 2, 5, 7)$

denote this orbifold. For example, Figure 1 is a picture of an orbifold of genus 3 with three cone points of orders 2, 5, and 7.

The fundamental group of an orbifold is, by definition, the group of deck transformations of the universal covering orbifold. That such an orbifold exists is a theorem which we will not prove, because in our case, the orbifolds are *good*, namely, they have a manifold as a (branched) orbifold cover. In fact, almost all of our examples are hyperbolic, namely their universal covering is H^2 and π_1^{orb} is a discrete subgroup of PSL(2,$\mathbf{R}$). Thus, we take as $\pi_1^{orb}(X)$ the group of deck transformations of the universal branched cover. The orbifold fundamental group can be computed in terms of curves on X. For consider a closed curve x going once around a cone point $c \in X$of order a.

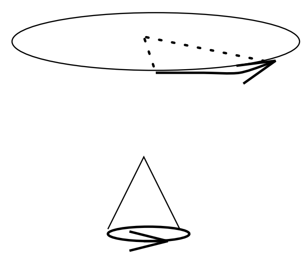

Figure 2: The curve x lifted to $\tilde{x}$.

Because c has order a, a neighborhood of c has an l.u.s. $\{\tilde{D}^2, \mathbf{Z}_a, \phi\}$. Lifting x to the path $\tilde{x}$ in $\tilde{D}^2$. we see that x^a lifts to a closed path in $\tilde{D}^2$, which is contractible.

Thus we have generators x_i of order a_i for each cone point $c_i \in X$. We also have the standard generatos A_i, B_i coming from the g handles of X.

Then the product of the x_i's is homotopic to π which itself is homotopic to the product of the $[A_i, B_i]$'s, i.e.

$$\prod_{i=1}^{n} x_i = \pi = \prod_{i=1}^{g} [A_i, B_i].$$

Thus, setting $X = X(g; a_1, \ldots, a_n)$, we get the group presentation

$$\pi_1^{orb}(X) = \langle A_1, B_1, \ldots, A_g, B_g, x_1, \ldots, x_n \mid x_i^{a_i} = 1 \text{ and } \prod_{i=1}^{n} x_i = \prod_{i=1}^{g} [A_i, B_i] \rangle.$$

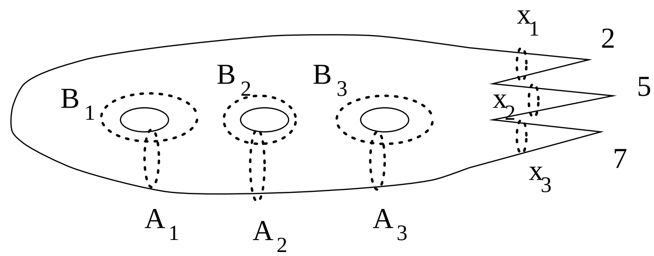

Figure 3: The generators for π_1^{orb}.

We shall often use the following smoothing procedure, which replaces an orbifold X with its underlying Riemann surface X_s. To do this, choose a collection of non-intersecting neighborhoods D_i^2 of the cone points $c_i \in X$. Dropping the subscripts, for $c \in D^2 \subset X$ we have an l.u.s. of the form $\{\tilde{D}^2, \mathbf{Z}_a, \phi\}$. Let $\tilde{c} = \phi^{-1}(c)$. We see that the action of $\mathbf{Z}_a$ is free on the punctured disk $\tilde{D}^2 \setminus \{\tilde{c}\}$. Thus, we can glue in a deleted holomorphic disk $\widehat{D}^2 = (\tilde{D}^2 \setminus \{\tilde{c}\})/\mathbf{Z}_a$, giving a holomorphic structure on $X_0 = X \setminus \{c_1, \ldots, c_n\}$. We compactify this by adding in the points $\{p_1, \ldots, p_n\}$ to obtain a smooth Riemann surface which we denote by X_s.

In § 4 we introduce a process of smoothing on the level of bundles. This replaces an orbifold bundle over X with a bundle over X_s with some additional data. Briefly, an orbifold bundle is locally a Γ-equivariant bundle.

Definition 2.2 *A complex orbifold bundle is a continuous map $E \xrightarrow{\theta} X$ between orbifolds such that for any $x \in X$, there is an open set U containing x with an l.u.s. $\{\tilde{U}, \Gamma, \phi\}$ and a compatible l.u.s. for $E_U = \theta^{-1}(U)$ of the form $\{\tilde{E}_U, \Gamma, \phi'\}$ where*

1. *$\tilde{E}_U = \tilde{U} \times \mathbf{C}^n$*
2. *the Γ action on $\tilde{E}_U$ is given by a representation $\rho : \Gamma \to GL(n,\mathbf{C})$.*
3. *θ is covered by $\tilde{\theta} : \tilde{E}_U \to \tilde{U}$ which is projection onto the first factor.*

Remark: The action of Γ on $\tilde{E}_U = \tilde{U} \times \mathbf{C}^n$ is the diagonal action. One does not need to assume, as we have done, that the bundle is proper, i.e. that the finite groups for the l.u.s. of $\tilde{U}$ and $\tilde{E}_U$ coincide, a surjection would suffice.

For us, an orbifold bundle $E \to X$ consists of an honest bundle $E_0 \to X_0$ along with "equivariant trivializations" over each cone point $c \in X$. That is, for $c \in D^2$, we have $\tilde{E}_{D^2} \approx \tilde{D}^2 \times \mathbf{C}^n$ with an action of $\mathbf{Z}_a$ given by a representation $\rho : \mathbf{Z}_a \to \mathrm{GL}(n, \mathbf{C})$. Such representations are determined by their characters.

In what follows, we use freely the many results of differential geometry for orbifolds. Namely, a version of the Atiyah-Singer index theorem holds (see [10]), and the Hodge decomposition theorem holds (see [2]). Of course, to make any sense of this, we need definitions of the following differential geometric gadgets.

Suppose $E \to X$ is an orbifold bundle with compatible l.u.s.'s $\{\tilde{U}, \Gamma, \phi\}$ for $U \subset X$ and $\{\tilde{E}_U, \Gamma, \phi'\}$ for $E_U \subset E$. Then a section $s : X \to E$ is an *orbifold section* if s

descends from a Γ-equivariant C^∞ section $\tilde{s} : \tilde{U} \to \tilde{E}_U$. Since an orbifold X has natural tangent bundle TX and cotangent bundle T^*X, we can construct the associated tensor bundles. Let $TX_c = TX \otimes_{\mathbf{R}} \mathbf{C}$ and $T^*X_c = T^*X \otimes_{\mathbf{R}} \mathbf{C}$ be the complexified tangent and cotangent bundles. We use $\bigwedge^k T^*X_c$ to denote the bundle of complex alternating k-tensors and $\Omega^k(X)$ the orbifold sections of $\bigwedge^k T^*X_c$. Notice that $\Omega^0(X)$ is just the smooth maps from X into $\mathbf{C}$, namely $C^\infty(X)$. Then the exterior derivative extends by complex linearity to give

$$d : \Omega^k(X) \to \Omega^{k+1}(X).$$

For the orbifold bundle $E \to X$, we denote by $\Omega^k(E)$ the orbifold sections of the bundle $E \otimes \bigwedge^k T^*X_c$. Then a *connection* on E is a $\mathbf{C}$ linear map

$$\nabla : \Omega^0(E) \to \Omega^1(E)$$

satisfying $\nabla(fs) = (df)s + f(\nabla s)$ for $f \in \Omega^0(X)$ and $s \in \Omega^0(E)$.

Thus, ∇ has a description locally as a Γ-invariant connection $\tilde{\nabla}$ in the Γ-bundle $\tilde{E}_U \to \tilde{U}$. With a connection ∇, we get the induced covariant derivative

$$d_\nabla : \Omega^k(E) \to \Omega^{k+1}(E).$$

A *hermitian metric* h is a Γ-invariant hermitian metric $\tilde{h}$ in $\tilde{E}_U \to \tilde{U}$. We call a bundle $E \to X$ with a hermitian metric a hermitian bundle. Given a hermitian bundle $E \to X$, the connection ∇ is hermitian if it satisfies

$$d(s_1, s_2) = (\nabla s_1, s_2) + (s_1, \nabla s_2),$$

for $s_i \in \Omega^0(E)$, where we have written $(\cdot, \cdot)$ for the metric.

Using the complex structure on X, we decompose the k-forms into

$$\Omega^k(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X).$$

The holomorphic structure on X gives the Dolbeault operator

$$\bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$$

and the exterior derivative decomposes into $d = \partial + \bar{\partial}$. Likewise, we decompose the bundle-valued forms into (p,q) components by

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Then a *holomorphic structure* for E is a map

$$\begin{aligned} d'' &: \Omega^0(E) \to \Omega^{0,1}(E) \text{ satisfying} \\ d''(fs) &= (\bar{\partial} f)s + f(d''s) \text{ for } f \in \Omega^0(X) \text{ and } s \in \Omega^0(E). \end{aligned}$$

Given the connection ∇, we can decompose it

$$d_\nabla = d'_\nabla + d''_\nabla$$

where d'_∇ is the (1,0)-component and d''_∇ the (0,1)-component of d_∇. We say a connection is compatible with the holomorphic structure d'' provided

$$d''_\nabla = d''.$$

Because X has dimension 2, any connection determines a holomorphic structure (the integrability condition is just $d''_\nabla \circ d''_\nabla = 0$). Likewise, given a hermitian bundle E with holomorphic structure, then there exists a unique hermitian connection compatible with the holomorphic structure.

The argument in § 5 minimizes the trace norm of a connection in a holomorphic bundle. We shall need the following description of the induced connections on sub- and quotient bundles. The underlying principle is that while exact sequences of C^∞ bundles always split, the same is not true of holomorphic bundles. The obstruction to their splitting is measured by an extension class, with representative the second fundamental form which we describe now.

Suppose $0 \to P \to E \to Q \to 0$ is a short exact sequence of holomorphic orbifold bundles. Then a hermitian metric on E determines a C^∞ splitting $E = P \oplus Q$. Let π_P and π_Q be the projections $E \overset{\pi_P}{\to} P$ and $E \overset{\pi_Q}{\to} Q$. The metric defines hermitian metrics on P and Q by restriction. This, together with the holomorphic structures, determine the connections A, A_P, and A_Q on the bundles E, P, and Q respectively. For $s \in \Omega^0(P)$, we have $A_P(s) = \pi_P(A(s))$. Likewise, for $s \in \Omega^0(Q)$, we have $A_Q(s) = \pi_Q(A(s))$. This follows by uniqueness of the metric connections, because one can check that $\pi_P \circ A$ and $\pi_Q \circ A$ satisfy the requirements for being metric connections on P and Q. For $s \in \Omega^0(P)$, consider the difference

$$\alpha(s) = A(s) - A_P(s) \in \Omega^1(Q).$$

If $f \in \Omega^0(X)$, then $\alpha(fs) = f\alpha(s)$, thus α is linear over $\Omega^0(X)$ and can be represented by a 1-form $\alpha \in \Omega^1(P^* \otimes Q)$. In fact, if s is a holomorphic section of P, $\alpha s \in \Omega^{1,0}(Q)$, thus $\alpha \in \Omega^{1,0}(P^* \otimes Q)$. Similarly, if $s \in \Omega^0(Q)$, then $\beta s = A(s) - A_Q(s) \in \Omega^1(P)$ for $\beta \in \Omega^{0,1}(Q^* \otimes P)$. In fact, β is the adjoint of $-\alpha$. To see this, take $s_1 \in \Omega^0(P)$ and $s_2 \in \Omega^0(Q)$, then in terms of the metric, we have

$$\begin{aligned} 0 &= (s_1, s_2) \\ &= d(s_1, s_2) \\ &= (A(s_1), s_2) + (s_1, A(s_2)) \\ &= (A_P(s_1) + \alpha s_1, s_2) + (s_1, A_Q(s_2) + \beta s_2) \\ &= (\alpha s_1, s_2) + (s_1, \beta s_2). \end{aligned}$$

Because the curvature of a metric connection is a (1,1)-form, we see that $\bar{\partial}\beta = 0$. Thus, β represents a homology class in $\mathrm{H}^{0,1}(Q^* \otimes P)$. The connection A has matrix description

$$A = \begin{pmatrix} A_P & \beta \\ -\beta^* & A_Q \end{pmatrix}.$$

Furthermore, $\beta = 0 \Leftrightarrow A$ preserves the splitting, i.e. the splitting is actually a splitting of holomorphic bundles. We call β the second fundamental form and its homology class $[\beta]$ the extension class. If $[\beta] = 0$, then for some choice of metric, the splitting $E = P \oplus Q$ is holomorphic.

3 Parabolic Bundles

In this section, we define the notion of a parabolic bundle over a Riemann surface X. A parabolic bundle $\mathcal{E}$ is just a holomorphic bundle over X with the additional structure of weighted flags (not necessarily full) in the fibers $\mathcal{E}_p$ over a (finite) set of points $p \in X$. We shall see in §4 that holomorphic orbifold bundles really are parabolic bundles in an explicite way. Before we proceed, we point out that already at least two excellent references exist for this material (see [14] or [18]).

Definition 3.1 *Given a compact Riemann surface X with a finite set of points $\{p_j\}_1^n \subset X$ (called* parabolic points*), a* **parabolic bundle** *over $(X, \{p_j\})$ is a holomorphic bundle $\mathcal{E}$ over X with* parabolic structure*, i.e. for each parabolic point $p \in \{p_j\}_1^n$, we have*

1. *$\mathcal{E}_p = F_{p,1} \supset F_{p,2} \supset \ldots \supset F_{p,r_p} \supset 0$, a descending flag and*

2. *$0 \le a_{p,1} < a_{p,2} < \ldots < a_{p,r_p} < 1$ associated weights.*

The multiplicity *of the weight $a_{p,i}$ is $m_{p,i} = \dim(F_{p,i}) - \dim(F_{p,i+1})$.*

For the purpose of clarity, we shall write "$\mathcal{E}$ is a parabolic bundle over X", when the parabolic points in X and the parabolic structure on $\mathcal{E}$ are understood.

Definition 3.2 *We define the* parabolic degree *of a parabolic bundle $\mathcal{E}$ by the formula*

$$\mathrm{pardeg}(\mathcal{E}) = \deg(\mathcal{E}) + \sum_{p\in\{p_j\}} \sum_{i=1}^{r_p} m_{p,i} a_{p,i}$$

and the parabolic slope *by*

$$\mu(\mathcal{E}) = \frac{\mathrm{pardeg}(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

Definition 3.3 *Given two parabolic bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ over X, a* **parabolic morphism** *is a map $\psi : \mathcal{E}_1 \to \mathcal{E}_2$ of holomorphic bundles which respects the parabolic structures. I.e. for each parabolic point p with the parabolic structures on $\mathcal{E}_k$ at p for $k = 1, 2$ given by*

$$\begin{gathered} \mathcal{E}_{kp} = F_1^k \supset F_2^k \supset \ldots \supset F_{r_k}^k \supset 0 \\ 0 \leq a_1^k < a_2^k < \ldots < a_{r_k}^k < 1, \end{gathered}$$

we require that ψ_p satisfies

$$a_i^1 > a_j^2 \Rightarrow \psi_p(F_i^1) \subseteq F_{j+1}^2. \tag{1}$$

We use the notation $\mathrm{ParHom}(\mathcal{E}_1, \mathcal{E}_2)$ *for the set of parabolic morphisms of two bundles. A bundle isomorphism ψ is a* **parabolic isomorphism** *if both ψ and ψ^{-1} are parabolic maps. We use* $\mathrm{ParAut}(\mathcal{E})$ *to denote the set of parabolic automorphisms of a bundle.*

Remark: We can replace condition (1) by the following equivalent condition on ψ_p. Given the weight a_i^1, let a_j^2 be the smallest weight such that $a_i^1 \leq a_j^2$, then we require

$$\psi_p(F_i^1) \subseteq F_j^2. \tag{2}$$

If there is no such a_j^2, then we demand that $\psi_p(F_i^1) = 0$. Because $a_i^1 > a_{j-1}^2$, we see that conditions (1) and (2) are equivalent.

Remark: Given the parabolic bundle $\mathcal{E}$, consider the group of parabolic bundle automorphisms $\psi : \mathcal{E} \to \mathcal{E}$ lying over the identity map of X, denoted by $\mathrm{ParAut}(\mathcal{E})$. Then since ψ is a parabolic map, we must have $\psi_p(F_{p,i}) = F_{p,i}$. Thus $\mathrm{ParAut}(\mathcal{E})$ is independent of the weights (i.e. it depends only on the *quasi-parabolic* structure of $\mathcal{E}$, namely the unweighted flag structure).

Lemma 3.4 *If $\mathcal{E}_1 \xrightarrow{\psi} \mathcal{E}_2 \xrightarrow{\phi} \mathcal{E}_3$ is a sequence of parabolic morphisms, then $\phi \circ \psi$ is a parabolic map.*

Proof: Suppose $p \in X$ is a parabolic point. We use the notation $\{F^\imath_j, a^\imath_j\}$ for the weighted flag in $\mathcal{E}_\imath$ at p for $\imath = 1, 2, 3$. Given the weight a^1_i, let a^2_j be the smallest weight with $a^1_i \leq a^2_j$. Then by condition (2), $\psi_p(F^1_i) \subseteq F^2_j$. Also, if a^3_k is the smallest weight with $a^2_j \leq a^3_k$, then (again by condition (2)) $\phi_p(F^2_j) \subseteq F^3_k$. Thus we see that $(\phi \circ \psi)_p(F^1_i) \subseteq F^3_k$. On the other hand, let $a^3_{k'}$ be the smallest weight with $a^1_i \leq a^3_{k'}$. Since $a^1_i \leq a^3_k$, we see that $a^3_{k'} \leq a^3_k$. Thus $F^3_k \subseteq F^3_{k'} \Rightarrow (\phi \circ \psi)_p(F^1_i) \subseteq F^3_{k'}$. A final application of condition (2) shows $\phi \circ \psi$ is parabolic. ■

Given a short exact sequence of holomorphic bundles over X

$$0 \to \mathcal{E}_1 \xrightarrow{\imath} \mathcal{E}_2 \xrightarrow{\pi} \mathcal{E}_3 \to 0,$$

then a parabolic structure on $\mathcal{E}_2$ determines a unique parabolic structure on $\mathcal{E}_1$ and $\mathcal{E}_3$ as we shall explain in short order. We first remark that the converse is true (namely that parabolic structures on $\mathcal{E}_1$ and $\mathcal{E}_3$ determine a parabolic structure on $\mathcal{E}_2$). The interested reader is referred to page 68 of [18].

