1 Introduction

In this thesis we develop an infinitary logic of metric structures and generalize some classical results on the spectra of various infinitary sentences. Expositions of continuous logic date back to [5], but the finitary variant used in this paper is most similar to [2]. The infinitary logic developed in [4] resembles ours in its definition of countable conjunction and disjunction; however, we include the additional symbol $\rho$, which allows us to take the distance to the zeroset of any definable predicate. The infinitary language without $\rho$ will be denoted $L_{\omega_1\omega}$, and the language with $\rho$ will be written $L_{\omega_1\omega}(\rho)$.

The main result is a generalization of the Scott isomorphism theorem to metric structures. To prove the theorem, we construct Scott sentences in $L_{\omega_1\omega}(\rho)$ for separable models. We then begin a stability-theoretic study of the number of models of such sentences in various powers. In particular, we generalize results in Shelah’s paper [11]. Introductions to stability in the discrete context are given in, for example, [7], [10], and [9]. A more specific area of study, classification theory, is surveyed in [1] and detailed in [12]. Stability theory in the continuous context is explored in [3] and [6].

2 Metric logic

In the familiar (henceforth called discrete) model theory, a structure in a language $\mathcal{L}$ is a set together with interpretations for function and relation symbols. For a given $\mathcal{L}$-structure $\mathcal{M}$, an element $a \in M$ satisfies a $\mathcal{L}$-formula $\phi(x)$ if $\mathcal{M} \models \phi(a)$ – in words, if $\phi(a)$ is true of the structure $\mathcal{M}$. For instance, if we choose $\mathcal{L}$ such that $\phi(x) : x + 2 = 5$ is a formula, then, interpreting with $\mathbb{R}$, $\phi$ holds when $x = 3$ and does not when $x \neq 3$. Sometimes, however, it is useful to use our understanding of distance when determining the validity of propositions. In the previous example, it is reasonable to say that $x = 3 + 10^{-20}$ is closer to the right answer than
$x = 9001$, since $5 + 10^{-20}$ is closer to $5$ than $9003$ is. This, of course, is because $3 + 10^{-20}$ is closer to $3$ than $9001$ is. A key role is played by addition here – it is uniformly continuous. In this example, therefore, the distance between elements of $\mathbb{R}$ is proportional to how much those elements differ in terms of uniformly continuously defined properties. The model theory of metric structures can deal with problems like the previous example – so let’s take a look at how it works.

To concoct a formal language in the model-theoretic style, we’ll need some function and relation symbols. In the above example, we saw that closeness in the metric space was carried over into closeness in the space of truth values. This gives us a hint of the structure our formulas will need to have: they will need to be continuous, and perhaps more than that. More concretely, let’s look at the conditions from discrete model theory we would like to carry over to the continuous case. For instance, categories of models of discrete theories are closed under ultraproducts. We would like the same to be true of continuous logic, so let’s try to come up with a notion of metric ultraproduct. Ultrafilters and set-theoretic ultraproducts are basic discrete model-theoretic constructions, and so those looking for more background in this area are directed to any one of [7], [9], [10]. A primer on metric ultraproducts, as well as a review of ultrafilters, is provided in the appendices.

**Note 2.1.** There are two appendices: (i) ultrafilters, ultraproducts, and ultralimits, and (ii) Jensen’s diamond principle.

Henceforth, when not otherwise noted, we intend $I$ to be an index set and $\mathcal{U}$ an ultrafilter on $I$.

**Definition 2.2 (Ultralimit).** Let $(r_i)_{i \in I}$ be a sequence of real numbers. $r \in \mathbb{R}$ is the ultralimit of $(r_i)_{i \in I}$ if for every $\epsilon > 0$, $\{i \in I : |r - r_i| \leq \epsilon\} \in \mathcal{U}$. In symbols, we write

$$r = \lim_{i \to \mathcal{U}} r_i$$

When the context is clear, we will write $\lim_{\mathcal{U}} r_i$ for $\lim_{i \to \mathcal{U}} r_i$.

In discrete model theory, *ultraproducts* are structure-preserving products over arbitrary index sets. In the metric case, these products are defined via ultralimits. Let $(M_i, d_i)_{i \in I}$ be a sequence of metric spaces.