Suppose we have a parabolic structure on $\mathcal{E}_2$. Then at each parabolic point $p \in X$, we have the weighted flag

$$\begin{gathered} \mathcal{E}_{2p} = F^2_1 \supset F^2_2 \supset \ldots \supset F^2_{r_2} \supset 0 \\ 0 \leq a^2_1 < a^2_2 < \ldots < a^2_{r_2} < 1. \end{gathered}$$

We define the parabolic structure on $\mathcal{E}_1$ first. Let $H_i = \imath^{-1}(F^2_i)$ (think of this as $\mathcal{E}_1 \cap F^2_i$). We get a flag from the non-increasing sequence of subspaces

$$H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{r_2}$$

by removing those terms for which the inclusion is not proper. The easiest way to do this is to choose a subsequence $\{i_1, \ldots, i_{r_1}\} \subset \{1, \ldots, r_2\}$ so that

$$H_1 = \cdots = H_{i_1} \supset H_{i_1+1} = \cdots = H_{i_2} \supset H_{i_2+1} = \cdots = H_{i_{r_1}}.$$

Set $F^1_j = H_{i_j}$ and $a^1_j = a^2_{i_j}$ for $j = 1, \ldots, r_1$. This gives the following flag for $\mathcal{E}_{1p}$

$$\begin{gathered} \mathcal{E}_{1p} = F^1_1 \supset F^1_2 \supset \ldots \supset F^1_{r_1} \supset 0 \\ 0 \leq a^1_1 < a^1_2 < \ldots < a^1_{r_1} < 1. \end{gathered}$$

To define a parabolic structure on $\mathcal{E}_3$, set $H_i = \pi(F^2_i)$ and use the same technique to get a flag from $H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{r_2}$, i.e. choose a subsequence $\{i_1, \ldots, i_{r_3}\} \subset \{1, \ldots, r_2\}$, and set $F^3_j = H_{i_j}$ and $a^3_j = a^2_{i_j}$ for $j = 1, \ldots, r_3$. This gives the weighted flag for $\mathcal{E}_{3p}$.

Remark: Notice that the weights are assigned to the $\mathcal{E}_1$ and $\mathcal{E}_3$ by forcing

1. $a^1_i = a^2_j$ where $j =$ greatest integer with $\imath(F^1_i) \subseteq F^2_j$
2. $a^3_k = a^2_j$ where $j =$ greatest integer with $\pi(F^2_j) \subseteq F^3_k$

If we give $\mathcal{E}_1$ and $\mathcal{E}_3$ these canonical parabolic structures, $\imath$ and π are parabolic morphisms.

We call $\mathcal{E}_1$ with this canonical parabolic structure, a **parabolic subbundle** of $\mathcal{E}_2$. Likewise, we call $\mathcal{E}_3$ a **parabolic quotient**.
Warning : The following (seemingly innocent) statements are **false**.
(1) A parabolic isomorphism is an isomorphism that is a parabolic map.
(2) A parabolic subbundle is given by an injection that is a parabolic map.
(3) A parabolic quotient is given by a surjection that is a parabolic map.
The trivial flag $\mathcal{E}_p \subset 0$ with weight $a_1 = 0$ provides an easy counterexample to (1). For (2) and (3), notice that the canonical procedure specifies exactly what the weights of the flags in a subbundle and quotient must be. With this in mind, we define

Definition 3.5 *A short exact sequence of* parabolic *bundles is a short exact sequence of bundles*

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0,$$

where $\mathcal{E}_1$ is a parabolic *subbundle of $\mathcal{E}_2$ and $\mathcal{E}_3$ is a* parabolic *quotient.*

Lemma 3.6 *Suppose $\mathcal{P}$ is a parabolic bundle over X and*

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0,$$

is a short exact sequence of parabolic bundles over X. Then

1. $\psi : \mathcal{P} \to \mathcal{E}_1$ *parabolic* $\Leftrightarrow \imath \circ \psi : \mathcal{P} \to \mathcal{E}_2$ *parabolic*
2. $\phi : \mathcal{E}_3 \to \mathcal{P}$ *parabolic* $\Leftrightarrow \phi \circ \pi : \mathcal{E}_2 \to \mathcal{P}$ *parabolic*
3. $\mathrm{pardeg}(\mathcal{E}_1) + \mathrm{pardeg}(\mathcal{E}_3) = \mathrm{pardeg}(\mathcal{E}_2)$

Proof : Lemma 3.4 and the observation that $\imath$ and π are parabolic proves ($\Rightarrow$) for both (1) and (2). Choose p parabolic. We will use the following notation for the flags of these bundles at p. Let $\{F_i^{\mathcal{P}}, a_i^{\mathcal{P}}\}$ be the weighted flag of $\mathcal{P}$ at p and $\{F_j^{\imath}, a_j^{\imath}\}$ be the weighted flag of $\mathcal{E}_\imath$ for $\imath = 1, 2, 3$.
(1) We must show that if $a_i^{\mathcal{P}} > a_j^1$, then $\psi(F_i^{\mathcal{P}}) \subseteq F_{j+1}^1$. But since $\mathcal{E}_1$ is a parabolic subbundle, we have F_k^2, a_k^2 with $F_j^1 = \imath^{-1}(F_k^2)$ and $a_j^1 = a_k^2$. Moreover k is the largest integer with this property, i.e. $F_j^1 \neq \imath^{-1}(F_{k+1}^2)$, in fact $F_{j+1}^1 = \imath^{-1}(F_{k+1}^2)$. Since $\imath \circ \psi$ is parabolic, $(\imath \circ \psi)_p(F_i^{\mathcal{P}}) \subseteq F_{k+1}^2$. Thus $\psi_p(F_i^{\mathcal{P}}) \subseteq \imath^{-1}(F_{k+1}^2)$.
(2) For this we must show that if $a_j^3 > a_k^{\mathcal{P}}$, then $\phi(F_j^3) \subseteq F_{k+1}^{\mathcal{P}}$. Because $\mathcal{E}_3$ is a parabolic quotient, $F_j^3 = \pi(F_i^2)$ and $a_j^3 = a_i^2$. Since $\phi \circ \pi$ is parabolic, $(\phi \circ \pi)_p(F_i^2) \subseteq F_{k+1}^{\mathcal{P}}$. The result (2) now follows.
(3) Clearly $\deg(\mathcal{E}_1) + \deg(\mathcal{E}_3) = \deg(\mathcal{E}_2)$. But in our description of the canonical procedure it is evident that the sets of weights of $\mathcal{E}_1$ and of $\mathcal{E}_3$ form a partition of the set of weights of $\mathcal{E}_2$ (taken with multiplicity). Thus

$$\sum_i n_i^1 a_i^1 + \sum_k n_k^3 a_k^3 = \sum_j n_j^2 a_j^2$$

where $n^\imath_l$ is the multiplicity of the weight $a^\imath_l$ in $F^\imath_l$ for $\imath = 1, 2, 3$. ■

By §4 of [15], any non-zero map $\alpha : \mathcal{E} \to \mathcal{F}$ of holomorphic bundles has a canonical factorization

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{P} & \to & \mathcal{E} & \xrightarrow{\pi} & \mathcal{Q} & \to & 0 \\ & & & & & & \downarrow_\beta & & \\ 0 & \leftarrow & \mathcal{N} & \leftarrow & \mathcal{F} & \xleftarrow{\imath} & \mathcal{M} & \leftarrow & 0 \end{array}$$

where $\alpha = \imath \circ \beta \circ \pi$ and β has maximal rank. In particular, $\mathrm{rank}(\mathcal{Q}) = \mathrm{rank}(\mathcal{M}) = n$. Maximal rank means that $\wedge^n(\beta) : \wedge^n(\mathcal{Q}) \to \wedge^n(\mathcal{M})$ is not the zero map. If $\wedge^n(\beta)$ is nowhere zero, β is said to be of full rank and then β is seen to be isomorphism. In any case, it follows that $\deg(\mathcal{Q}) \le \deg(\mathcal{M})$ with equality $\Leftrightarrow$ β is an isomorphism.

We are interested in the analogous statement for parabolic bundles. Suppose $\mathcal{E}$ and $\mathcal{F}$ are parabolic bundles and that α is a parabolic map. Then $\mathcal{M}$, being a subbundle of $\mathcal{F}$, and $\mathcal{Q}$, a quotient of $\mathcal{E}$, inherit canonical parabolic structures. By Lemma 3.6, β is a parabolic map. The next lemma shows that $\mathrm{pardeg}(\mathcal{Q}) \le \mathrm{pardeg}(\mathcal{M})$.

Lemma 3.7 *If $\beta : \mathcal{E}_1 \to \mathcal{E}_2$ is a maximal rank, parabolic map between parabolic bundles. Then* $\mathrm{pardeg}(\mathcal{E}_1) \le \mathrm{pardeg}(\mathcal{E}_2)$.

Proof: We first show how the result follows if β has full rank, and then we address the more general case.

Consider a parabolic point $p \in X$. Since β has full rank, β is an isomorphism. In particular, $\deg(\mathcal{E}_1) = \deg(\mathcal{E}_2)$. Writing out the two weighted flags

$$\begin{gathered} \mathcal{E}_{1p} = F^1_1 \supset F^1_2 \supset \ldots \supset F^1_r \supset 0 \\ 0 \le a_1 < a_2 < \ldots < a_r < 1 \end{gathered}$$

$$\begin{gathered} \mathcal{E}_{2p} = F^2_1 \supset F^2_2 \supset \ldots \supset F^2_s \supset 0 \\ 0 \le b_1 < b_2 < \ldots < b_s < 1 \end{gathered}$$

with multiplicities $m_1, \ldots, m_r$ and $n_1, \ldots, n_s$, respectively, we see that the result will follow if we show that, for each parabolic point, we have the inequality

$$\sum_{i=1}^s n_i b_i \ge \sum_{i=1}^r m_i a_i.$$

In order to prove this, we write out each sum and claim

$$b_1 + \overset{n_1}{\cdots} + b_1 + \cdots + b_s + \overset{n_s}{\cdots} + b_s \ge a_1 + \overset{m_1}{\cdots} + a_1 + \cdots + a_r + \overset{m_r}{\cdots} + a_r.$$

There are $N = \dim(\mathcal{E}_{1p}) = \dim(\mathcal{E}_{2p})$ terms in each expression, so we prove the claim by simply showing that the i^{th} term on the left (b_{k_i}) is greater than or equal to the i^{th} term on the right (a_{j_i}), where $j : \{1, \ldots, N\} \to \{1, \ldots, r\}$ and $k : \{1, \ldots, N\} \to \{1, \ldots, s\}$ are the choice functions.

Suppose not, namely that $a_{j_i} > b_{k_i}$ for some i. Since β is parabolic, it follows that $\beta(F^1_{j_i}) \subseteq F^2_{k_i+1}$. Further, since β_p is an isomorphism, $\mathrm{rank}\beta(F^1_{j_i}) = \mathrm{rank}(F^1_{j_i})$. But $\mathrm{rank}(F^1_{j_i}) \geq N - i + 1$ and $\mathrm{rank}(F^2_{k_i+1}) < N - i + 1$, which gives the desired contradiction.

Now we prove the propostion in the general case (β is maximal rank). This means that for generic points $q \in X$, β_q is an isomorphism. The problem : there is no reason parabolic points must be generic. Call nongeneric points singular. Because X is a compact Riemann surface, there is a finite number of singular points $\{q_i\}$. Further, we have an exact sequence of sheaves

$$0 \to \mathcal{E}_1 \xrightarrow{\beta} \mathcal{E}_2 \to \mathcal{S} \to 0,$$

where $\mathcal{S}$ is a sum of skyscraper sheaves with support on $\{q_i\}$. We are tempted to call $\mathcal{S}$ a skyline sheaf! In any case, $\deg(\mathcal{S}) = \deg(\mathcal{E}_2) - \deg(\mathcal{E}_1)$ by the short exact sequence. For each q_i, let γ_i be the amount which β drops rank at q_i. More precisely, $\gamma_i \stackrel{def}{=} \mathrm{codim}(\beta(\mathcal{E}_{1q_i}))$ in $\mathcal{E}_{2p}$. Since $\mathcal{S}$ is a skyline sheaf, $\deg(\mathcal{S}) = \sum_i \gamma_i$. Since the previous argument will apply to the generic parabolic points, it suffices to show that, for any singular parabolic point p with $\gamma =$ the amount that β drops rank at p, then

$$\sum_{i=1}^{s} n_i b_i + \gamma \geq \sum_{i=1}^{r} m_i a_i \tag{3}$$

where we use the same notation for the multiplicities and weights as before.

To prove (3), we note that because each $a_i < 1$, we have

$$\gamma > a_{j_{(N-\gamma+1)}} + \cdots + a_{j_N}.$$

Here, as before, a_{j_i} means the i^{th} term in the expanded sum (j is a choice function). Writing out the remaining terms in each sum, we claim

$$b_1 + \stackrel{n_1}{\cdots} + b_1 + \cdots + b_s + \stackrel{n_s}{\cdots} + b_s \geq a_1 + \cdots + a_{j_{(N-\gamma)}}. \tag{4}$$

There are N terms on the left of (4) and $N - \gamma$ terms on the right. Comparing the $(\gamma + i)^{th}$ term on the left ($b_{k_{(\gamma+i)}}$) with the i^{th} term on the right (a_{j_i}), we claim that $b_{k_{(\gamma+i)}} \geq a_{j_i}$. For otherwise $a_{j_i} > b_{k_{(\gamma+i)}} \Rightarrow \beta_p(F^1_{j_i}) \subseteq F^2_{k_{(\gamma+i)}+1}$ because β is parabolic. But this forces β to drop rank more than γ at p, a contradiction. ∎

In summary,

Proposition 3.8 *Any nonzero parabolic map $\alpha : \mathcal{E} \to \mathcal{F}$ has the following canonical factorization*

$$\begin{array}{ccccccc} 0 \to \mathcal{P} \to \mathcal{E} & \xrightarrow{\pi} & \mathcal{Q} & \to 0 \\ & & \downarrow_{\beta} & \\ 0 \leftarrow \mathcal{N} \leftarrow \mathcal{F} & \xleftarrow{\imath} & \mathcal{M} & \leftarrow 0 \end{array}$$

where

1. *the two rows are short exact sequences of parabolic bundles,*

2. *β is a parabolic map and satisfies $\alpha = \imath \circ \beta \circ \pi$,*

3. $\mathrm{rank}(\mathcal{Q}) = \mathrm{rank}(\mathcal{M})$,

4. $\deg(\mathcal{Q}) \leq \deg(\mathcal{M})$ *with equality* $\Leftrightarrow$ *β is a bundle isomorphism.*

5. $\mathrm{pardeg}(\mathcal{Q}) \leq \mathrm{pardeg}(\mathcal{M})$ *with equality here and in (4)* $\Leftrightarrow$ *β is a parabolic isomorphism.*

We close this section with the following

Definition 3.9 *Suppose $\mathcal{E}$ is a parabolic bundle. Then*

1. *$\mathcal{E}$ is parabolic stable if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ for all proper subbundles $\mathcal{F}$.*

2. *$\mathcal{E}$ is parabolic semistable if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ for all proper subbundles $\mathcal{F}$.*

4 Push Forward Construction

Suppose X is a holomorphic 2-dimensional orbifold and E is a $\mathbf{C}^n$ holomorphic orbifold bundle over X. As in section 2, we construct X_s, the smoothing of X, with holomorphic structure. Further if $\{c_1, c_2, \ldots, c_N\}$ is the set of cone points of X, their image under our topological identification $X \approx X_s$ is a set of distinguished points $\{p_1, p_2, \ldots, p_N\}$ which we call parabolic points. We will show how to use the holomorphic structure of E to obtain a holomorphic bundle over X_s with the additional data of partial flags over each parabolic points p_i.

Proposition 4.1 *Given a holomorphic orbifold bundle E over X, there is a natural parabolic bundle $\mathcal{E}$ over X_s. Here, by natural, we mean that given a holomorphic map of orbifold bundles $\phi : E_1 \to E_2$, there is an associated parabolic morphism of the parabolic bundles $\tilde{\phi} : \mathcal{E}_1 \to \mathcal{E}_2$ (see Proposition 4.5).*

Proof: We construct the sheaf of sections of $\mathcal{E}$. It will follow from our description that this sheaf is actually locally free and hence describes a vector bundle. First consider the situation over a nonsingular neighborhood U of X. Then E_U is a (regular) holomorphic bundle over U. Thus, sections of E_U are in an obvious way sections of $\mathcal{E}_U$. Of course we are using the fact that U is simultaneously a smooth neighborhood for both X and X_s.

Next, consider the situation over a cone point c_i of X. Choose a neighborhood $U \approx \tilde{U}/\Gamma_U$ of c_i not containing any other cone points. We may assume that E has the trivialization $E_U \approx \tilde{E_U}/\Gamma_U$ where $\tilde{E_U} \approx \tilde{U} \times \mathbf{C}^n$. Sections of E_U over U are just Γ_U-invariant sections of $\tilde{E_U}$ over $\tilde{U}$. Taking holomorphic coordinates $\tilde{U} \approx D^2 \approx \{z \in$

$\mathbf{C}\ |z| \leq 1\}$ and $\tilde{E_U} \approx D^2 \times \mathbf{C}^n$, then local sections of E are holomorphic, Γ_U-invariant maps

$$\begin{aligned} s \quad : \quad & D^2 \to D^2 \times \mathbf{C}^n \\ & z \mapsto (z, f(z)) \end{aligned}$$

Because the action of Γ_U is necessarily holomorphic, we know Γ_U is cyclic. We can choose a generator σ for $\Gamma_U \cong \mathbf{Z}_m$ so that

$$\begin{aligned} \sigma \quad : \quad & D^2 \to D^2 \\ & z \mapsto \omega z \end{aligned}$$

where $\omega = e^{2\pi i/m}$. Since the action of $\mathbf{Z}_m$ on the bundle is the diagonal action, σ acts on $\mathbf{C}^n$ by a matrix $\rho(\sigma)$ By choosing a basis $\{e_1, \ldots, e_n\}$ of eigenvectors for $\rho(\sigma)$, we see

$$\rho(\sigma) = \begin{pmatrix} \omega^{k_1} & & 0 \\ & \ddots & \\ 0 & & \omega^{k_n} \end{pmatrix}$$

Because s is Γ_U-invariant, s satisfies $\sigma(s) := \sigma s \sigma^{-1} = s$.

$$\text{But } \sigma(s\sigma^{-1}(z)) = \sigma(s(\overline{\omega}z)) = \sigma(\overline{\omega}z, f(\overline{\omega}z)) = (z, \rho(\sigma)f(\overline{\omega}z)).$$

So we see that since s is Γ_U-invariant

$$\rho(\sigma)f(\overline{\omega}z) = f(z) \tag{5}$$

Writing $f(z) = f_1(z)e_1 + \ldots + f_n(z)e_n$ in terms of the basis $\{e_1, \ldots, e_n\}$, we see that

$$\begin{aligned} \rho(\sigma)f(\overline{\omega}z) &= \rho(\sigma)(f_1(\overline{\omega}z)e_1, \ldots, f_n(\overline{\omega}z)e_n) \\ &= \omega^{k_1}f_1(\overline{\omega}z)e_1 + \ldots + \omega^{k_n}f_n(\overline{\omega}z)e_n. \end{aligned}$$

In these coordinates for $\tilde{E_U}$, equation (5) becomes

$$\omega^{k_i}f_i(\overline{\omega}z) = f_i(z) \text{ for } i = 1, \ldots, n \tag{6}$$

Now we use the holomorphicity of s. This implies that each f_i is a holomorphic map, i.e.

$$f_i(z) = \sum_{j=0}^{\infty} a_j z^j.$$

Taking j^{th} derivatives of both sides in equation 6 and evaluationg at $z = 0$, it follows that $a_j = 0$ unless $j \equiv k_i (\bmod\ m)$. Thus

$$f_i = z^{k_i} \sum_{j=0}^{\infty} b_j (z^m)^j = z^{k_i} \hat{f}_i(z^m),$$

where $\hat{f}_i(z^m)$ is a holomorphic function on $U \approx \tilde{U}/\Gamma_U \approx D^2/\mathbf{Z}_m$. Thus

$$f(z) = z^{k_1}\hat{f}_1(z^m)e_1 + \ldots + z^{k_n}\hat{f}_n(z^m)e_n$$

and

$$\hat{f}(z^m) = \hat{f}_1(z^m)e_1 + \ldots + \hat{f}_n(z^m)e_n$$

are local holomorphic sections. The sheaf of sections is, by construction, locally free, and we call the associated bundle $\mathcal{E}$ the push forward bundle and $\hat{f}$ the push forward section.