Let $M = \prod_{i \in I} M_i$. For $\bar{x}, \bar{y} \in M$,

$$d(\bar{x}, \bar{y}) = \lim_{\mathcal{U}} d_i(x_i, y_i)$$
Suppose we wish to take the ultraproduct of an $I$-indexed sequence of unbounded metric spaces $(M_i, d_i)$. Then we can construct sequences $(x_i), (y_i)$ such that $d_i(x_i, y_i) \geq 2^{n_i}$, where $n_i$ is the $n^{th}$ natural number, for each $i \in I$. Then $d^{M_i}(\bar{x}, \bar{y}) = \infty$. We don’t want this, so it is assumed that all metric spaces are bounded. Indeed, the construction of the sequences $\bar{x}, \bar{y}$ is possible in any product of metric spaces of increasing (finite) diameter, so we require the metric spaces to be uniformly bounded.

**Lemma 2.3.** $d$ is a pseudo-metric on $\prod_{i \in I} M_i$.

**Proof.** The pseudometric requirements follow from the fact that the $d_i$ are metrics:

1. $d(\bar{x}, \bar{x}) = \lim_{i \to U} d_i(x_i, x_i) = 0$.
2. $d(\bar{x}, \bar{y}) = \lim_{i \to U} d_i(x_i, y_i) = \lim_{i \to U} d_i(y_i, x_i) = d(\bar{y}, \bar{x})$.
3. $d(\bar{x}, \bar{z}) = \lim_{i \to U} d_i(x_i, z_i) \leq \lim_{i \to U} (d_i(x_i, y_i) + d_i(y_i, z_i)) = d(\bar{x}, \bar{y}) + d(\bar{y}, \bar{z})$.

\[ \square \]

However, $d$ is not a metric. For example, let $I = \mathbb{N}, X_i = \mathbb{R}$ for all $i$, and consider the sequences $\bar{x} = (\frac{1}{n})_{n \in \mathbb{N}}, \bar{y} = 0$. $\bar{x} \neq \bar{y}$, but $d(\bar{x}, \bar{y}) = 0$. So we do the obvious and quotient by $d$. The ultraproduct is then written:

\[ \prod_{i \in I} (M_i, d_i) / U \]

**Note 2.4.** In model theory, it is typical to study $n$-ary operations and relations on $M$; this is no problem in the discrete case, as equality is canonically defined in $M^n$ for all $n$. This is not the case for metrics. Suppose $(M, d), (N, \rho)$ are metric spaces: there are all sorts of ways we could define the metric on $M \times N$. So let’s go by convention: we always regard the product metric as being the maximum of the two component metrics, i.e. for $(x, y), (u, v) \in M \times N$,

\[ d \times \rho((x, y), (u, v)) = \max\{d(x, u), \rho(y, v)\} \]

Let’s define metric languages such that their classes of models are always closed under this ultraproduct. The ultraproduct requirement provides us with a way to decide what functions and relations we would like to admit into metric structures. Functions, for instance, should be defined componentwise.
Proposition 2.5. Suppose we have metric spaces \((M_i, d_i)\) with \(I\) as before and \(M\) the resulting ultraproduct. Consider an \(I\)-indexed family of functions \(f_i : M_i \to M_i\). If \(f(\bar{x}) = \lim U f_i(x_i)\) is well-defined, then \(f\) is uniformly continuous.

Proof. Suppose not. Then there is some \(\epsilon > 0\) such that for all \(i < \omega\) we can find \(y_i, z_i \in M_i\) for which \(d^{M_i}(y_i, z_i) \leq \frac{1}{i}\) and \(d^{M_i}(f(y_i), f(z_i)) > \epsilon\). In \(M\), the ultralimit \(d^M(\bar{y}, \bar{z}) = 0\). \(d\) is a metric on \(M\), so \(\bar{y} = \bar{z}\). However, \(d^M(f(\bar{y}), f(\bar{z})) > \epsilon\), so \(f(\bar{y}) \neq f(\bar{z})\) and so \(f\) is not well-defined. \(\square\)

In light of proposition 2.5, every admissible function must be uniformly continuous and so the notion of uniform continuity must be built into the definition of metric languages. The same is true for relations: an \(n\)-ary relation on a domain \(M\) is a function from \(M^n\) to a metric space \(N^1\).

We want the “truth values” to converge in the ultraproduct, and so \(N\) is required to be compact. For the rest of this article, it is assumed that \(N\) is \([0, 1]\) with the metric inherited from the absolute value.