This bundle has additional structure of a descending (partial) flag at the parabolic point $p \in U$. Order the basis $\{e_1, \ldots, e_n\}$ so that

$$\rho(\sigma) = \begin{pmatrix} \omega^{k_1} & & 0 \\ & \ddots & \\ 0 & & \omega^{k_n} \end{pmatrix} \text{ satisfies } 0 \leq k_1 \leq k_2 \leq \ldots \leq k_n < m.$$

By reindexing, we can write

$$\rho(\sigma) = \begin{pmatrix} \omega^{k'_1} & & & & & & 0 \\ & \ddots & & & & & \\ & & \omega^{k'_1} & & & & \\ & & & \ddots & & & \\ & & & & \omega^{k'_r} & & \\ & & & & & \ddots & \\ 0 & & & & & & \omega^{k'_r} \end{pmatrix} \text{ where } 0 \leq k'_1 < k'_2 < \ldots < k'_r < m$$

and are repeated according to their multiplicities $n_1, \ldots, n_r$. Let W_i be the $\omega^{k'_i}$-eigenspace of $\rho(\sigma)$ and define

$$F_{p,i} = W_i \oplus \cdots \oplus W_r \text{ with associated weight } a_i = k'_i/m \text{ for } i = 1, \ldots, r.$$

$$\begin{array}{rl} \text{Then} & \mathcal{E}_p = F_1 \supset F_2 \supset \ldots \supset F_r \supset 0 \quad \text{is a flag} \\ \text{with weights} & 0 \leq a_1 < a_2 < \ldots < a_r < 1 \end{array}$$

We will see from Proposition 4.5 that this correspondence is natural and from its corollary (Corollary 4.6) that the parabolic bundle is canonical. This ends the proof of the proposition. ∎

Remark: Although there is no canonical choice for the basis $\{e_1, \ldots, e_n\}$, the eigenspaces W_i are canonical. And so the flag $F_1 \supset F_2 \supset \ldots \supset F_r \supset 0$ is canonical.

Definition 4.2 *Given a flag $F_1 \supset F_2 \supset \ldots \supset F_r \supset 0$, whose successive quotients F_i/F_{i+1} are of dimension n_i, then a basis $\{e_1, \ldots, e_n\}$ for F_1 is a* **flag basis** *if*

$$\{e_{n_1+1}, \ldots, e_n\} \textit{ is a basis for } F_2,$$

$$\{e_{n_1+n_2+1}, \ldots, e_n\} \text{ is a basis for } F_3,$$

$$\vdots$$

$$\{e_{n_1+\ldots+n_{r-1}+1}, \ldots, e_n\} \text{ is a basis for } F_r.$$

Remark: Occasionally it will be convenient to list the weights repeated according to their multiplicities. Then we will write

$$0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n < 1$$

where the α_i are the a_j repeated $\dim(F_j/F_{j+1})$ times. For example, in the construction above, it is clear that $\alpha_i = k_i/m$.

From a parabolic bundle $\mathcal{E}$ over X_s, we construct the pull back bundle E over X, an orbifold bundle which pushes forward to $\mathcal{E}$. Roughly, we use the flag data to construct local representations of cyclic groups on $\mathbf{C}^n$. Notice that any parabolic bundle which is a push forward has rational weights of the form k/m where m is the order of the cone point in X. Thus, not every parabolic bundle can be pulled back to an orbifold bundle. We begin with a definition. Suppose $X_s \xrightarrow{\psi} X$ is our topological identification, with parabolic points $p_i \in X_s$ corresponding to cone points $c_i \in X$, i.e. $\psi(p_i) = c_i$.

Definition 4.3 *Given a parabolic bundle $\mathcal{E}$ over X_s. We say a parabolic bundle $\mathcal{E}$ over X_s is* **commensurate** *with X if the weights of the flag over each parabolic point p_i are rational numbers of the form k/m_i where m_i is the order of the cone point $c_i \in X$.*

Proposition 4.4 *If $\mathcal{E}$ is commensurate with X, then there exists a holomorphic orbifold bundle E over X so that $\mathcal{E}$ is the push forward of E.*

Proof: For each parabolic point p_i, choose small 2-disk neighborhoods D_i of p_i so that $D_i \cap D_j = \emptyset$. Let $X_{s0} = X_s \setminus \cup_i D_i$. Let $U_i = \psi(D_i)$ be the corresponding neighborhoods for each cone point $c_i \in X$. We have $U_i \approx \tilde{D^2}_i/\mathbf{Z}_{m_i}$. Setting $X_0 = X \setminus \cup_i U_i$, we have a diffeomorphism $X_{s0} \overset{\psi_0}{\approx} X_0$. We define the bundle over the nonsingular part of X by $E_0 = (\psi_0)^*(\mathcal{E}_0)$.

Now we need to define E over each U_i. We choose a particular U_i and drop the i-subscripts in what follows. Let $\{e_1, \ldots, e_n\}$ be a flag basis for the flag

$$\mathcal{E}_p = F_1 \supset F_2 \supset \ldots \supset F_r \supset 0$$
$$0 \leq a_1 < a_2 < \ldots < a_r < 1$$

and $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n < 1$ be the weights repeated according to their multiplicities. We may assume $\mathcal{E} \approx D^2 \times \mathbf{C}^n$ where $\{e_1, \ldots, e_n\}$ is our basis for $\mathbf{C}^n$. Since we assumed $\mathcal{E}$ is commensurate with X, there exists $k_i \in \mathbf{Z}$ with $0 \leq k_i < m$ so that $\alpha_i = k_i/m$ for all i. We define a function

$$\Delta : \mathbf{C}^* \to \mathrm{GL}(n, \mathbf{C}) \text{ by } \Delta(z) = \begin{pmatrix} z^{k_1} & & 0 \\ & \ddots & \\ 0 & & z^{k_n} \end{pmatrix}$$

Notice that Δ is independent of choice of flag basis. Let $\omega = e^{2\pi i/m}$ and choose a generator $\sigma \in \mathbf{Z}_m$ so that

$$\begin{aligned} \sigma \quad : \quad & \tilde{D}^2 \to \tilde{D}^2 \\ & z \mapsto \omega z \end{aligned}$$

Define the action of $\mathbf{Z}_m$ on $\tilde{D}^2 \times \mathbf{C}^n$ by $\sigma(z, v) = (wz, \Delta(\omega)v)$ and set

$$\tilde{E}_U \approx \tilde{D}^2 \times \mathbf{C}^n.$$

Now we check that on the intersection $U_i \cap X_0 = S^1$, there is an equivariant patching map. Clearly, since $\tilde{S}^1 = \partial \tilde{D}^2$ is the $\mathbf{Z}_m$-cover of S^1, the action is free. We have to patch together the two $\mathbf{Z}_m$ actions on the bundle $\tilde{S}^1 \times \mathbf{C}^n$, one which is trivial on the second factor, the other nontrivial (twist by $\Delta(\omega)$). Let σ_0 denote the first action and σ_1 the second. We need to construct a map F so that

$$\begin{array}{ccc} \tilde{S}^1 \times \mathbf{C}^n & \xrightarrow{\sigma_0} & \tilde{S}^1 \times \mathbf{C}^n \\ \downarrow \mathrm{F} & & \downarrow \mathrm{F} \\ \tilde{S}^1 \times \mathbf{C}^n & \xrightarrow{\sigma_1} & \tilde{S}^1 \times \mathbf{C}^n \end{array}$$

commutes. Defining F by

$$\begin{aligned} F \quad : \quad & \tilde{S}^1 \times \mathbf{C}^n \to \tilde{S}^1 \times \mathbf{C}^n \\ & (z, v) \mapsto (z, \Delta(z)v) \end{aligned}$$

we check that it is our required equivariant patching map. ∎

Suppose E_1 and E_2 are holomorphic orbifold bundles over X. Given a holomorphic orbifold bundle morphism

$$\begin{array}{c} E_1 \xrightarrow{\phi} E_2 \\ \searrow \swarrow \\ X \end{array}$$

we show how to construct the push forward morphism of parabolic bundles

$$\begin{array}{c} \mathcal{E}_1 \xrightarrow{\hat{\phi}} \mathcal{E}_2 \\ \searrow \swarrow \\ X_s \end{array}$$

We outline the idea informally. Suppose s_1 is a local holomorphic section of E_{1U}, and let $s_2 = \phi(s_1)$ be the local section of E_{2U}. Then we have the push forward sections $\hat{s}_1$

and $\hat{s}_2$ of $\mathcal{E}_1$ and $\mathcal{E}_2$. We define $\hat{\phi}(\hat{s}_1) = \hat{s}_2$. Of course, to see that this is well-defined, we need to know that every local section $\hat{s}_1$ of $\mathcal{E}_1$ is the push forward of a canonical section s_1 of E_1. This is the content of proposition 4.4. We are interested in proving a stronger result, namely that $\hat{\phi}$ is a parabolic morphism (recall definition 3.3).

We formulate this statement in terms of unitary connections. Suppose E_1, E_2 are unitary orbifold bundles over X, and A_1, A_2 are unitary orbifold connections in E_1, E_2 respectively. We push forward the holomorphic structures d''_{A_1}, d''_{A_2}, to obtain parabolic bundles $\mathcal{E}_1, \mathcal{E}_2$ over X_s. Let d''_{12} be the (0,1)-component of the connection $A_1^* \otimes 1 + 1 \otimes A_2$ on $E_1^* \otimes E_2$. So

$$d''_{12} : \Omega^0(E_1^* \otimes E_2) \to \Omega^{0,1}(E_1^* \otimes E_2)$$

and $d''_{12}(\phi) = 0 \Leftrightarrow \phi$ is a holomorphic orbifold morphism. Then

Proposition 4.5 *Given $\phi : E_1 \to E_2$, then $d''_{12}(\phi) = 0 \Leftrightarrow \hat{\phi} : \mathcal{E}_1 \to \mathcal{E}_2$ is a parabolic morphism.*

Proof: Let $p \in X_s$ be a parabolic point and

$$\begin{gathered} \mathcal{E}_{1p} = F_1^1 \supset F_2^1 \supset \ldots \supset F_{r_1}^1 \supset 0 \\ 0 \le a_1^1 < a_2^1 < \ldots < a_{r_1}^1 < 1 \end{gathered}$$

$$\begin{gathered} \mathcal{E}_{2p} = F_1^2 \supset F_2^2 \supset \ldots \supset F_{r_2}^2 \supset 0 \\ 0 \le a_1^2 < a_2^2 < \ldots < a_{r_2}^2 < 1 \end{gathered}$$

be the weighted flags for $\mathcal{E}_1$ and $\mathcal{E}_2$.We must show that $\hat{\phi}_p$ satisfies condition

$$\hat{\phi}_p(F_i^1) \subset F_{j+1}^2 \text{ whenever } a_i^1 > a_j^2.$$

Equivalently, writing $\hat{\phi}_p = \hat{\phi}_{ij}(p)$ in terms of flag bases

$$\{e_1^1, \ldots, e_{n_1}^1\} \text{ for } \mathcal{E}_{1p} \text{ and } \{e_1^2, \ldots, e_{n_2}^2\} \text{ for } \mathcal{E}_{2p},$$

this requires

$$\hat{\phi}_{ij}(p) = 0 \text{ whenever } \alpha_i^1 > \alpha_j^2 \tag{7}$$

where

$$0 \le \alpha_1^1 \le \alpha_2^1 \le \ldots \le \alpha_{n_1}^1 < 1 \text{ and } 0 \le \alpha_1^2 \le \alpha_2^2 \le \ldots \le \alpha_{n_2}^2 < 1$$

are the weights repeated according to their multiplicities.

Let $c \in X$ be the cone point associated to p, and suppose $c \in U \approx \tilde{U}/\Gamma_U$ over which the bundles E_1 and E_2 have trivializations

$$E_{1U} \approx \tilde{E}_{1U}/\Gamma_U \text{ where } \tilde{E}_{1U} \approx \tilde{U} \times \mathbf{C}^{n_1}$$

and

$$E_{2U} \approx \tilde{E}_{2U}/\Gamma_U \text{ where } \tilde{E}_{2U} \approx \tilde{U} \times \mathbf{C}^{n_2}.$$

Further, we may assume $\tilde{U} \approx D^2$ and $\Gamma_U \cong \mathbf{Z}_m$, where σ a generator for $\mathbf{Z}_m$ gives the standard elliptic action, which is just multiplication by $\omega = e^{2\pi i/m}$

$$\begin{aligned} \sigma : \quad D^2 &\to D^2 \\ z &\mapsto \omega z \end{aligned}$$

Let ρ_1 and ρ_2 be the representaions of the $\mathbf{Z}_m$ actions on $\tilde{E}_{1U} \approx \tilde{U} \times \mathbf{C}^{n_1}$ and $\tilde{E}_{2U} \approx \tilde{U} \times \mathbf{C}^{n_2}$. Then

$$\begin{aligned} \sigma : \quad \tilde{U} \times \mathbf{C}^{n_1} &\to \tilde{U} \times \mathbf{C}^{n_1} \\ (z, v_1) &\mapsto (\omega z, \rho_1(\sigma) v_1) \end{aligned}$$

and

$$\begin{aligned} \sigma : \quad \tilde{U} \times \mathbf{C}^{n_2} &\to \tilde{U} \times \mathbf{C}^{n_2} \\ (z, v_2) &\mapsto (\omega z, \rho_2(\sigma) v_2) \end{aligned}$$

Choose bases $\{e^1_1, \ldots, e^1_{n_1}\}$ for $\mathbf{C}^{n_1}$ and $\{e^2_1, \ldots, e^2_{n_1}\}$ for $\mathbf{C}^{n_2}$ so that

$$\rho_1(\sigma) = \begin{pmatrix} \omega^{k_1} & & 0 \\ & \ddots & \\ 0 & & \omega^{k_{n_1}} \end{pmatrix} \text{ where } 0 \le k_1 \le k_2 \le \ldots \le k_{n_1} < m$$

and

$$\rho_2(\sigma) = \begin{pmatrix} \omega^{h_1} & & 0 \\ & \ddots & \\ 0 & & \omega^{h_{n_2}} \end{pmatrix} \text{ where } 0 \le h_1 \le h_2 \le \ldots \le h_{n_2} < m.$$

As can be seen from definition 4.2,

$$\{e^1_1, \ldots, e^1_{n_1}\} \text{ and } \{e^2_1, \ldots, e^2_{n_2}\}$$

give flag bases for $\mathcal{E}_1$ and $\mathcal{E}_2$. A local section s^1 of E_{1U} is a Γ_U-equivariant map $\tilde{U} \to \tilde{E}_{1U}$, which is given by a $\mathbf{Z}_m$-equivariant map $D^2 \to \mathbf{C}^{n_1}$ in these coordinates. As in the proof of 4.1 write $s^1(z) = \sum s^1_i(z) e^1_i$ in terms of the basis $\{e^1_1, \ldots, e^1_{n_1}\}$. Then each $s^1_i(z)$ satisfies

$$s^1_i(z) = z^{k_i} \hat{s}^1_i(z^m)$$

where $\hat{s}^1_i(z^m)$ is the push forward section on $\mathcal{E}_1$. Applying the same considerations to the local section s^2 of E_{2U} gives $s^2(z) = \sum s^2_j(z) e^2_j$ where

$$s^2_j(z) = z^{h_j} \hat{s}^2_j(z^m).$$

$\hat{s}^2_j(z^m)$ is the push forward section on $\mathcal{E}_2$. Now we write $\phi_U : \tilde{E}_{1U} \to \tilde{E}_{2U}$ as a matrix (ϕ_{ij}). Since $d''_{12}(\phi) = 0$, we know that ϕ is holomorphic, i.e. that $\phi(s^1)$ is holomorphic. Thus we can apply the above considerations to $s^2 = \phi(s^1)$. Further, we define $\hat{\phi} : \mathcal{E}_1 \to \mathcal{E}_2$ by sending the section $\hat{s}^1$ to $\hat{s}^2$. Writing $\hat{\phi}$ also as a matrix $(\hat{\phi}_{ij})$, then

$$\begin{aligned} \sum_i \phi_{ij}(z) s_i^1(z) &= s_j^2(z) = z^{h_j} \hat{s}_j^2(z^m) \\ \sum_i \phi_{ij}(z) z^{k_i} \hat{s}_i^1(z^m) &= \sum_i z^{h_j} \hat{\phi}_{ij}(z^m) \hat{s}_i^1(z^m) \end{aligned}$$

It follows that

$$\phi_{ij}(z) = z^{h_j - k_i} \hat{\phi}_{ij}(z^m) \tag{8}$$

But since $\phi_{ij}(z)$ is bounded as $z \to 0$, we see that

$$\hat{\phi}_{ij} = 0 \text{ whenever } k_i > h_j \tag{9}$$

But since $\alpha_i^1 = k_i/m$ and $\alpha_j^2 = h_j/m$, clearly condition (9) is equivalent to condition (7). This proves ($\Rightarrow$) of the claim. To see ($\Leftarrow$), notice that if $\hat{\phi}$ is parabolic, then we can define ϕ via equation (8). Thus ϕ will be well-defined precisely when $\hat{\phi}$ is parabolic. By its definition, ϕ is holomorphic, thus $d''_{12}(\phi) = 0$. This completes the proof. ■

An easy consequence of Proposition 4.5 is

Corollary 4.6 *If $g \in \mathcal{G}_{orb}^{\mathbf{C}}(E)$, then $\hat{g} \in \mathrm{ParAut}(\mathcal{E})$*

Proof: Let d_{AA^g} be the orbifold connection on $E^* \otimes E$ induced by A^* and A^g. Then $g \in \mathcal{G}_{orb}(E) \Rightarrow d_{AA^g}(g) = 0$, and $g \in \mathcal{G}_{orb}^{\mathbf{C}}(E) \Rightarrow d''_{AA^g}(g) = 0$. ■

5 Main Theorem

At this point we have developed the tools for orbifold and parabolic bundles necessary for the following generalization of [4]. As the argument in this case is very similar, we pay particular attention to those steps of the argument which are not found in [4]. We remind the reader of the definition of *parabolic slope* for a parabolic bundle $\mathcal{E}$

$$\mu(\mathcal{E}) = \frac{\mathrm{pardeg}(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

Recall further that $\mathcal{E}$ is *parabolic stable* if, for every proper subbundle $\mathcal{F} \subset \mathcal{E}$, we have

$$\mu(\mathcal{F}) < \mu(\mathcal{E}).$$

Theorem 5.1 *Given an indecomposable holomorphic orbifold bundle E over X, let $\mathcal{E}$ be the parabolic bundle over X_s obtained by pushing forward E, then $\mathcal{E}$ is parabolic stable $\Leftrightarrow$ $\exists$ unitary orbifold connection A compatible with E with constant central curvature, i.e. $*F_A = -2\pi i\mu \cdot I$, where $\mu = \mu(\mathcal{E})$ and I denotes the identity matrix. This connection is unique up to isomorphism.*

In order to prove this, we define a functional $J(A)$ on connections A as follows. For any $n \times n$ hermitian matrix M, let

$$\tau(M) = \sqrt{tr(M^*M)} = \sum_{i=1}^{n} |\lambda_i|$$

where λ_i are the eigenvalues of M. We can define τ equivalently by,

$$\tau(M) = \max_{\{e_i\}} \sum_{i=1}^{n} |(Me_i, e_i)|, \text{ where } \{e_i\} \text{ is an orthonormal basis for } \mathbf{C}^n$$

since this max will be obtained by a basis of eigenvectors for M. It is easy to check that τ is a norm from this characterization. Also, we see that if M is written in the block form $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$, then $\tau(M) \geq |tr(A)| + |tr(B)|$. This follows since $\sum |(Me_i, e_i)| = |tr(A)| + |tr(D)|$ for the standard basis. We can extend this to smooth self-adjoint sections $s \in \Omega^0(\text{End}E)$ by

$$N(s) = \left(\int_X \tau(s)^2 \right)^{1/2}$$

where orbifold integration is understood, i.e. over a neighborhood of the form $U \approx \tilde{U}/\Gamma_U$, we integrate by

$$\frac{1}{|\Gamma_U|} \int_{\tilde{U}} \tau(\tilde{s})^2.$$

Since N is norm equivalent to the usual L^2 norm, it extends to L^2 sections. If $\{s_i \in L^2(\Omega^0(\text{End}E)\}$ is a sequence and $\tau_\infty = \liminf \tau(s_i)$, then by Fatou's Lemma, we observe that $\|\tau_\infty\|_{L^2} \leq \liminf N(s_i)$. Define $J(A)$ for an L^2_1 connection A by

$$J(A) = N\left(\frac{*F_A}{2\pi i} + \mu \cdot I \right).$$

By the previous observation J is upper-semicontinuous, i.e. if $A_i \to B$ weakly in L^2_1, then $J(B) \leq \liminf J(A_i)$.

Also $J(A) = 0 \Leftrightarrow A$ is of the type required by the theorem. We will minimize $J(A)$ along a gauge orbit to obtain a connection A with $J(A) = 0$. The pertinent gauge group here is the complexified gauge group $\mathcal{G}^{\mathbf{C}}_{orb}$ of orbifold gauge transformations which are general linear in each fiber. These are precisely the bundle automorphisms of E preserving its holomorphic structure. Consider a connection A, and decompose d_A into the $(1,0)$ and $(0,1)$ components

$$d_A = d'_A + d''_A.$$

If $g \in \mathcal{G}^{\mathbf{C}}_{orb}$, then it acts on a connection d_A by

$$d'_{g(A)} = g \circ d'_A \circ g^{-1} = d'_A + g(d'_A g^{-1})$$

$$d''_{g(A)} = g^{*-1} \circ d''_A \circ g^* = d''_A + g^{*-1}(d''_A g^*).$$

Thus, $g(A) = A + a$, where

$$a = gd'_A g^{-1} + g^{*-1} d''_A g^*.$$

The curvature transforms by $F_{A+a} = F_A + d_A a + a \wedge a$. This gives

$$F_{g(A)} = F_A + d''_A(gd'_A g^{-1}) + d'_A(g^{*-1} d''_A g^*) + g(d'_A g^{-1}) g^{*-1}(d''_A g^*) + g^{*-1}(d''_A g^*) g(d'_A g^{-1}).$$

Using $F_A = d'_A d''_A + d''_A d'_A$, we can write this more conveniently as

$$\begin{aligned} g^{-1} F_{g(A)} g &= F_A + d''_A(h^{-1} d'_A h) \\ &= F_A + h^{-1}(d''_A d'_A h - d''_A h h^{-1} d'_A h) \end{aligned}$$

where $h = g^* g$.

First, we need the following theorem of Uhlenbeck (adapted to orbifolds, see [5])

Proposition 5.2 *Suppose A_i is a sequence of L^2_1 connections with $\|F_{A_i}\|_{L^2}$ bounded. Then $\exists$ a subsequence $\{i'\}$ and L^2_2 gauge transformations $g_{i'}$ so that $g_{i'}(A_{i'})$ converges weakly in L^2_1.*

Suppose E is a holomorphic orbifold bundle and A any connection compatible with E. Let $\mathcal{G}^{\mathbf{C}}_{orb}(A)$ be the gauge orbit of A in $\mathcal{A}$. For any orbifold connection A' on E (not necessarily compatible with the holomorphic structure), let $\mathcal{E}_{A'}$ be the parabolic bundle obtained by pushing forward E with holomorphic structure induced by A', namely $d''_{A'}$. With this notation, we are ready to prove the following consequence of Proposition 5.2,

Lemma 5.3 *Either* $\inf\{J(A')|A' \in \mathcal{G}^{\mathbf{C}}_{orb}(A)\}$ *is obtained in $\mathcal{G}^{\mathbf{C}}_{orb}(A)$, or $\exists$ a unitary connection B on E so that $\mathcal{E}_A$ and $\mathcal{E}_B$ are not isomorphic, but have the same rank, degree, and parabolic degree, and satisfy*

1. $J(B) \leq \inf\{J(A')|A' \in \mathcal{G}^{\mathbf{C}}_{orb}(A)\}$
2. $\mathrm{ParHom}(\mathcal{E}_A, \mathcal{E}_B) \neq 0$

Proof: Choose $A_i \in \mathcal{G}^{\mathbf{C}}_{orb}(A)$ a minimizing sequence for J. Because N is norm-equivalent to the L^2 norm, it follows that $\|F_{A_i}\|_{L^2}$ is bounded. Applying proposition 5.2 (with a mild abuse of notation), we obtain a subsequence of connections A_i and gauge transformations g_i so that $g_i(A_i) \to B$ weakly in L^2_1. Since J is upper-semicontinuous, we have

$$J(B) \leq \liminf J(A_i) = \inf\{J(A')|A' \in \mathcal{G}^{\mathbf{C}}_{orb}(A)\}.$$

To complete the proof, we need to show that $\mathrm{ParHom}(\mathcal{E}_A, \mathcal{E}_B) \neq 0$, the conclusion of the theorem being established if $\mathcal{E}_A \approx \mathcal{E}_B$ or not. Using A^* on E^* and B on E we construct the connection $A^* \otimes 1 + 1 \otimes B$ on $E^* \otimes E = \mathrm{Hom}(E, E)$. Consider the $(0,1)-$component of this, namely

$$d''_{AB} : \Omega^0(\mathrm{Hom}(E, E)) \longrightarrow \Omega^{0,1}(\mathrm{Hom}(E, E)).$$

Then by Proposition 4.5, $s \in \ker(d''_{AB}) \Leftrightarrow \hat{s} \in \mathrm{ParHom}(\mathcal{E}_A, \mathcal{E}_B)$, so we need to show

Claim: $\ker(d''_{AB}) \neq 0$.
Suppose otherwise. Since d''_{AB} is first order elliptic, we have

$$\|d''_{AB}(s)\|_{L^2} \geq c\|s\|_{L^2_1} \text{ for some } c > 0, \text{ and all s.}$$

By the Sobolev inequalities $L^2_1 \hookrightarrow L^4$, we have $\|s\|_{L^4} \leq c_1\|s\|_{L^2_1} \Rightarrow$

$$\|d''_{AB}(s)\|_{L^2} \geq c_2\|s\|_{L^4}$$

Now $A_i \to B$ converges weakly in L^2_1, and so by the Sobolev inequalities, it converges in L^4. Thus

$$\|d''_{AB}(s)\|_{L^2} - \|d''_{AA_i}(s)\|_{L^2} \leq \|d''_{AB}(s) - d''_{AA_i}(s)\|_{L^2} \leq c_3\|B - A_i\|_{L^4}\|s\|_{L^4},$$

where the first estimate is just the triangle inequality, and the second is seen by noticing that $d''_{AB} - d''_{AA_i}$ is the $(0,1)-$component of $B - A_i$. It follows that

$$\begin{aligned}\|d''_{AA_i}(s)\|_{L^2} &\geq \|d''_{AB}(s)\|_{L^2} - c_3\|B - A_i\|_{L^4}\|s\|_{L^4} \\ &\geq (c_2 - c_3\|B - A_i\|_{L^4})\|s\|_{L^4} \\ &\geq c\|s\|_{L^4} \text{ for some } c > 0\end{aligned}$$

where the last inequality follows by choosing i large. This holds for all s, contradicting the fact that $\ker(d''_{AA_i}) \neq 0$. ∎

We now need two estimates (Lemmas 5.4 & 5.5) to show that if $\mathcal{E}$ is parabolic *stable*, then the second case of Lemma 5.3 cannot hold. To this end, recall from §2 that given any short exact sequence of holomorphic orbifold bundles

$$0 \to P \to E \to Q \to 0 \tag{10}$$

then a hermitian structure on E determines a C^∞ splitting of (10), and the second fundamental form $\beta \in \Omega^{0,1}(Q^* \otimes P)$ is the obstruction to this splitting being holomorphic. In terms of a unitary connection A on E and the induced connections on P, Q denoted by A_P, A_Q, we see that in this splitting, A has the form

$$A = \begin{pmatrix} A_P & \beta \\ -\beta^* & A_Q \end{pmatrix}$$

and the curvature F_A has the form

$$F_A = \begin{pmatrix} F_P - \beta \wedge \beta^* & d\beta \\ -d\beta^* & F_Q - \beta^* \wedge \beta \end{pmatrix},$$

where $d : \Omega^1(P^* \otimes Q) \to \Omega^2(P^* \otimes Q)$ is the covariant derivative of the connection $A_P^* \otimes 1 + 1 \otimes A_Q$. Of course we can push forward the entire sequence in (10) to obtain a short exact sequence of parabolic bundles

$$0 \to \mathcal{P} \to \mathcal{E} \to \mathcal{Q} \to 0. \tag{11}$$

It follows that β is the obstruction to this sequence admitting a parabolic splitting.

Remark: This notation for the second fundamental form is the adjoint of that in [8]. The important point here is that $\beta^* \wedge \beta$ and $-\beta \wedge \beta^*$ are positive semidefinite (1,1)-forms. Since $*(d\bar{z} \wedge dz) = 2i = - * (dz \wedge d\bar{z})$ we can normalize so that

$$*tr(\beta \wedge \beta^*) = - * tr(\beta^* \wedge \beta) = 2\pi i |\beta|^2$$

Lemma 5.4 *Suppose E is a holomorphic orbifold bundle with parabolic push forward $\mathcal{F}$. Then if $0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{N} \to 0$ is any short exact sequence with $\mu(\mathcal{M}) \geq \mu(\mathcal{F})$ ($\Rightarrow \mu(\mathcal{F}) \geq \mu(\mathcal{N})$), then for any unitary connection B compatible with E, we have*

$$J(B) \geq \mathrm{rank}(\mathcal{M})(\mu(\mathcal{M}) - \mu(\mathcal{F})) + \mathrm{rank}(\mathcal{N})(\mu(\mathcal{F}) - \mu(\mathcal{N})) \stackrel{def}{=} J_0$$

with equality $\Leftrightarrow$ the sequence splits.

Remark: Note that by hypothesis, $J_0 \geq 0$. We first show how this lemma proves ($\Rightarrow$) of theorem 5.1. For suppose E is an indecomposable holomorphic orbifold bundle with unitary connection A and $J(A) = 0$. Then if $\mathcal{M}$ is a proper parabolic subbundle of $\mathcal{E}$ we have $\mu(\mathcal{M}) < \mu(\mathcal{E})$. Otherwise, by the lemma $J_0 = 0 = J(A) \Rightarrow \mathcal{E}$ decomposes, which is a contradiction. Thus $\mathcal{E}$ is stable.

Proof: Set $\mu = \mu(\mathcal{F})$. Following the notation introduced above, for any B, we have

$$F_B = \begin{pmatrix} F_{\mathcal{M}} - \beta \wedge \beta^* & d\beta \\ -d\beta^* & F_{\mathcal{N}} - \beta^* \wedge \beta \end{pmatrix}$$

where $F_{\mathcal{M}} = F_{B_{\mathcal{M}}}$ and $F_{\mathcal{N}} = F_{B_{\mathcal{N}}}$. Note that $B_{\mathcal{M}}$ and $B_{\mathcal{N}}$ are the induced connections on the pullbacks of $\mathcal{M}$ and $\mathcal{N}$ respectively. ¿From the properties of τ on block matrices, it follows that

$$\tau\left(\frac{*F_B}{2\pi i} + \mu \cdot I_{\mathcal{F}}\right) \geq \left| tr\left(\frac{*(F_{\mathcal{M}} - \beta \wedge \beta^*)}{2\pi i} + \mu \cdot I_{\mathcal{M}}\right)\right| + \left| tr\left(\frac{*(F_{\mathcal{N}} - \beta \wedge \beta^*)}{2\pi i} + \mu \cdot I_{\mathcal{N}}\right)\right|$$

Thus, by Cauchy-Schwarz we see

$$\begin{aligned} J(B) &= \left(\int_X \tau\left(\frac{*F_B}{2\pi i} + \mu \cdot I_{\mathcal{F}}\right)^2\right)^{1/2} \\ &\geq \left|\int_X tr\left(\frac{*(F_{\mathcal{M}} - \beta \wedge \beta^*)}{2\pi i} + \mu \cdot I_{\mathcal{M}}\right)\right| + \left|\int_X tr\left(\frac{*(F_{\mathcal{N}} - \beta^* \wedge \beta)}{2\pi i} + \mu \cdot I_{\mathcal{N}}\right)\right| \\ &= \left|\int_X tr\left(\frac{*F_{\mathcal{M}}}{2\pi i} + \mu \cdot I_{\mathcal{M}}\right) - |\beta|^2\right| + \left|\int_X tr\left(\frac{*F_{\mathcal{N}}}{2\pi i} + \mu \cdot I_{\mathcal{N}}\right) + |\beta|^2\right| \\ &= \left|\int_X tr\left(\frac{*F_{\mathcal{M}}}{2\pi i} + \mu \cdot I_{\mathcal{M}}\right)\right| + \left|\int_X tr\left(\frac{*F_{\mathcal{N}}}{2\pi i} + \mu \cdot I_{\mathcal{N}}\right)\right| + 2\|\beta\|^2 \\ &= \mathrm{rank}(\mathcal{M})(\mu(\mathcal{M}) - \mu(\mathcal{F})) + \mathrm{rank}(\mathcal{N})(\mu(\mathcal{F}) - \mu(\mathcal{N})) + 2\|\beta\|^2 \end{aligned}$$

where the last two steps hold because

$$\begin{aligned}
\int_X tr\left(\frac{*F_{\mathcal{M}}}{2\pi i}+\mu\cdot I_{\mathcal{M}}\right) &= \operatorname{rank}(\mathcal{M})(\mu(\mathcal{F})-\mu(\mathcal{M}))\le 0\\
\int_X tr\left(\frac{*F_{\mathcal{N}}}{2\pi i}+\mu\cdot I_{\mathcal{N}}\right) &= \operatorname{rank}(\mathcal{N})(\mu(\mathcal{F})-\mu(\mathcal{N}))\ge 0
\end{aligned}$$

by hypothesis. Furthermore, equality above implies $\beta=0$, which is equivalent to a holomorphic splitting of the sequence. ∎

For the second estimate, we again look at short exact sequences of holomorphic orbifold bundles, except that now we assume the middle term E has parabolic stable push forward $\mathcal{E}$.

Lemma 5.5 *Suppose E is a holomorphic orbifold bundle of rank n, and that its push forward bundle $\mathcal{E}$ is parabolic stable. Assuming (by induction) that theorem 5.1 is true for bundles with rank $< n$, then given any short exact sequence*

$$0\to\mathcal{P}\to\mathcal{E}\to\mathcal{Q}\to 0$$

of parabolic bundles, then ∃ an orbifold connection A compatible with E (i.e. $\mathcal{E}\approx\mathcal{E}_A$) so that

$$J(A)<\operatorname{rank}\mathcal{P}(\mu(\mathcal{E})-\mu(\mathcal{P}))+\operatorname{rank}\mathcal{Q}(\mu(\mathcal{Q})-\mu(\mathcal{E}))\stackrel{def}{=}J_1.$$

Note: since $\mathcal{E}$ is parabolic stable, $\mu(\mathcal{P})<\mu(\mathcal{E})$ and $\mu(\mathcal{E})<\mu(\mathcal{Q})$, thus J_1 is positive.