Definition 2.6 (Metric language). A metric language \(L\) is a triple \((\mathcal{S}, \mathcal{F}, \mathcal{R})\), where:

1. \(\mathcal{S}\) is a collection of sorts. For each sort \(s\), there is a natural number \(K_s\) bounding the metric symbol \(d_s : s \times s \to [0, K_s]\), which has the identity as its modulus of uniform continuity.

2. \(\mathcal{F}\) is a set of function symbols. To each \(f \in \mathcal{F}\) is associated a domain of sorts \(\prod s_i\) and a range sort \(s\). Furthermore, for each sort in the domain of \(f\), there is a modulus of uniform continuity \(\delta_{f,i}\).

3. \(\mathcal{R}\) is a set of relation symbols. To each \(R \in \mathcal{R}\) is associated a domain of sorts \(\prod s_i\) and a range, which we take to be \([0, 1]\) from above. As with functions, there is a modulus of uniform continuity \(\delta_{R,i}\) corresponding to the relation \(R\) on sort \(s_i\).

Given a language, we can talk about structures.

Definition 2.7 (Metric structure). A metric structure is a triple \(((M_s)_{s \in \mathcal{S}}, \mathcal{F}^M, \mathcal{R}^M)\), where:

1. \(M_s\) is a complete metric space \((M_s, d_s)\) bounded by \(K_s\).

\(^1\)The discrete correspondent of \(N\) is \(\{0, 1\}\), so \(N\) is in some sense a space of truth values.
2. $F^M$ is a set of functions interpreting the symbols of $F$. So for each $n$-ary function symbol $f \in F$ with domain sorts $\prod s_i$ and range sort $s$, $F^M$ contains an uniformly continuous $n$-ary function $f^M : \prod M_{s_i} \to M_s$, along with corresponding moduli of uniform continuity $\delta_{f,i}$.

3. $R^M$ is a set of relations interpreting the symbols of $R$. So for each $n$-ary relation symbol $R \in R$ with domain sorts $\prod s_i$, $R^M$ contains an uniformly continuous $n$-ary relation $R^M : \prod M_{s_i} \to [0,1]$, along with the corresponding moduli of uniform continuity $\delta_{R,i}$.

Notice, in particular, that any discrete metric structure is, as the use of the word “discrete” up to this point has indicated, a metric structure with the discrete metric.

**Definition 2.8** (Terms). The set of terms in a metric language $\mathcal{L}$ is defined as follows:

1. Constants and variable symbols are terms.
2. If $\tau_1, \ldots, \tau_n$ are terms and $f$ is an $n$-ary function symbol, then $f(\tau_1, \ldots, \tau_n)$ is a term.

**Definition 2.9** (Formulas). The set of formulas in a metric language $\mathcal{L}$, denoted $\mathcal{F}(\mathcal{L})$, is defined as follows:

1. If $\tau_1, \ldots, \tau_n$ are terms and $R$ is an $n$-ary relation symbol, then $R(\tau_1, \ldots, \tau_n)$ is a (atomic) formula.
2. If $\phi_1(x_1, \ldots, x_m), \ldots, \phi_n(x_1, \ldots, x_m)$ are formulas and $f : [0,1]^n \to [0,1]$ is an uniformly continuous function, then $f(\phi_1(x_1, \ldots, x_m), \ldots, \phi_n(x_1, \ldots, x_m))$ is a formula.
3. If $\phi(x_1, \ldots, x_n)$ is a formula, then $\inf_y \phi(y, x_2, \ldots, x_n)$ and $\sup_y \phi(y, x_2, \ldots, x_n)$ are formulas.

Details are given in [2] and [4]. Morphisms in categories of models are often taken to be elementary embeddings.

**Definition 2.10** (Elementary embedding). Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures. A map $f : \mathcal{M} \to \mathcal{N}$ is elementary if it preserves all the $\mathcal{L}$-structure; that is, if $\phi^\mathcal{M}(\bar{a}) = \phi^\mathcal{N}(f(\bar{a})$ for all formulas $\phi$ in variables $\bar{x}$ of length $n$, and all $\bar{a} \in M^n$. We then write $\mathcal{M} \hookrightarrow \mathcal{N}$.
Proof. By induction on the complexity of $\phi$.

The ultraproduct defined above commutes with the quantifiers sup, inf, as detailed by the following proposition.