Proof: To any parabolic bundle, we have a canonical (Harder-Narasimhan-parabolic) filtration (see [18]). Applying this to $\mathcal{P}$, we get

$$0\subset\mathcal{P}_1\subset\mathcal{P}_2\subset\ldots\subset\mathcal{P}_p=\mathcal{P}$$

so that each quotient $\mathcal{M}_i=\mathcal{P}_i/\mathcal{P}_{i-1}$ is semistable with decreasing slopes $\mu_i=\mu(\mathcal{M}_i)$. Note that $\mu_i\le\mu_1=\mu(\mathcal{P}_1)<\mu(\mathcal{E})$ by stability. Now $\mathcal{M}_i$ is semistable, thus has a filtration of the form

$$0\subset(\mathcal{M}_i)_1\subset(\mathcal{M}_i)_2\subset\ldots\subset(\mathcal{M}_i)_{m_i}=(\mathcal{M}_i),$$

each of whose quotients $\mathcal{C}_{ij}=(\mathcal{M}_i)_j/(\mathcal{M}_i)_{j-1}$ is stable with slope $\mu(\mathcal{C}_{ij})=\mu_i$. Although this filtration is not canonical, the isomorphism class of

$$\operatorname{Gr}(\mathcal{M}_i)\stackrel{def}{=}\bigoplus_{j=1}^{m_i}\mathcal{C}_{ij}$$

depends only on that of $\mathcal{M}_i$ (see p.71 of [18] for details). Since $\operatorname{rank}(\mathcal{C}_{ij})<\operatorname{rank}(\mathcal{E})$, we can apply the inductive hypothesis to each $\mathcal{C}_{ij}$. To facilitate our discussion, we will adopt the following breach of ethics, namely we will say "A is a connection on $\mathcal{E}$" when we really mean that A is an orbifold connection on E whose push forward is $\mathcal{E}$, i.e. $\mathcal{E}_A\approx\mathcal{E}$. With the aforementioned amoralities, we apply theorem 5.1 to get a connection A_{ij} on $\mathcal{C}_{ij}$ whose curvature F_{ij} satisfies $*F_{ij}=-2\pi i\mu_i$. Since $\mathcal{C}_{i1}=(\mathcal{M}_i)_1$, this gives a connection $A_{(\mathcal{M}_i)_1}=A_{i1}$ on $(\mathcal{M}_i)_1$ for each i. Furthermore, given connections $A_{(\mathcal{M}_i)_j}$ on $(\mathcal{M}_i)_j$ and A_{ij+1} on $\mathcal{C}_{ij+1}$ along with a choice of second fundamental form $\beta_{ij} \in \Omega^{0,1}(\mathcal{C}^*_{ij+1} \otimes (\mathcal{M}_i)_j)$ for the exact sequence

$$0 \to (\mathcal{M}_i)_j \to (\mathcal{M}_i)_{j+1} \to \mathcal{C}_{ij+1} \to 0,$$

these determine a connection on $(\mathcal{M}_i)_{j+1}$ by the formula

$$A^t_{(\mathcal{M}_i)_{j+1}} = \begin{pmatrix} A_{(\mathcal{M}_i)_j} & t\beta_{ij} \\ -t\beta^*_{ij} & A_{ij+1} \end{pmatrix}.$$

Proceeding inductively, we get connections A^t_i on each $\mathcal{M}_i$. Since $\mathcal{P}_1 = \mathcal{M}_1$, this gives a connection on $\mathcal{P}_1$. The same argument applied to the short exact sequences

$$0 \to \mathcal{P}_i \to \mathcal{P}_{i+1} \to \mathcal{M}_i \to 0$$

gives, at long last, a connection $A^t_{\mathcal{P}}$ on $\mathcal{P}$. Then $A^t_{\mathcal{P}} \to A^0_{\mathcal{P}}$ as $t \to 0$, where $A^0_{\mathcal{P}}$ is a connection on $\bigoplus_i \mathrm{Gr}(\mathcal{M}_i)$ whose curvature satisfies $*F^0_{\mathcal{P}} = -2\pi i \Lambda_{\mathcal{P}}$. Here $\Lambda_{\mathcal{P}}$ is the diagonal matrix

$$\Lambda_{\mathcal{P}} = \begin{pmatrix} \mu_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \mu_1 & & & & \\ & & & \ddots & & & \\ & & & & \mu_p & & \\ & & & & & \ddots & \\ 0 & & & & & & \mu_p \end{pmatrix}.$$

where μ_i is repeated $\dim(\mathcal{M}_i)$ times. Notice that by construction

$$\begin{aligned} tr(\frac{*F^0_{\mathcal{P}}}{2\pi i} + \mu(\mathcal{E}) \cdot I_{\mathcal{P}}) &= tr(\mu(\mathcal{E}) \cdot I_{\mathcal{P}} - \Lambda_{\mathcal{P}}) \\ &= \mathrm{rank}(\mathcal{P})\mu(\mathcal{E}) - \sum_i \mathrm{pardeg}(\mathcal{M}_i) \\ &= \mathrm{rank}(\mathcal{P})\mu(\mathcal{E}) - \mathrm{pardeg}(\mathcal{P}) \\ &= \mathrm{rank}(\mathcal{P})(\mu(\mathcal{E}) - \mu(\mathcal{P})) \end{aligned}$$

The same considerations applied to $\mathcal{Q}$ yield the filtration

$$0 \subset \mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \ldots \subset \mathcal{Q}_q = \mathcal{Q}$$

with semistable quotients $\mathcal{N}_i = \mathcal{Q}_i/\mathcal{Q}_{i-1}$ whose slopes $\lambda_i = \mu(\mathcal{N}_i)$ are decreasing. Note that $\lambda_i \geq \lambda_q > \mu(\mathcal{E})$ by stability. A construction similar to that given above builds connections $A^t_{\mathcal{Q}}$ on $\mathcal{Q}$ so that $A^t_{\mathcal{Q}} \to A^0_{\mathcal{Q}}$ as $t \to 0$, where $A^0_{\mathcal{Q}}$ is a connection on $\bigoplus_i \mathrm{Gr}(\mathcal{N}_i)$ whose curvature satisfies $*F^0_{\mathcal{Q}} = -2\pi i \Lambda_{\mathcal{Q}}$ where

$$\Lambda_{\mathcal{Q}} = \begin{pmatrix} \lambda_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_q & & \\ & & & & & \ddots & \\ 0 & & & & & & \lambda_q \end{pmatrix}.$$

Again notice that

$$tr(\frac{*F^0_{\mathcal{Q}}}{2\pi i} + \mu(\mathcal{E}) \cdot I_{\mathcal{Q}}) = \mathrm{rank}(\mathcal{Q})(\mu(\mathcal{E}) - \mu(\mathcal{Q})) \tag{12}$$

Using the connections $A^t_{\mathcal{P}}$ and $A^t_{\mathcal{Q}}$ constructed above, we get an operator d_t on $\mathcal{Q}^* \otimes \mathcal{P}$. (Actually, the breach of ethics is rather severe here, since d_t is really the covariant derivative on the orbifold bundle corresponding to the pullback of $\mathcal{Q}^* \otimes \mathcal{P}$.) For each t, we choose a harmonic (with respect to d_t) representative β_t of the extension class of $\mathcal{E}$. By changing the scale, we may assume $\|\beta_t\|_{L^2} = 1$ Also,

$$\|\beta_t\|_{L^2_{k+1}} \leq c_t(\|d_t\beta_t\|_{L^2_k} + \|\beta_t\|_{L^2}).$$

The c_t can be uniformly bounded, since $d_t \to d_0$. Thus there is a uniform bound for $\|\beta_t\|_{C^o}$. This gives the connection A_{st} on $\mathcal{E}$

$$A_{st} = \begin{pmatrix} A^t_{\mathcal{P}} & s\beta_t \\ -s\beta^*_t & A^t_{\mathcal{Q}} \end{pmatrix}$$

with curvature

$$F_{st} = \begin{pmatrix} F^t_{\mathcal{P}} - s^2\beta_t \wedge \beta^*_t & 0 \\ 0 & F^t_{\mathcal{Q}} - s^2\beta^*_t \wedge \beta_t \end{pmatrix}.$$

Now $-\Lambda_{\mathcal{P}} + \mu(\mathcal{E}) \cdot I_{\mathcal{P}}$ has all positive eigenvalues, and so do sufficiently close operators. For these operators $\tau(\cdot) = tr(\cdot)$. Furthermore, $*tr(\beta_t \wedge \beta^*_t) = -2\pi i|\beta_t|^2$. Using this and formula (12) we see for small s and t,

$$\begin{aligned} \tau(\frac{*(F^t_{\mathcal{P}} - s^2\beta_t \wedge \beta^*_t)}{2\pi i} + \mu(\mathcal{E}) \cdot I_{\mathcal{P}}) &= tr(\frac{*F^t_{\mathcal{P}}}{2\pi i} + \mu(\mathcal{E}) \cdot I_{\mathcal{P}}) - s^2|\beta_t|^2 \\ &= tr(\frac{*F^0_{\mathcal{P}}}{2\pi i} + \mu(\mathcal{E}) \cdot I_{\mathcal{P}}) - s^2|\beta_t|^2 + \delta_1(t) \\ &= \mathrm{rank}(\mathcal{P})(\mu(\mathcal{E}) - \mu(\mathcal{P})) - s^2|\beta_t|^2 + \delta_1(t) \end{aligned}$$

where $\delta_1(t) \to 0$ as $t \to 0$.

Similarly, $-\Lambda_{\mathcal{Q}} + \mu(\mathcal{E}) \cdot I_{\mathcal{Q}}$ has all negative eigenvalues. And so for sufficiently close operators, $\tau(\cdot) = -tr(\cdot)$. Also $*tr(\beta_t^* \wedge \beta_t) = 2\pi i|\beta_t|^2$, and so by formula (12)

$$\begin{aligned}
\tau(\frac{*(F_{\mathcal{Q}}^t - s^2\beta_t^* \wedge \beta_t)}{2\pi i} + \mu(\mathcal{E}) \cdot I_{\mathcal{Q}}) &= -tr(\frac{*F_{\mathcal{Q}}^t}{2\pi i} + \mu(\mathcal{E}) \cdot I_{\mathcal{Q}}) - s^2|\beta_t|^2 \\
&= -tr(\frac{*F_{\mathcal{Q}}^0}{2\pi i} + \mu(\mathcal{E}) \cdot I_{\mathcal{Q}}) - s^2|\beta_t|^2 + \delta_2(t) \\
&= \mathrm{rank}(\mathcal{Q})(\mu(\mathcal{Q}) - \mu(\mathcal{E})) - s^2|\beta_t|^2 + \delta_2(t)
\end{aligned}$$

Putting this all together, we see that for small s and t,

$$\tau(\frac{*F_{st}}{2\pi i} + \mu(\mathcal{E}) \cdot I_{\mathcal{E}}) = J_1 - 2s^2|\beta_t|^2 + \delta(t).$$

It follows that

$$\begin{aligned}
J(A_{st})^2 &= \int_X (J_1 - 2s^2|\beta_t|^2 + \delta(t))^2 \\
&= J_1^2 + 4\int_X s^4|\beta_t|^4 - J_1 s^2|\beta_t|^2 + \delta'(t)
\end{aligned}$$

Since we have a uniform bound on $\|\beta_t\|_{C^o}$, we can choose s small enough so that

$$J_1 s^2 = J_1 s^2 \int_X |\beta_t^2| > s^4 \int_X |\beta_t|^4.$$

Then by choosing t so that $\delta(t)$ is negligible, then $J(A_{st}) < J_1$ as required. ∎

We are now ready to prove ($\Leftarrow$) of theorem 5.1. Suppose that $\mathcal{E}$ is stable and that the theorem has been proved for bundles of lower rank. Let A' be a unitary connection in E. Then

Claim: $\inf\{J(A)|A \in \mathcal{G}_{orb}^{\mathbf{C}}(A')\}$ is attained in $\mathcal{G}_{orb}^{\mathbf{C}}(A')$.

For if not, then by Lemma 5.3, we have a connection B and parabolic bundle $\mathcal{F} \stackrel{def}{=} \mathcal{E}_B$ with same rank, degree, parabolic structure as $\mathcal{E}$ so that $J(B) < \inf\{J(A)|A \in \mathcal{G}_{orb}^{\mathbf{C}}(A')\}$ and $\mathrm{ParHom}(\mathcal{E},\mathcal{F}) \neq 0$. Choosing $\alpha \neq 0 \in \mathrm{ParHom}(\mathcal{E},\mathcal{F})$, by Proposition 3.8 we have the canonical factorization of α

$$\begin{array}{ccc}
0 \to \mathcal{P} \to \mathcal{E} \stackrel{\pi}{\to} & \mathcal{Q} & \to 0 \\
& \downarrow_{\beta} & \\
0 \leftarrow \mathcal{N} \leftarrow \mathcal{F} \stackrel{\imath}{\leftarrow} & \mathcal{M} & \leftarrow 0
\end{array}$$

where $\alpha = \imath \circ \beta \circ \pi$, $\mathrm{rank}(\mathcal{M}) = \mathrm{rank}(\mathcal{Q})$ and $\mathrm{pardeg}(\mathcal{M}) \geq \mathrm{pardeg}(\mathcal{Q})$. Notice that

$$\mu(\mathcal{M}) \geq \mu(\mathcal{Q}) > \mu(\mathcal{E}) = \mu(\mathcal{F}).$$

¿From Lemma 5.4 applied to the bottom row we get that

$$J(B) \geq J_0.$$

Moreover, applying Lemma 5.5 to the top row we get a connection A on $\mathcal{E}$ with

$$J(A) < J_1.$$

But since $\mu(\mathcal{Q}) \leq \mu(\mathcal{M}), \mu(\mathcal{E}) = \mu(\mathcal{F})$ and $\mu(\mathcal{P}) \geq \mu(\mathcal{N})$, we see $J_0 \geq J_1$ and so

$$J(B) \geq J_0 \geq J_1 > J(A),$$

a contradiction. This proves the claim.

Now we must show that $J(A) = 0$ for this minimizing connection $A \in \mathcal{G}^{\mathbf{C}}_{orb}(A')$. Suppose not. Then, because E is indecomposable, $\ker d_A^* d_A =$ constant scalars, for if $s \in \ker d_A^* d_A$ is a self-adjoint section of $\mathrm{End}(E)$, then the eigenspaces of s give a holomorphic splitting of E. Projecting $*F_A/2\pi i$ onto $\ker d_A^* d_A$, we get

$$Proj(\frac{*F_A}{2\pi i}) = -\mu(\mathcal{E}) \cdot I.$$

Using the Inverse Function Theorem (working $\perp \ker d_A^* d_A$) we get a self-adjoint section of $h \in \Omega^0(\mathrm{End}(E))$ with $id_A^* d_A(h) = *F_A + 2\pi i \mu \cdot I$. Set $g_t = 1 - th$. Then for t small, $g_t \in \mathcal{G}^{\mathbf{C}}_{orb}$. If $A_t = g_t(A)$, then

$$A_t = A + g_t d'_A g_t^{-1} + g_t^{-1} d''_A g_t.$$

and

$$\begin{aligned} F_{A_t} &= F_A + d''_A(g_t d'_A g_t^{-1}) + d'_A(g_t^{-1} d''_A g_t) + g_t(d'_A g_t^{-1}) g_t^{-1}(d''_A g_t) + g_t^{-1}(d''_A g_t) g_t (d'_A g_t^{-1}) \\ &= F_A + t(d''_A d'_A - d'_A d''_A)h + q(t,h) \end{aligned}$$

where $\|q(t,h)\|_{L_2} \leq c_0 t^2 \|h\|$. Using the fact that $*(d''_A d'_A - d'_A d''_A) = -id_A^* d_A$ we see

$$\begin{aligned} \frac{*F_{A_t}}{2\pi i} + \mu \cdot I &= \frac{*F_A - itd_A^* d_A(h)}{2\pi i} + \mu \cdot I + q(t,h)/2\pi i \\ &= \left(\frac{*F_A}{2\pi i} + \mu \cdot I\right)(1-t) + q(t,h)/2\pi i \end{aligned}$$

And it follows that

$$J(A_t) = J(A)(1-t) + O(t^2).$$

So in order for $J(A)$ to be a minimum, we must have $J(A) = 0$.

As for uniqueness, suppose A and $B = g(A)$ are two connections so that $F_A = F_B = \mu \cdot I$. Writing $g = u \cdot g'$ where $u \in \mathcal{G}_{orb}$ and g' is self-adjoint, by unitary invariance of $J(A)$, we can assume $g = g'$. We see that

$$F_A = F_{g(A)} = g^{-1} F_{g(A)} g \Rightarrow d''_A d'_A g^* g = d'' g^* g g^{-1} g^{*-1} d'_A g^* g.$$

Now, using the fact that $g = g^*$, we get

$$d''_A d'_A g^2 = d'' g^2 g^{-2} d'_A g^2 = -\{(d'' g^2) g^{-1}\}\{(d'' g^2) g^{-1}\}^*.$$

Taking the trace $\tau = tr(g^2)$, it follows that $\Delta\tau \leq 0$. Now by the maximum principle, we get that $\Delta\tau = 0$ and so $d''_A g^2 = 0 = d'_A g^2$. Thus, since the bundle is indecomposable, it follows that g is a constant scalar, and so $A = B$. ∎

6 Applications

Using the inductive procedure of Atiyah and Bott, adapted to parabolic bundles as in [17], we compute $\mathrm{H}^*(\mathcal{S})$, where $\mathcal{S}$ is the moduli of stable parabolic bundles. For simplicity, we assume the bundles are rank 2 and parabolically flat. In applications, we often restrict further to the cases where the underlying Riemann surface X is either the Riemann sphere or the torus. This is because by computing $\mathrm{H}^*(\mathcal{S})$, we can deduce the cohomology of the SU(2) representation space of any torsion free Seifert fibration over S^2 or T^2 (see Theorem 6.4 and formula (20)) . This includes all the Seifert fibered homology spheres, for example. As a consequence of this and [6], we get information about Casson's invariant and so also the Floer homology of these homology spheres.

For starters observe that as a consequence of Grothendieck's theorem [9], the assumption genus = 0 gives a rather dull moduli space in the case of non-parabolic bundles (this is because only line bundles are stable). In fact, the case of parabolic bundles over S^2 is only interesting when there are many (i.e. > 2) parabolic points. In the rank 2 case, $\mathcal{S}$ is a smooth complex manifold of complex dimension $n-3$, where $n =$ the number of parabolic points. Because the authors of [14] concentrate on the $n = 1$ case, they assume genus ≥ 2, which is necessary for a nontrivial moduli space. We developed Theorem 5.1, the natural generalization of [4] and [14], because we wanted a representation theoretic interpretation for $\mathcal{S}$ for all genus (compare Theorem 4.1 of [14]).