**Proposition 2.11.** Let $(Y, d, f)$ denote the ultraproduct $\prod_{i \in I} (X_i, d_i, f_i)/U$ and $\bar{a}_2, \ldots, \bar{a}_n \in Y$. Then:

$$\sup_{x \in Y} f(x, \bar{a}_2, \ldots, \bar{a}_n) = \lim_{U} \sup_{x \in X_i} f_i(x, a^i_2, \ldots, a^i_n)$$

**Proof.** We will show first that $LHS \geq RHS$ and then that $LHS \leq RHS$.

Suppose that $\sup_{x \in Y} f(\bar{x}, \bar{a}_2, \ldots, \bar{a}_n) \geq r$. Fix $\epsilon > 0$ so that $\sup_{x \in Y} f(\bar{x}, \bar{a}_2, \ldots, \bar{a}_n) > r - \epsilon$. Witness $\bar{x}$ by $\bar{b}$. Then $f(\bar{b}, \bar{a}_2, \ldots, \bar{a}_n) > r - \epsilon$, so $\{i \in I : f_i(b_i, a^i_1, \ldots, a^i_n) > r - \epsilon\} \in U$. Then $\{i \in I : \sup_{x_i \in X_i} f_i(x_i, a^i_n) > r - \epsilon\} \in U$; that is to say,

$$\lim_{i \in U} \sup_{x_i \in X_i} f_i(x_i, \ldots, a^i_n) \geq r - \epsilon$$

Similarly, if $\lim_{i \in U} \sup_{x_i \in X_i} f_i(x_i, \ldots, a^i_n) > r - \epsilon$, then $S = \{i \in I : \sup_{x_i \in X_i} f_i(x_i, \ldots, a^i_n) > r - \epsilon\} \in U$. For $i \in S$, fix $b_i \in X_i$ such that $f_i(b_i, \ldots, a^i_n) > r - \epsilon$. Form $\bar{b} = (b_i)_{i \in U}$, so that in the ultraproduct $f(\bar{b}, \ldots, \bar{a}_n) \geq r - \epsilon$.

We can now generalize some basic theorems of discrete model theory to the continuous case. First, the metric ultraproduct as defined above really does act like an ultraproduct:

**Theorem 2.12** (Los). Let $(M_i, d_i)_{i \in I}$ be a sequence of metric spaces whose ultraproduct is $M$. Then for all formulas $\phi(x_1, \ldots, x_m)$ and $\bar{a}_1, \ldots, \bar{a}_m \in M$,

$$\phi^M(\bar{a}_1, \ldots, \bar{a}_m) = \lim_{U} \phi^M_i(a^i_1, \ldots, a^i_m)$$

**Proof.** By induction on the complexity of $\phi(\bar{x})$, where $\ell(\bar{x}) = m$.

1. If $\phi(\bar{x})$ is atomic, then without loss of generality we can assume that $\phi(\bar{x}) = R \circ (f_1(\bar{x}), \ldots, f_n(\bar{x}))$. For any given $\bar{a}_1, \ldots, \bar{a}_m \in M$ we have

$$\phi^M(\bar{a}_1, \ldots, \bar{a}_m) = (R \circ (f_1, \ldots, f_n))^M(\bar{a}_1, \ldots, \bar{a}_m)$$

$$= \lim_{U} R^M_i(f_1, \ldots, f_m)(\bar{a}_1, \ldots, \bar{a}_m)$$

$$= \lim_{U} R^M_i \circ (f^M_i(a^i_1, \ldots, a^i_m), \ldots, f^M_n(a^i_1, \ldots, a^i_m))$$

$$= \lim_{U} \phi^M_i(a^i_1, \ldots, a^i_m)$$
2. If the result holds for $\phi_1(x_1, ..., x_n), ..., \phi_m(x_1, ..., x_n)$ and $f : [0, 1]^m \to [0, 1]$ is a uniformly continuous function, then for all $\bar{a}_1, ..., \bar{a}_n \in M$:

$$f^M(\phi_1(\bar{a}_1, ..., \bar{a}_n), ..., \phi_m(\bar{a}_1, ..., \bar{a}_n)) = f \left( \lim_{\mathcal{U}} \phi_1^M(a_1, ..., a_n), ..., \lim_{\mathcal{U}} \phi_m^M(a_1, ..., a_n) \right)$$

3. Suppose the hypothesis holds for $\phi(x_1, ..., x_m)$. So for any $\bar{a}_1, ..., \bar{a}_{m-1} \in M$, $\inf_y \phi^M_i(\bar{a}_1, ..., y) = \lim_{\mathcal{U}} \inf_y \phi^M_i(a_1, ..., y)$ by the previous proposition.

\[ \square \]

Note 2.13. Let $M$ be any metric structure and $M^\mathcal{U}$ an ultrapower of $M$. Consider the diagonal embedding $j : x \mapsto (x, x, x, ...)$. By Los' theorem, for all $\mathcal{L}$-formulas $\phi(x_1, ..., x_n)$ in all arities $n$ and for all $\bar{a}_1, ..., \bar{a}_n \in M$, $\phi^M_i(\bar{a}_1, ..., \bar{a}_n) = \lim_{\mathcal{U}} \phi^M(a_1^i, ..., a_n^i)$. But $M^\mathcal{U}$ is an ultrapower, so for all $k \leq n$, $\bar{a}_k = a_k \in M$. In other words, $\bar{a}_k = j(a_k)$. So

$$\phi^M_i(\bar{a}_1, ..., \bar{a}_n) = \phi^M_i(j(a_1), ..., j(a_n)) = \lim_{\mathcal{U}} \phi^M(a_1, ..., a_n)$$

Thus the diagonal embedding of a structure in an ultrapower is elementary.

Fixing a metric language $\mathcal{L}$ and a $\mathcal{L}$-sentence $\phi$, a condition is an inequality $\phi \leq r$ or $\phi \geq r$ for $r \in \mathbb{R}$. For $\epsilon > 0$, an $\epsilon$-approximation of a condition $\phi \leq r$ is $\phi \leq r + \epsilon$ or $\phi \leq r - \epsilon$. The $\epsilon$-approximation of a set $\Sigma$ is the set of $\epsilon$-approximations of all $\sigma \in \Sigma$. We say that a set of conditions is approximately finitely satisfiable if every finite subset thereof has a satisfiable $\epsilon$-approximation for every $\epsilon > 0$. The various notions of satisfiability are related by the following theorem, which is proved in the usual manner via ultraproducts:

Theorem 2.14 (Compactness). Let $\Gamma$ be a set of sentences in a metric language. The following are equivalent:

1. $\Gamma$ is satisfiable.
2. $\Gamma$ is finitely satisfiable.
3. $\Gamma$ is approximately finitely satisfiable.

Another cornerstone of discrete model theory carries over to the continuous case, whose proof on fragments of $L_{\omega_1\omega}$ is given in the next section:

**Theorem 2.15** (Downward Löwenheim-Skolem). *Given a metric language $\mathcal{L}$, an infinite cardinal $\kappa \leq |\mathcal{L}|$, a $\mathcal{L}$-structure $\mathcal{M}$ and a subset $A$ of $\chi(A) \leq \kappa$, there exists a substructure $\mathcal{N}$ of $\mathcal{M}$ for which:

1. $\mathcal{N} \preceq \mathcal{M}$.
2. $A \subseteq N \subseteq M$.
3. $\chi(N) \leq \kappa$.*

### 3 Infinitary metric logic

I write $F_n(\mathcal{L})$ for the set of $\mathcal{L}$-formulas in $n$ free variables, and $F(\mathcal{L})$ for the set of all $\mathcal{L}$-formulas. In discrete model theory, the notion of *definability* is very important. The metric provides us with an extension of this notion: given a formula $\phi$, we can ask “how far from $\phi$-definable” an element is. As the definable elements are exactly those in the zeroset of $\phi$, answering the preceding question is equivalent to checking the distance from the chosen element to the zeroset of $\phi$. We append a symbol $\rho$ (to our given continuous language) with which we intend to denote the distance-to-zeroset operator. To do this, we require some notation for the zeroset.

**Definition 3.1.** Let $\mathcal{L}$ be a metric language and $\mathcal{M} \in \text{Mod}(\mathcal{L})$. The zeroset in $\mathcal{M}$ of any $\mathcal{L}$-formula in $n$ variables $\phi(x_1, \ldots, x_n)$ is

$$Z^\mathcal{M}\phi = \{ \bar{y} \in M^n : \phi^\mathcal{M}(\bar{y}) = 0 \}$$

The distance to the zeroset of $\phi(x_1, \ldots, x_n)$ is defined

$$\rho^\mathcal{M}(\bar{y}, Z^\mathcal{M}\phi) = \inf \{ d^\mathcal{M}(\bar{y}, \bar{a}) : \bar{a} \in Z^\mathcal{M}\phi \}$$