This section is divided into eight parts. The first section gives a brief account of equivariant cohomology. The second describes the stratification on the space $\mathcal{C}$ of holomorphic structures arising from the Harder-Narasimhan parabolic filtration. The third introduces the gauge groups $\mathcal{G}^{\mathbf{C}}$ and $\mathcal{P}$. In the fourth, using a fact (due to Nitsure) that the filtration is equivariantly perfect, we derive a formula for the equivariant homology of the semistable bundles. The fifth section shows how to deduce the singular homology of the moduli space $\mathcal{S}$ of stable bundles in the case when semistable = stable. The main issue is that $\mathrm{H}^*(\mathcal{S})$ is torsion free. In the sixth section, we interpret these formulas in the case where X has genus 0 and 1. The seventh section shows how this relates to the cohomology of the $SU(2)$ representation space of cerain Seifert-fibered spaces. And in the last section, we perform explicit computations of $\mathrm{H}^*(\mathcal{S})$.

6.1 Equivariant cohomology

For a topological group G and any G-space Y, consider the universal bundle $G \to EG \to BG$. Let

$$Y_G = EG \times_G Y = EG \times Y/\sim \text{ where } (eg, y) \sim (e, gy).$$

Then we have the fibration $Y \to Y_G \to BG$, and the equivariant cohomology of Y is defined by $\mathrm{H}^*_G(Y) = \mathrm{H}^*(Y_G)$. If the G-action on Y is *free*, then $Y_G \simeq Y/G$. It follows that

$$\mathrm{H}^*_G(Y) = \mathrm{H}^*(Y/G).$$

On the other hand, if the action is trivial, then $Y_G \simeq BG \times Y$ and so

$$\mathrm{H}^*_G(Y) = \mathrm{H}^*(BG \times Y).$$

Also, if Y is contractible, then $Y_G \simeq BG$ and so

$$\mathrm{H}^*_G(Y) = \mathrm{H}^*(BG).$$

In the course of the argument, we will need the following

Proposition 6.1 *Suppose H is a normal subgroup of G which acts trivially on Y so that the quotient $\bar{G} = G/H$ acts freely. Suppose further that the fibration $BH \to BG \to B\bar{G}$ is trivial. Then, $Y_G = BH \times Y/G$. If, in addition, BH and Y_G are torsion-free, then Y/G is torsion-free and* $\mathrm{H}^*_G(Y) = \mathrm{H}^*(BH) \otimes \mathrm{H}^*(Y/G)$.

Proof: Since $BG = BH \times B\bar{G}$, we see that $EG = EH \times E\bar{G}$. So

$$Y_G = EG \times_G Y = (EH \times E\bar{G}) \times_G Y = BH \times (E\bar{G} \times_{\bar{G}} Y)$$

because the action of H is trivial on both Y and $E\bar{G}$. So, $Y_G = BH \times Y_{\bar{G}}$. Now since $\bar{G}$ acts freely, $Y_{\bar{G}} = Y/\bar{G} = Y/G$. The rest now follows from the Kunneth theorem.∎

6.2 The filtration on $\mathcal{C}$

Fix E a rank 2, C^∞ bundle over a Riemann surface X of genus g. Suppose that E has a topological parabolic structure, i.e. over the finite set $\{p_i\}_1^n \subset X$ of parabolic points, we have weighted flags

$$E_{p_i} = F_1^i \supset F_2^i$$
$$0 \leq \alpha_1^i < \alpha_2^i < 1.$$

Further assume that E is parabolically flat i.e.

$$\mathrm{pardeg}(E) \stackrel{def}{=} \deg(E) + \sum_{i=1}^{n} (\alpha_1^i + \alpha_2^i) = 0.$$

Remark: Temporarily ignore the possibility of trivial flags, which are ones of the form $E_p = F_1$ with one weight of multiplicity 2 because trivial flags impose no restictions on parabolic automorphisms of E, and in fact, their sole effect is that they contribute to the parabolic degree when the weight is nontrivial.

Consider all holomorphic structures d'' on E, namely $\mathbf{C}-$linear operators

$$d'' : \Omega^0(E) \to \Omega^{0,1}(E)$$

satisfying $d''(fs) = (\overline{\partial} f)s + f(d''s)$ for $f \in C^\infty(X)$ and $s \in \Omega^0(E)$. Because X is a complex curve, the integrability condition $d'' \circ d'' = 0$ is automatically satisfied, thus by the Newlander-Nirenberg theorem, each d'' determines a holomorphic bundle (with parabolic structure) which is denoted by $\mathcal{E}$. Let $\mathcal{C}$ be the space of all holomorphic structures. Then $\mathcal{C}$ is an ∞-dimensional affine space modeled on $\Omega^{0,1}(\mathrm{End}\, E)$. To see this, consider two operators $d_1'', d_2'' \in \mathcal{C}$. Then the difference $d_1'' - d_2'' : \Omega^0(E) \to \Omega^{0,1}(E)$ is linear over $C^\infty(X)$, thus $d_1'' - d_2'' \in \Omega^{0,1}(\mathrm{End}\, E)$.

Recalling Definition 3.9, let $\mathcal{C}_s$ and $\mathcal{C}_{ss}$ be the subspaces of $\mathcal{C}$ of parabolic stable and semistable structures. For any bundle $\mathcal{E} \in \mathcal{C} \setminus \mathcal{C}_{ss}$, there is a unique destabilizing line subbundle L of $\mathcal{E}$, where

$$0 \to L \to \mathcal{E} \to Q \to 0$$

is a short exact sequence of parabolic bundles and

$$\mathrm{pardeg}(L) > 0 \ \ (\Leftrightarrow \mathrm{pardeg}(Q) < 0).$$

Set $\lambda = \deg(L)$ and $e_i = \dim(L_{p_i} \cap F_2^i)$ for each parabolic point p_i. Then the parabolic degree is determined by λ and $e = (e_1, \ldots, e_n)$ by the formula

$$\mathrm{pardeg}(L) = \lambda + \sum_i [(1 - e_i)\alpha_1^i + e_i \alpha_2^i]. \tag{13}$$

We say that $\mathcal{E}$ is of type (λ, e). Bundles of type (λ, e) form a locally closed, connected submanifold $\mathcal{C}_{\lambda,e}$ of finite codimension in $\mathcal{C}$. Note further that each $\mathcal{C}_{\lambda,e}$ is nonempty. This is because given (λ, e), we can build a bundle of this type by taking a direct sum. The argument given in [17] carries over to show that the stratification

$$\mathcal{C} = \mathcal{C}_{ss} \cup \bigcup_{\lambda,e} \mathcal{C}_{\lambda,e}$$

is equivariantly perfect (in a sense we shall explain shortly).
Remark: Nitsure restricts attention to the case where the genus g ≥ 2. The only reason for this is that for higher rank and genus 0, it is not clear (in fact not true!) that each strata is nonempty. In fact, we shall see that for certain parabolic structures, there are no semistable rank 2 bundles. Keeping track of "empty" strata is one of the difficulties in generalizing this procedure to rank 3 and higher.

6.3 The gauge groups $\mathcal{G}^{\mathbf{C}}$ and $\mathcal{P}$

We now define the two "gauge groups" with natural actions on $\mathcal{C}$. The complexified gauge group

$$\mathcal{G}^{\mathbf{C}} = \mathrm{Aut}\, E = \{g : E \to E \text{ over X with } g_x \in \mathrm{GL}(2, \mathbf{C}) \text{ for all } x \in X\},$$

and the parabolic gauge group

$$\mathcal{P} = \mathrm{ParAut}\, E = \{g \in \mathcal{G}^{\mathbf{C}} \text{ with } g_{p_i}(F_2^i) = F_2^i \text{ for } 1 \leq i \leq n\}.$$

These act on $\mathcal{C}$ by

$$g(d'') = g^{-1} \circ d'' \circ g = d'' + g^{-1} d'' g.$$

The $\mathcal{G}^{\mathbf{C}}$ orbits are isomorphism classes of holomorphic structures on E, and the $\mathcal{P}$ orbits are parabolic isomorphism classes of parabolic holomorphic structures on E. We use the method to identify the tangent and normal spaces to the gauge orbits. Suppose g_t is a curve in $\mathcal{G}^{\mathbf{C}}$ with $g_0 = 1$. Then $g_t(d'') = d'' + g_t^{-1}d''g_t$. Taking the derivative and evaluating at 0, we get $d''g'$, where $g' \in \Omega^0(\operatorname{End} E)$ is the derivative of g_t at 0. Thus, the tangents to the gauge orbits at d'' are elements of $\operatorname{im} d''$, where

$$d'' : \Omega^0(\operatorname{End} E) \to \Omega^{0,1}(\operatorname{End} E).$$

Also, the normal bundle at d'' is just $\operatorname{coker} d''$ and we identify the tangent space of $\mathcal{C}/\mathcal{G}^{\mathbf{C}}$ at $[d'']$ with $\mathrm{H}^1(X, \operatorname{End} E)$. Similarly, the tangent space to $\mathcal{C}/\mathcal{P}$ at $[d'']$ is $\mathrm{H}^1(X, \operatorname{ParEnd} E)$ where $\operatorname{ParEnd} E$ is the sheaf of parabolic endomorphisms of E.

Remark: This is actually quite tricky, requiring Sobolev completions and all. To treat this right, we must descend into the nether-world of sheaf theory. We refer the adventuresome to [17].

Atiyah and Bott prove that the stratification on $\mathcal{C}$ induced by the Harder-Narasimhan filtration is $\mathcal{G}^{\mathbf{C}}$ perfect, and Nitsure proves that the stratification on $\mathcal{C}$ induced by the parabolic filtration is $\mathcal{P}$ perfect. In either case, you can deduce the equivariant cohomology of the top stratum ($\mathcal{C}_{ss}$) from that of the unstable strata ($\mathcal{C}_{\lambda,e}$) along with the equivariant cohomology of the whole space.

6.4 The equivariant cohomology of $\mathcal{C}_{ss}$

Because the stratification on $\mathcal{C}$ is perfect, we have the formula for the equivariant Poincare polynomials (where we use $\tilde{P}$ for equivariant H^*)

$$\tilde{P}_t(\mathcal{C}) = \tilde{P}_t(\mathcal{C}_{ss}) + \sum_{(\lambda,e)} t^{2d_{\lambda,e}} \tilde{P}_t(\mathcal{C}_{\lambda,e}) \tag{14}$$

where $d_{\lambda,e} = \operatorname{codim}(\mathcal{C}_{\lambda,e})$. We calculate the various pieces of the above formula. First, since $\mathcal{C} \simeq *$,

$$\mathrm{H}^*_{\mathcal{P}}(\mathcal{C}) = \mathrm{H}^*(B\mathcal{P}).$$

To calculate this, we use the fibration

$$\mathcal{P} \to \mathcal{G}^{\mathbf{C}} \to \mathcal{F}$$

where $\mathcal{F}$ is the flag variety, which in this case is $\mathbf{CP}^1 \times \overset{n}{\cdots} \times \mathbf{CP}^1$ (n is the number of nontrivial flags).

Remark: For rank 2, a (nontrivial) flag is just a point in $\mathbf{CP}^1$. Thus, a choice of parabolic structure is an element of $\mathcal{F} = \mathbf{CP}^1 \times \overset{n}{\cdots} \times \mathbf{CP}^1$. A partition of unity argument shows that the action of $\mathcal{G}^{\mathbf{C}}$ is transitive on parabolic structures, giving a surjection $\mathcal{G}^{\mathbf{C}} \to \mathcal{F}$ with fiber the subgroup $\mathcal{P}$ of parabolic gauge transformations.

On the level of classifying spaces, we get a fibration

$$\mathcal{F} \to B\mathcal{P} \to B\mathcal{G}^{\mathbf{C}}. \tag{15}$$

This is a sequence of pull backs of the following fibration

$$F \to BP \to BG,$$

where $G = U(n)$, P is a parabolic subgroup, and F is the corresponding flag. Both F and BP are torsion free with cohomology in only even dimensions. It follows that the Leray-Serre spectral sequence collapses at the E_2 term (since d : even $\to$ odd), and therefore this fibration is cohomologically trivial. Consequently, the fibration (15) is also cohomologically trivial. Now, by Theorem 2.15 of [1], $B\mathcal{G}^{\mathbf{C}}$ is torsion free with homology given by

$$P_t(B\mathcal{G}^{\mathbf{C}}) = \frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^2)^2(1-t^4)}.$$

So $B\mathcal{P}$ has no torsion and

$$\begin{aligned} P_t(B\mathcal{P}) &= P_t(B\mathcal{G}^{\mathbf{C}}) \otimes P_t(\mathcal{F}) \\ &= \frac{(1+t)^{2g}(1+t^3)^{2g}(1+t^2)^{n-1}}{(1-t^2)^3}. \end{aligned}$$

Having computed $\mathrm{H}^*_{\mathcal{P}}(\mathcal{C})$, we turn our attention to the other terms in formula (14). We compute $d_{\lambda,e}$ in terms of λ and e. This, with $\tilde{P}_t(\mathcal{C}_{\lambda,e})$, will yield the equivariant cohomology of $\mathcal{C}_{ss}$.

Now each strata $\mathcal{C}_{\lambda,e}$ is a union of orbits, thus the normal to $\mathcal{C}_{\lambda,e}$ is a quotient of $\mathrm{H}^1(X, \mathrm{ParEnd}\ E)$. Given $\mathcal{E} \in \mathcal{C}_{\lambda,e}$, let $\mathrm{ParEnd}'E$ denote those endomorphisms which preserve the filtration $0 \subset L \subset \mathcal{E}$. Then the tangent space to $\mathcal{C}_{\lambda,e}$ contains $\mathrm{H}^1(X, \mathrm{ParEnd}'E)$. Letting $\mathrm{ParEnd}''E$ be the quotient

$$\mathrm{ParEnd}'E \hookrightarrow \mathrm{ParEnd}\ E \to \mathrm{ParEnd}''E,$$

we can identify the normal to $\mathcal{C}_{\lambda,e}$ with $\mathrm{H}^1(X, \mathrm{ParEnd}''E)$. ¿From the exact sequence

$$0 \to L \to \mathcal{E} \to Q \to 0$$

and the fact that $\mathrm{pardeg}(L) > \mathrm{pardeg}(Q) \Rightarrow \mathrm{ParHom}(L, Q) = 0$, it follows that $\mathrm{H}^0(X, \mathrm{ParEnd}''E) = 0$.

We may now calculate the precise value of $d_{\lambda,e} = -\chi(\mathrm{ParEnd}''E)$ by Riemann-Roch. Let $\mathrm{End}'E$ be the endomorphisms (not necessarily parabolic) which preserve the filtration and $\mathrm{End}''E$ the quotient

$$\mathrm{End}'E \hookrightarrow \mathrm{End}\ E \to \mathrm{End}''E.$$

Then we have a short exact sequence of sheaves

$$0 \to \mathrm{ParEnd}''E \to \mathrm{End}''E \to S \to 0$$

where S is a skyscraper sheaf with a one dimensional stalk over each parabolic point p_i with $e_i = 1$. Thus,

$$\begin{aligned} d_{\lambda,e} &= \mathrm{h}^1(X, \mathrm{ParEnd}''E) \\ &= -\chi(\mathrm{ParEnd}''E) \\ &= -\chi(\mathrm{End}''E) + \chi(S) \end{aligned}$$

But we calculate $\chi(\mathrm{End}''E) = k - 2\lambda + 1 - g$ by Riemann-Roch, where $k = \deg(E)$. Since S is a skyscraper sheaf, $\chi(S) = \mathrm{h}^0(X, S) = \sum_i e_i$. Thus,

$$d_{\lambda,e} = 2\lambda - k + (g-1) + \sum_i e_i \tag{16}$$

To complete the calculation, we find $\tilde{P}_t(\mathcal{C}_{\lambda,e})$ for all the unstable strata. It is shown in 3.4 of [17] (or see 7.12 of [1]) that

$$\mathrm{H}^*_{\mathcal{P}}(\mathcal{C}_{\lambda,e}) = \mathrm{H}^*_{\mathcal{P}(L)}(\mathcal{C}_{ss}(L)) \otimes \mathrm{H}^*_{\mathcal{P}(Q)}(\mathcal{C}_{ss}(Q))$$

But, $\mathcal{P}(L) = \mathcal{P}(Q) = \mathbf{C}^*$, and $\mathcal{C}_{ss}(L) = \mathcal{C}_{ss}(Q) = J(X)$, the Jacobian. Since $\mathbf{C}^*$ acts trivially, $\mathrm{H}^*_{\mathbf{C}^*}(J(X)) = \mathrm{H}^*(BU(1)) \otimes \mathrm{H}^*(J(X))$. Thus $\mathrm{H}^*_{\mathcal{P}}(\mathcal{C}_{\lambda,e}) = \mathrm{H}^*_{\mathbf{C}^*}(J(X)) \otimes \mathrm{H}^*_{\mathbf{C}^*}(J(X))$ and so

$$\tilde{P}_t(\mathcal{C}_{\lambda,e}) = \frac{(1+t)^{4g}}{(1-t^2)^2}.$$

Putting it all together, equation (14) implies

$$\tilde{P}_t(\mathcal{C}_{ss}) = \frac{(1+t)^{4g}}{(1-t^2)^3}\left((1-t+t^2)^{2g}(1+t^2)^{n-1} - (1-t^2)\sum_{\lambda,e} t^{2d_{\lambda,e}}\right) \tag{17}$$

6.5 The cohomology of $\mathcal{S}$ in the case $\mathcal{C}_{ss} = \mathcal{C}_s$

Now, because we are interested in $\mathrm{H}^*(\mathcal{S})$, we assume that semistable bundles are in fact stable. This assumption holds for our application (torsion free Seifert fibrations) and boils down to an arithmetic requirement on the weights (for example, that the nontrivial denominators are relatively prime). In order to compare the parabolic and the nonparabolic cases, we first give an outline for (regular) stable bundles. In [1] it is proved that $\mathcal{C}_{ss} = \mathcal{C}_s$ whenever the rank and degree of the bundle are coprime. Another consequence of $(\mathrm{rank}, \deg) = 1$ is that $\mathrm{H}^*(\mathcal{S})$ is torsion free. This follows by considering the sequence

$$1 \to U(1) \to \mathcal{G} \to \bar{\mathcal{G}} \to 1.$$

If $(\mathrm{rank}, \deg) = 1$, then the corresponding fibration

$$BU(1) \to B\mathcal{G} \to B\bar{\mathcal{G}}$$

is trivial. Moreover, $\mathrm{H}^*(BU(1))$ and $\mathrm{H}^*_{\mathcal{G}}(\mathcal{C}_{ss})$ are torsion-free. It now follows from Proposition 6.1 that

$$\mathrm{H}^*_{\mathcal{G}}(\mathcal{C}_{ss}) = \mathrm{H}^*(BU(1)) \otimes \mathrm{H}^*(\mathcal{S})$$

taken with $\mathbf{Z}$ coefficients.

For general parabolic bundles, it is observed in [17] that $\mathrm{H}^*(\mathcal{S})$ is torsion free provided $(\mathrm{rank}, \deg) = 1$. Now we prove the stronger result that for rank 2 parabolic bundles with at least one nontrivial flag, $\mathrm{H}^*(\mathcal{S})$ is torsion free. ¿From the short exact sequence

$$1 \to \mathbf{C}^* \to \mathcal{P} \to \bar{\mathcal{P}} \to 1$$

we get the fibration of classifying spaces

$$BU(1) \to B\mathcal{P} \to B\bar{\mathcal{P}}.$$

In order to prove $\mathrm{H}^*(\mathcal{S})$ is torsion free, we need to show that this bundle is trivial. Because the fiber is a $K(\mathbf{Z}, 2)$, this bundle is classified by an element of

$$\mathrm{Map}(B\bar{\mathcal{P}}, K(\mathbf{Z}, 3)) = \mathrm{H}^3(B\bar{\mathcal{P}}, \mathbf{Z}).$$

We want to see that the bundle is trivial; it is enough to show that the map

$$\mathrm{H}^2(B\mathcal{P}, \mathbf{Z}) \xrightarrow{\imath^*} \mathrm{H}^2(BU(1), \mathbf{Z})$$

is onto. But since $B\mathcal{P}$ and $BU(1)$ are torsion free,

$$\mathrm{H}^2(B\mathcal{P}, \mathbf{Z}) \cong \mathrm{H}_2(B\mathcal{P}, \mathbf{Z}) \overset{h}{\cong} \pi_2 B\mathcal{P} \cong \pi_1\mathcal{P}$$

and similarly, $\mathrm{H}^2(BU(1), \mathbf{Z}) \cong \pi_1 U(1)$. Thus, it suffices to show that the map $\pi_1 U(1) \to \pi_1\mathcal{P}$ coming from the inclusion $\mathbf{C}^* \hookrightarrow \mathcal{P}$ induces a direct sum. The fibration $\mathcal{P} \to \mathcal{G}^{\mathbf{C}} \to \mathcal{F}$ gives the long exact sequence in homotopy

$$\cdots \xrightarrow{0} \pi_2\mathcal{F} \to \pi_1\mathcal{P} \to \pi_1\mathcal{G}^{\mathbf{C}} \to 0.$$

Both $\pi_2\mathcal{F} \cong \mathbf{Z}\oplus \overset{n}{\cdots} \oplus\mathbf{Z}$ and $\pi_1\mathcal{G}^{\mathbf{C}} \cong \mathbf{Z} \oplus \mathbf{Z}$ are free abelian, and because $\pi_1\mathcal{P}$ is abelian, we have $\pi_1\mathcal{P} \cong \mathbf{Z}\oplus \overset{n+2}{\cdots} \oplus\mathbf{Z}$. Composing with the inclusion gives the commutative triangle

$$\begin{array}{ccc} \mathbf{C}^* & & \\ \imath\downarrow & \searrow \jmath & \\ \mathcal{P} & \hookrightarrow & \mathcal{G}^{\mathbf{C}}, \end{array}$$

which, on the level of homotopy, gives

$$\begin{array}{ccccccccc} & & & & \pi_1 U(1) & & & & \\ & & & & \imath_*\downarrow & \searrow \jmath_* & & & \\ 0 & \to & \pi_2\mathcal{F} & \to & \pi_1\mathcal{P} & \to & \pi_1\mathcal{G}^{\mathbf{C}} & \to & 0, \end{array}$$

Atiyah and Bott prove that $\mathrm{im}(\jmath_*)$ is a direct summand of $\pi_1\mathcal{G}^{\mathbf{C}}$ in case $(\mathrm{rank}, \deg) = 1$. But it is possible (in fact likely) that $\mathrm{im}(\imath_*)$ is a direct summand of $\pi_1\mathcal{P}$ even though $\mathrm{im}(\jmath_*)$ is not. This is the content of

Proposition 6.2 *Suppose that $\mathcal{E}$ is a rank 2, parabolic bundle with at least one non-trivial flag. Then the image of the map $\pi_1 U(1) \to \pi_1 \mathcal{P}$ is a direct summand.*

Proof: The general statement follows easily from the case where there is exactly one nontrivial flag so that $\mathcal{F} = \mathbf{CP}^1$. Let r be the map which restricts an automorphism to the nontrivial parabolic point p. Then on the level of homotopy, since we may replace the groups with their maximal compact subgroups, r_* maps the sequence

$$0 \to \pi_2 \mathbf{CP}^1 \to \pi_1 \mathcal{P} \to \pi_1 \mathcal{G}^{\mathbf{C}} \to 0 \tag{18}$$

$$\Downarrow r_*$$

$$0 \to \pi_2 \mathbf{CP}^1 \to \pi_1 U(1) \oplus \pi_1 U(1) \xrightarrow{\phi_*} \pi_1 U(2) \to 0. \tag{19}$$

Here ϕ_* is induced by the natural inclusion ϕ of the maximal torus of $U(2)$, i.e.

$$\phi(z_1, z_2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \text{ for } z_1, z_2 \in U(1).$$

In $U(2)$, the curves $\phi(e^{i\theta}, 1)$ and $\phi(1, e^{i\theta})$ are homotopic to the generator for $\pi_1 U(2)$, so $\ker \phi_*$ is generated by $(1, -1)$, where we have identified (19) with

$$0 \to \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \to 0.$$

Now, a splitting σ' of (19) determines a splitting σ of (18) by $\sigma(x) = \sigma' \circ r_*(x)$ for $x \in \pi_1 \mathcal{P}$. We check that the map $\sigma'(1,0) = 1, \sigma'(0,1) = 0$ is a splitting. Further, since $r \circ \imath(z) = (z, z)$ for $z \in U(1)$, we see that $r_* \circ \imath_*(1) = (1,1) \in \pi_1 U(1) \oplus \pi_1 U(1)$. It follows that $\sigma \circ \imath_*(1) = 1 \in \pi_2 \mathbf{CP}^1$. Thus $\mathrm{im}(\imath_*)$ is indeed a direct sum. This argument carries over to more parabolic points without difficulty. ■

In the previous section, we saw that $\mathrm{H}^*_{\mathcal{P}}(\mathcal{C}_{ss})$ is torsion free. Further, if $\mathcal{C}_{ss} = \mathcal{C}_s$, since $\mathbf{C}^* \subset \mathcal{P}$ acts acts trivially and $\bar{\mathcal{P}}$ acts freely, we may apply Proposition 6.1 to conclude that $\mathrm{H}^*_{\mathcal{P}}(\mathcal{C}_{ss}) = \mathrm{H}^*(BU(1)) \otimes \mathrm{H}^*(\mathcal{S})$. It follows from $P_t(\mathcal{S}) = (1-t^2)\tilde{P}_t(\mathcal{C}_{ss})$ and formula (17) that

$$P_t(\mathcal{S}) = \frac{(1+t)^{4g}}{(1-t^2)^2} \left((1-t+t^2)^{2g}(1+t^2)^{n-1} - (1-t^2) \sum_{\lambda,e} t^{2d_{\lambda,e}} \right). \tag{20}$$

In the cases of genus 0 and 1, this equation gives the cohomology of any torsion free Seifert fibered three manifold over S^2 and T^2, which will follow from the next two sections.

6.6 Results for genus 0 and 1

a: <u>Genus 0</u>

Assuming that X has genus 0 and rewriting equation (20), we get

$$P_t(\mathcal{S}) = \frac{1}{(1-t^2)^2} \left((1+t^2)^{n-1} - (1-t^2) \sum_{\lambda,e} t^{2d_{\lambda,e}} \right). \tag{21}$$

It follows that $\mathrm{H}^i(\mathcal{S}) = 0$ for i odd. In the next section we shall see how $\mathcal{S}$ is related to $\mathcal{R}(\Sigma)$, the representation space of Seifert fibered homology spheres Σ. Thus, we conclude that $\mathcal{R}(\Sigma)$ has only cohomology in the even dimensions, which we expect based on a conjecture of Fintushel and Stern. This conjecture was proved by Kirk and Klassen in [11] (see also [3] and [7]). With additional results about $\pi_1(\mathcal{S})$, we would have an independent proof of that conjecture (based on the work of Smale). Unfortunately, our method is homological in nature. For example, we could conclude (as was done in [1]) that $\pi_1(\mathcal{S}) = 0$ if we knew that all the codimensions $d_{\lambda,e} \geq 2$. Unfortunately, this is not the case. Another way around this is to prove that $\mathcal{S}$ is a rational variety as in [3] and [7]. Anyway, formula (21) is a useful and fairly simple tool for computation. For example, one can calculate the possible codimensions and their multiplicities to deduce the cohomology of $\mathcal{S}$. First, consider the case when $\mathcal{S}$ is empty, so that $(1-t^2)\sum_{\lambda,e} t^{2d_{\lambda,e}} = (1+t^2)^{n-1}$. For a given n, we can solve this to find $\sum_{\lambda,e} t^{2d_{\lambda,e}}$. For example, if $n = 3$, we get

$$\sum_{\lambda,e} t^{2d_{\lambda,e}} = 1 + 3t^2 + 4t^4 + \cdots \Rightarrow P_t(\mathcal{S}) = 0.$$

Since $(1-t^2)\sum_{\lambda,e} t^{2d_{\lambda,e}}$ is the polynomial $(1+t^2)^{n-1} - (1-t^2)P_t(\mathcal{S})$, the power series must be of the form $q(t) + \sum_{i=n}^{\infty} at^{2i}$ where $q(t)$ is some polynomial. In fact, for each n, there is a finite list of possibilities for this power series.

For $n = 3$, then the only nontrivial case is

$$\sum_{\lambda,e} t^{2d_{\lambda,e}} = 4t^2 + 4t^4 + \cdots \Rightarrow P_t(\mathcal{S}) = 1.$$

This reflects the fact that $\mathcal{S}$ is either empty or a point [6]. If $n = 4$, then

$$\sum_{\lambda,e} t^{2d_{\lambda,e}} = 4t^2 + 8t^4 + \cdots \Rightarrow P_t(\mathcal{S}) = 1 + t^2.$$

This reflects the fact that $\mathcal{S}$ is either empty or an S^2 [6]. If $n = 5$, then

$$\sum_{\lambda,e} t^{2d_{\lambda,e}} = (6-b)t^2 + (10+b)t^4 + 16t^6 + \cdots \Rightarrow P_t(\mathcal{S}) = 1 + bt^2 + t^4.$$

It follows immediately that $0 \leq b \leq 6$. In fact, $b \neq 0$. This is observed by Kirk and Klassen [11], where they prove that these four dimensional components are either $S^2 \times S^2$ or $\mathbf{CP}^2 \# h\overline{\mathbf{CP}}^2$ where $0 \leq h \leq 5$. We now list the possibilities for $n = 6$ where we have

$$\sum_{\lambda,e} t^{2d_{\lambda,e}} = (7-b)t^2 + 16t^4 + (25+b)t^6 + 32t^8 \cdots \Rightarrow P_t(\mathcal{S}) = 1 + bt^2 + bt^4 + t^8.$$

Again, it is immediate that $b \leq 7$, but it is not clear (although true because $\mathcal{S}$ is Kähler) that $b = 0$ is not realized. In the last section, we will explicitely compute an $n = 5$ and $n = 6$ example, showing that the bound on b, in these cases, is sharp. These bounds on the second Betti number b^2 generalize as follows. Since $\mathcal{S}$ is a

$2n-6$ manifold, set $P_t(\mathcal{S}) = \sum_{i=0}^{n-3} b^{2i}t^{2i}$ and solve for the power series $\sum_{\lambda,e} t^{2d_{\lambda,e}}$. For instance, if n = 7 , then dim $\mathcal{S} = 8$, and so $P_t(\mathcal{S}) = 1 + t^8 + b^2(t^2+t^6) + b^4t^4$. Solving (21), we get

$$\sum_{\lambda,e} t^{2d_{\lambda,e}} = (8-b^2)t^2 + (22+b^2-b^4)t^4 + (42+b^4-b^2)t^6 + (56+b^2)t^8 + 64t^{10} + \cdots.$$

But of course, the coefficients must all be nonnegative and we conclude that $b^2 \leq 8$ and $b^4 \leq 22 + b^2 \leq 30$. This process extends to the general case of n nontrivial flags to give that $b^2 \leq n+1$. Moreover, we get the recursive relation

$$b^{2i} - b^{2i-2} \leq \sum_{r=0}^{i} \binom{n}{r}$$

among the Betti numbers b^{2i}. This, in turn, yields bounds on the Euler characteristic $\chi(\mathcal{S})$. For instance,

1. $\chi(\mathcal{S}) \leq 8$ for n = 5,
2. $\chi(\mathcal{S}) \leq 16$ for n = 6,
3. $\chi(\mathcal{S}) \leq 48$ for n = 7.

These give bounds for Casson's invariant of Seifert-fibered homology spheres, which follows from the next section. Before we address the genus 1 case, we comment that this information for genus 0 gives us much information for the higher genus cases. In fact, fixing the weights and parabolic structure of the bundle, but allowing the genus of the underlying surface to increase, we notice that by knowing the series $\sum_{\lambda,e} t^{2d_{\lambda,e}}$ for genus 0, we know the corresponding series for genus g; it is obtained by simply multiplying the genus 0 series by t^{2g}. This is because the same unstable strata occur but their codimensions $d_{\lambda,e}$ have increased by g (see formula (16)).

b: <u>Genus 1</u>

Assuming now that X has genus 1 we rewrite equation (20) to get

$$P_t(\mathcal{S}) = \frac{(1+t)^2}{(1-t)^2}\left((1-t+t^2)^2(1+t^2)^{n-1} - (1-t^2)\sum_{\lambda,e} t^{2d_{\lambda,e}}\right). \qquad (22)$$

We introduce the notation $\mathcal{S}^0$ for the stable bundles of fixed determinant. While $\mathcal{S}$ corresponds to $U(n)$ representations, $\mathcal{S}^0$ corresponds to $SU(n)$ representations. It is easy to show, using the fibration $SU(n) \to U(n) \to U(1)$, that $\mathcal{S} = \mathcal{S}^0 \times J(X)$, where $J(X)$ denotes the Jacobian. Thus, in the genus 0 case, $\mathcal{S}$ and $\mathcal{S}^0$ coincide. In general, we have

$$P_t(\mathcal{S}) = (1+t)^{2g} P_t(\mathcal{S}^0).$$

Using (22), we get

$$P_t(\mathcal{S}^0) = \frac{1}{(1-t)^2}\left((1-t+t^2)^2(1+t^2)^{n-1} - (1-t^2)\sum_{\lambda,e} t^{2d_{\lambda,e}}\right). \qquad (23)$$

The series $\sum_{\lambda,e} t^{2d_{\lambda,e}}$ differs from that of the previous (genus 0) case by a factor of t^2, coming from the fact that the codimensions $d_{\lambda,e}$ have increased by 1. So, for example, the trivial case in genus 0 (when $(1-t^2)\sum_{\lambda,e} t^{2d_{\lambda,e}} = (1+t^2)^{n-1}$) now gives

$$\sum_{\lambda,e} t^{2d_{\lambda,e}} = t^2(1+t^2)^{n-1}.$$

Using formula (23) we get

$$P_t(\mathcal{S}^0) = (1+t^2)^n.$$

In fact, $\mathcal{S}^0 \approx S^2 \times \overset{n}{\cdots} \times S^2$. This follows by considering SU(2) representations of the following group presentation:

$$\pi_1^{orb}(X) = \langle a, b, x_1, \ldots, x_n \mid x_i^{a_i} = 1,\ [a,b]x_1 \cdots x_n = 1\rangle.$$

We use capital letters for the images of the corresponding elements in $SU(2)$. Thus, in $SU(2)$, X_i is required to lie in the set of a_i^{th} roots of unity. The set of a^{th} roots of unity is a disjoint union of S^2's. Picking a connected component of the representation space means choosing a specific copy of S^2 for each X_i. Because the corresponding component of the genus 0 representation space is trivial, it follows that $X_1 \cdots X_n \neq 1$. Thus, applying Corollary 1 of [16], we see that A and B, the images of the other two generators, are determined up to conjugation. We conclude that this component of the genus 1 representation space is in fact $S^2 \times \overset{n}{\cdots} \times S^2$.

Just as in the previous case, there is a finite list of possibilities for $\sum_{\lambda,e} t^{2d_{\lambda,e}}$ for each n. For example, if $n = 3$, then the only other case besides that already mentioned is

$$\sum_{\lambda,e} t^{2d_{\lambda,e}} = 4t^4 + 4t^6 + \cdots \Rightarrow P_t(\mathcal{S}) = 1 + 4t^2 + 2t^3 + 4t^4 + t^6.$$

Likewise, for $n = 4$, the only other possibility is

$$\sum_{\lambda,e} t^{2d_{\lambda,e}} = 4t^4 + 8t^6 + \cdots \Rightarrow P_t(\mathcal{S}) = 1 + 5t^2 + 2t^3 + 8t^4 + 2t^5 + 5t^6 + t^8.$$

We now show that $\mathcal{S}^0$ is simply connected. In the genus zero case, $\mathcal{S}$ nonempty $\Rightarrow$ all $d_{\lambda,e} \geq 1$. In this case, either $\mathcal{S}^0 \approx S^2 \times \overset{n}{\cdots} \times S^2$, or all $d_{\lambda,e} \geq 2$. In the second case we argue just as in Theorem 9.12 of [1] to show that $\mathcal{S}^0$ is simply connected. For higher genus, namely $g \geq 2$, this argument carries over immediately to give simple connectivity of $\mathcal{S}^0$.

6.7 Relationship between $\mathcal{S}$ and $\mathcal{R}(\Sigma)$.

We explain what this all has to do with representation spaces of Seifert-fibered spaces Σ following the ideas of [6] and [3]. First, we introduce the notation for the SU(2)-representation space. In particular, recall that

$$\mathcal{R}(\Sigma) = \mathrm{Hom}^*(\pi_1\Sigma, SU(2))/SO(3) \text{ for manifolds } \Sigma$$

$$\text{and } \mathcal{R}(X) = \mathrm{Hom}^*(\pi_1^{orb}(X), SU(2))/SO(3) \text{ for orbifolds X.}$$

Here, Hom* indicates the nontrivial representations. We prove that if Σ is a torsion free Seifert fibration, then there is a two dimensional orbifold X so that $\mathcal{R}(\Sigma) \cong \mathcal{R}(X)$. Although there is no well-defined homomorphism $\pi_1\Sigma \to \pi_1^{orb}(X)$, $\pi_1\Sigma$ and $\pi_1^{orb}(X)$ have a common quotient Γ.

So suppose Σ is the a torsion free Seifert fibration over F_g, the genus g surface. Then Σ has the Seifert invariants $\{b_0, (a_1, b_1), \ldots, (a_n, b_n)\}$, where the b_i are not unique, but, because $\mathrm{H}_1(\Sigma)$ is torsion free, must satisfy

$$a(-b_0 + \sum_{i=1}^{n} \frac{b_i}{a_i}) = 1 \tag{24}$$

where $a = a_1 \cdots a_n$. We use $\Sigma(g; a_1, \ldots, a_n)$ to denote this Seifert fibration. It follows from (24) that the $\{a_1, \ldots, a_n\}$ are pairwise relatively prime, and so we may order them so that the only possibly even a_i is a_1.

The following argument shows that we may assume that b_i is even for $i \neq 1$ and that b_1 is odd. Because we can change each b_i by a multiple of a_i at the expense of changing b_0, and because each a_i is odd for $i > 1$, we have b_i even for $i > 1$. Further, we may assume b_0 is even by adding a_1 to b_1, which, though it may not affect b_1's parity, certainly affects b_0's. Finally, if b_1 is even, then each term in equation (24) is even, which is a contradiction.

In the following group presentations, we adopt the convention that $i = 1, \ldots, n$ and $j = 1, \ldots, g$. Then $\pi = \pi_1(\Sigma)$ has the presentation

$$\pi = \langle A_j, B_j, x_i, h \mid h \text{ central},\ x_i^{a_i} = h^{-b_i},\ \prod[A_j, B_j] \prod x_i = h^{-b_0} \rangle.$$

Now consider the orbifold $X = X(g; 2a_1, \ldots, a_n)$ and the presentation of its fundamental group $\pi_1^{orb} = \pi_1^{orb}(X)$ (see §2)

$$\pi_1^{orb} = \langle A_j, B_j, y_i \mid y_1^{2a_1} = 1, y_i^{a_i} = 1 \text{ for } i > 1,\ \prod[A_j, B_j] \prod y_i = 1 \rangle.$$

The groups π and π_1^{orb}, have the common quotient group Γ defined by

$$\Gamma = \langle A_j, B_j, z_i \mid z^{a_1} \text{ central},\ z_1^{2a_1} = 1,\ z_i^{a_i} = 1 \text{ for } i > 1, \prod[A_i, B_i] \prod z_i = 1 \rangle.$$

There is an obvious map $\phi : \pi_1^{orb} \longrightarrow \Gamma$. Define the map $\psi : \pi \longrightarrow \Gamma$ by making the following assignments:

$$\begin{aligned} \psi(A_i) &= A_i \text{ and } \psi(B_i) = B_i \\ \psi(x_i) &= z_i \text{ and } \psi(h) = z_1^{a_1}. \end{aligned}$$

To check that ψ is well-defined, use the fact that $\psi(h)^{2n} = 1$ and $\psi(h)^{2n+1} = z_1^{a_1}$. Then it follows that

$$\begin{aligned} \psi(x_i)^{a_i} &= z_i^{a_i} = 1 = \psi(h)^{-b_i} \text{ for } i > 1 \text{ since } b_i \text{ is even,} \\ \psi(x_1)^{a_1} &= z_1^{a_1} = \psi(h)^{-b_1} \text{ since } b_1 \text{ is odd, and} \\ \psi(\prod[A_j, B_j] \prod x_i) &= 1 = \psi(h)^{-b_0} \text{ since } b_0 \text{ is even.} \end{aligned}$$

Clearly both ϕ and ψ are onto. Consider the maps

$$\begin{aligned}\phi^* &: \mathrm{Hom}(\Gamma, SU(2)) \longrightarrow \mathrm{Hom}(\pi_1^{orb}, SU(2)) \text{ and} \\ \psi^* &: \mathrm{Hom}(\Gamma, SU(2)) \longrightarrow \mathrm{Hom}(\pi, SU(2))\end{aligned}$$

defined by precomposition. Then ϕ^* and ψ^* are one-to-one because ϕ and ψ are onto. In fact, both ϕ^* and ψ^* are onto. This is obvious for ϕ^*, the reason being that if $\rho \in \mathrm{Hom}(\pi_1^{orb}, SU(2))$ then since $\rho(x_1)^{2a_1} = 1$ we must have $\rho(x_1)^{a_1} = \pm 1$, which is central in $SU(2)$. As for ψ^*, notice that for any element $\rho \in \mathrm{Hom}(\pi, SU(2))$, we have $\rho(h) = \pm 1$. This follows by considering the two cases: ρ is either reducible or irreducible. First, if ρ is irreducible, then h central $\Rightarrow \rho(h) = \pm 1$. On the other hand, if ρ is reducible, then

$$\rho(\prod_{i=1}^{g}[A_i, B_i]) = 1.$$

This shows that the last relation in the presentation of π gives that

$$\rho(\prod_{i=1}^{n} x_i) = \rho(h)^{-b_0}.$$

Raising this relation to the power $a = a_1 \cdots a_n$, and noticing that

$$x_i^a = (x_i^{a_i})^{a/a_i} = (h^{-b_i})^{a/a_i},$$

we get

$$1 = \rho(h)^{a(-b_0 + \sum_{i=1}^{n} \frac{b_i}{a_i})}.$$

By equation (24), it follows that $\rho(h) = 1$. Now consider $\rho \in \mathrm{Hom}(\pi, SU(2))$. We can define $\gamma \in \mathrm{Hom}(\Gamma, SU(2))$ by setting $\gamma(A_j) = \rho(A_j)$, $\gamma(B_j) = \rho(B_j)$, and $\gamma(z_i) = \rho(x_i)$. Then γ is well-defined because $\rho(h) = \pm 1$ and b_i is even for $i \neq 1$. Clearly the assignment $\rho \longmapsto \gamma$ gives an inverse to ψ^*. We conclude

$$\mathrm{Hom}(\pi_1^{orb}, SU(2)) \overset{\phi^*}{\cong} \mathrm{Hom}(\Gamma, SU(2)) \overset{\psi^*}{\cong} \mathrm{Hom}(\pi, SU(2))$$

Since conjugation commutes with the above isomorphisms, we see

Theorem 6.3 $\mathcal{R}(\Sigma(g; a_1, \ldots, a_n)) \cong \mathcal{R}(X(g; 2a_1, \ldots, a_n))$

We now investigate the method for computing $\mathrm{H}^*(\mathcal{R}(X))$. Suppose $X = X(g; 2a_1, \ldots, a_n)$. We decompose the representation space into its connected components

$$\mathcal{R}(X) = \coprod_{\bar\alpha} \mathcal{R}_{\bar\alpha}(X)$$

where $\bar\alpha = (\alpha_1, \ldots, \alpha_n)$ are the rotation numbers. More precisely,

$$\rho \in \mathcal{R}_{\bar\alpha}(X) \text{ if } \rho(x_j) = M_j \begin{pmatrix} e^{2\pi i \alpha_j} & 0 \\ 0 & e^{-2\pi i \alpha_j} \end{pmatrix} M_j^{-1} \text{ for all } j,$$

where x_j refers to the generator in the presentation of $\pi_1^{orb}(\mathrm{X})$. It is obvious that these components are in fact disjoint. Each α_i is a fraction with denominator a_i (for $i = 1$, α_1 has denominator $2a_1$). Further we can assume that $0 \leq \alpha_i \leq 1/2$ by conjugating, if necessary. The sequence $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ determines n pairs of weights by setting the weight at p_i equal to $(\alpha_i, 1-\alpha_i)$ if $\alpha_i \neq 0$ and $(0,0)$ if $\alpha_i = 0$. By abuse of notation, we denote the weights again by $\bar{\alpha}$. Let $\mathcal{S}(2,k,0,\bar{\alpha})$ denote the moduli of stable parabolic bundles over the genus g surface X_s of rank 2, degree k, parabolic degree 0, and weights $\bar{\alpha}$. The degree k is equal to the number of nontrivial weights, i.e. the order of the set $\{j \mid \alpha_j \neq 0\}$. This justifies shortening $\mathcal{S}(2,k,0,\bar{\alpha})$ to $\mathcal{S}_{\bar{\alpha}}$. We further introduce $\mathcal{S}^0_{\bar{\alpha}}$ as the corresponding moduli of stable bundles with fixed determinant. A consequence of the main theorem is

Corollary 6.4

$$\mathcal{R}_{\bar{\alpha}}(X) \cong \mathcal{S}^0_{\bar{\alpha}}$$

Remark : For $\bar{\alpha}$ nontrivial, $\dim(\mathcal{S}^0_{\bar{\alpha}}) = 2n + 6(g-1)$, where n is equal to the number of nontrivial flags, i.e. the order of the set $\{j \mid \alpha_j \neq 0 \text{ and } \alpha_j \neq 1/2\}$.

The idea is to use formula (20) to compute $\mathrm{H}^*(\mathcal{S}_{\bar{\alpha}})$ which computes $\mathrm{H}^*(\mathcal{R}(\Sigma))$ one component at a time. In order to do this, we need to check that $\mathcal{C}_s = \mathcal{C}_{ss}$. This is equivalent to requiring that there are no reducibles in $\mathcal{R}_{\bar{\alpha}}(X)$. This holds provided $\bar{\alpha}$ is nontrivial. Since the weights are fractions with denominators a_i which are relatively prime and at least one of them is nonzero, for any line subbundle L, $\mathrm{pardeg}(L)$ is not an integer. In particular, $\mathrm{pardeg}(L) \neq 0$. This verifies that $\mathcal{C}_s = \mathcal{C}_{ss}$ for $\bar{\alpha}$ nontrivial. On the other hand, if $\bar{\alpha} = \bar{0}$, then $\mathcal{S}_{\bar{0}} = \mathcal{R}(F_g)$, representations of the surface of genus g. If $g = 0$, then this component consists soley of the trivial representation. If $g = 1$, then this component consists entirely of reducibles and is a quotient of $S^1 \times S^1$ by a $B^{Z}{}_2$ action, just S^2. If $g \geq 2$, then the reducibles form a subvariety of $\mathcal{R}(F_g)$, which is no longer smooth. This component is the only one of $\mathcal{R}(X)$ where our technique fails and is the only reason we restrict to the cases where $g = 0$ or 1. In fact, Kirwan explicitely computes the intersection Betti numbers of this component for higher genus (see Proposition 5.9 of [12]), giving a complete answer modulo 2-torsion. For genus 2, this component turns out (by accident) to be smooth with Poincare polynomial

$$P_t(\mathcal{S}_{\bar{0}}) = (1+t)^4(1 + 2t^2 + 2t^4 + t^6).$$

Our work, along with the results contained in [12], give a complete description of the cohomology of the SU(2) representation space of any torsion free Seifert fibred 3-manifold.

A computer is helpful because there are potentially so many components. For example, the easiest example of a homology sphere with five fibers is $\Sigma(2,3,5,7,11)$. To calculate $\mathcal{R}(\Sigma)$, we have to check over 150 components. Luckily, computers are more patient than graduate students. We have a program that performs this calculation for $n \leq 7$, and theoretically we could do it for any number of fibers.

6.8 Explicit computations

Assuming $g = 0$, consider the orbifold $X = X(4, 3, 5, 7, 11)$. We first compute the cohomology of $\mathcal{R}_{\bar{\alpha}}(X) = \mathcal{S}_{\bar{\alpha}}$ where $\bar{\alpha} = (\frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{2}{7}, \frac{2}{11})$. Listing all possible destabilizing line bundles $L \to E$ with $\mathrm{pardeg}(L) > 0$, we compute the codimensions $d = d_{\lambda,e}$ of their strata. Because there may be several different strata $\mathcal{C}_{\lambda,e}$ with the same codimension d, we introduce the multiplicity m_d of d, which is the number of times a strata with $d_{\lambda,e} = d$ occurs. In terms of the power series

$$\sum_{\lambda,e} t^{2d_{\lambda,e}} = \sum_{d\geq 0} m_d t^{2d}.$$

Each $m_d = \sum_\lambda m_{\lambda,d}$ where $m_{\lambda,d}$ is the number of times a strata with $\deg(L) = \lambda$ and $d_{\lambda,e} = d$ occurs.

In order to keep track of all the fractions, we will use the notation $\bar{\beta} = (\beta_1, \ldots, \beta_5)$ for the larger weights, i.e. $\beta_i = 1 - \alpha_i$. So the flag at p_i has the two weights α_i, β_i. Notice that in this case, $1 < \sum \alpha_i < 2$ and $3 < \sum \beta_i < 4$. Setting $\bar{e} = (e_1, \ldots, e_5)$ equal to the intersection numbers of L, we can check the condition $\mathrm{pardeg}(L) > 0$ with formula (13) and compute $d_{\lambda,e}$ with formula (16). Notice that the different ways for L to intersect the flags are enumerated by the $2^5 = 32$ ways of choosing a five bit word $\bar{e}$. Since $\bar{e}$ contributes $\sum e_i$ to the codimension, we partition the set of all five bit words W into the subsets $W_h = \{\bar{e} \mid \sum e_i = h\}$ for $h = 0, \ldots, 5$.

Because $\mathrm{pardeg}(E) = 0$, we must have $\deg(E) = -5$. Now, E could have destabilizing subbundles L only if $\lambda = \deg(L) \geq -3$. (If $\lambda \leq -4$, then the parabolic degree of L is at most $-4 + \sum \beta_i < 0$ which is not destabilizing). On the other hand, if $\lambda \geq -1$, then the parabolic degree of L is at least $-1 + \sum \alpha_i > 0$, so no matter what the intersection numbers $\bar{e}$ are, L is destabilizing. So we just check the two cases $\lambda = -3, -2$.

For $\lambda = -3$, the following intersection numbers give $\mathrm{pardeg}(L) > 0$:

- any $\bar{e} \in W_4$, giving $d_{\lambda,e} = 2$,
- and also $\bar{e} = (1, 1, 1, 1, 1)$, giving $d_{\lambda,e} = 3$.

We can list this in the table

$\lambda = -3$

d	$m_{-3,d}$
2	5
3	1

For $\lambda = -2$, the following intersection numbers give $\mathrm{pardeg}(L) > 0$:

- any $\bar{e} \in W_2$, giving $d_{\lambda,e} = 2$,
- any $\bar{e} \in W_3$, giving $d_{\lambda,e} = 3$,
- any $\bar{e} \in W_4$, giving $d_{\lambda,e} = 4$,

- and $\bar{e} = (1,1,1,1,1)$, giving $d_{\lambda,e} = 5$.

Summarizing this in the table

$$\lambda = -2$$

d	$m_{-2,d}$
2	10
3	10
4	5
5	1

Likewise, for each $\lambda \geq -1$, we get a table of the form

d	$m_{\lambda,d}$
$2\lambda+4$	1
$2\lambda+5$	5
$2\lambda+6$	10
$2\lambda+7$	10
$2\lambda+8$	5
$2\lambda+9$	1

Computing $m_d = \sum_\lambda m_{\lambda,d}$, we find that

$$\sum_{d\geq 0} m_d t^{2d} = 16t^4 + 16t^6 + \cdots$$

and conclude

$$P_t(\mathcal{S}_{\bar{\alpha}}) = 1 + 6t^2 + t^4.$$

Now consider the six-dimensional component where $n = 6$ and $\bar{\alpha} = (\frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{2}{7}, \frac{3}{11}, \frac{4}{13})$. Here, there are $2^6 = 64$ possible ways for a line bundle to intersect the flags, and we keep track of them all with the six bit word $\bar{e}$. Again, we partition the set of all words W into the subsets $W_h = \{\bar{e} \,|\, \sum e_i = h\}$ for $h = 0, \ldots, 6$.

Using the same notation, we see $\bar{\beta} = (\frac{3}{4}, \frac{2}{3}, \frac{3}{5}, \frac{5}{7}, \frac{8}{11}, \frac{9}{13})$. We have $\deg(E) = -6$, and because $\sum \beta_i < 5$, a destabilizing line bundle L must have $\deg(L) \geq -4$. Furthermore, since $\sum \alpha_i > 1$, if $\deg(L) \geq -1$, then no matter what the intersection numbers are, L is destabilizing. Thus, we need to check the cases $\deg(L) = \lambda =$ -2, -3, and -4.

For $\lambda = -4$, only $\bar{e} = (1,1,1,1,1,1)$ is destabilizing, contributing one term

d	$m_{-4,d}$
3	1

For $\lambda = -3$, L is destabilizing for the following intersection numbers:

- 10 of the 20 $\bar{e} \in W_3$, giving $d_{\lambda,e} = 2$,
- any $\bar{e} \in W_4$, giving $d_{\lambda,e} = 3$,
- any $\bar{e} \in W_5$, giving $d_{\lambda,e} = 4$,

- and $\bar{e} = (1,1,1,1,1)$, giving $d_{\lambda,e} = 5$.

Summarizing this in the table

$\lambda = -3$

d	$m_{-3,d}$
2	10
3	15
4	6
5	1

For $\lambda = -2$, L is destabilizing for the following intersection numbers:

- any $\bar{e} \in W_1$, giving $d_{\lambda,e} = 2$,
- any $\bar{e} \in W_2$, giving $d_{\lambda,e} = 3$,
- any $\bar{e} \in W_3$, giving $d_{\lambda,e} = 4$,
- any $\bar{e} \in W_4$, giving $d_{\lambda,e} = 5$,
- any $\bar{e} \in W_5$, giving $d_{\lambda,e} = 6$,
- and $\bar{e} = (1,1,1,1,1)$, giving $d_{\lambda,e} = 7$.

Summarizing this in the table

$\lambda = -2$

d	$m_{-2,d}$
2	6
3	15
4	20
5	15
6	6
7	1

For any $\lambda \geq -1$ we have the table

d	$m_{\lambda,d}$
$2\lambda + 5$	1
$2\lambda + 6$	6
$2\lambda + 7$	15
$2\lambda + 8$	20
$2\lambda + 9$	15
$2\lambda + 10$	6
$2\lambda + 11$	1

Computing $m_d = \sum_\lambda m_{\lambda,d}$, we find that

$$\sum_{d \geq 0} m_d t^{2d} = 16t^4 + 32t^6 + \cdots$$

and conclude

$$P_t(\mathcal{S}_{\bar{\alpha}}) = 1 + 7t^2 + 7t^4 + t^6.$$

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