For any $\mathcal{M}$ and $\phi(\bar{x})$, we have

$$\rho^\mathcal{M}(\bar{y}, Z^\mathcal{M}\phi) \leq \rho^\mathcal{M}(\bar{z}, Z^\mathcal{M}\phi) + d^\mathcal{M}(\bar{y}, \bar{z})$$

and thus, for any $\epsilon > 0$,

$$|\rho^\mathcal{M}(\bar{y}, Z^\mathcal{M}\phi) - \rho^\mathcal{M}(\bar{z}, Z^\mathcal{M}\phi)| \leq d^\mathcal{M}(\bar{y}, \bar{z}) \leq \delta_\rho(\epsilon) = \epsilon$$

We now define the language $L_{\omega_1\omega}(\rho)$, obtained by appending $\rho$ to the language $L_{\omega_1\omega}$. The construction of $L_{\omega_1\omega}$ without $\rho$ can be found in [4].
Definition 3.2. Given a separable metric language \( L \), we construct \( L_{\omega_1^\infty}(\rho) \). Simultaneously, for each variable \( x \) and formula \( \phi \), we define moduli of uniform continuity \( \delta_{\phi,x} \).

1. Any \( L \)-formula is a \( L_{\omega_1^\infty}(\rho) \)-formula.

2. For any \( L_{\omega_1^\infty}(\rho) \)-formulas \( \phi_1, \ldots, \phi_n \) and uniformly continuous function \( f : [0, 1]^n \to [0, 1] \), \( f(\phi_1, \ldots, \phi_n) \) is a \( L_{\omega_1^\infty}(\rho) \)-formula with range and modulus of uniform continuity given by composition.

3. If \( \phi(\vec{x}) \) is any \( L_{\omega_1^\infty}(\rho) \)-formula with free variables \( \vec{x} \) in the sorts \( \vec{s} \), then \( \rho(\vec{y}, \mathcal{Z}\phi) \) is a \( L_{\omega_1^\infty}(\rho) \)-formula with free variables \( \vec{y} \) in the sorts \( \vec{s} \). The modulus of uniform continuity is given by the identity.

4. Suppose \( \Phi \) is a countable subset of \( \mathcal{F}_m(L_{\omega_1^\infty}(\rho)) \) and \( \delta_{\phi,i} \) is the modulus of uniform continuity for \( \phi \) with respect to \( x_i \). Then the countable conjunction

\[
\bigwedge \Phi = \bigwedge_{\phi \in \Phi} \phi(x_1, \ldots, x_m) := \sup_{\phi \in \Phi} \phi \in \mathcal{F}_m(L_{\omega_1^\infty}(\rho))
\]

if and only if for every \( \epsilon > 0 \),

\[
\min_{i \leq m} \inf_{\phi \in \Phi} \delta_{\phi,i}(\epsilon) > 0
\]

Furthermore,

\[
\delta_{\lambda \Phi, i} = \sup_{0 < \eta < \epsilon} \inf_{\phi \in \Phi} \delta_{\phi,i}(\eta)
\]

5. Countable disjunction is defined as an abbreviation:

\[
\bigvee_{\phi \in \Phi} \phi = 1 - \bigwedge_{\phi \in \Phi} (1 - \phi)
\]

Some care with the size of the space of formulas is necessitated for our later concerns. In particular, we will require our language to be separable. In order for this notion to make sense, the obvious topology is defined for \( L \)-formulas:

Definition 3.3. Let \( \vec{x} \) be an \( n \)-tuple of variables corresponding to the \( n \)-tuple of sorts \( \vec{s} \). Given \( L \)-formulas \( \phi, \psi \) in the free variables \( \vec{x} \), define:

\[
D(\phi(\vec{x}), \psi(\vec{x})) = \sup \{ |\phi^\mathcal{M}(\vec{a}) - \psi^\mathcal{M}(\vec{a})| : \mathcal{M} \in \text{Mod}(L) \text{ and } \vec{a} \in M^n \}
\]
Proposition 3.4. $D$ as defined above is a pseudometric on $\mathcal{F}(\mathcal{L}, \bar{s})$ the space of $\mathcal{L}$-formulas over the sorts $\bar{s}$.

To preserve separability of the language $\mathcal{L}$, we typically work within a separable fragment of $\mathcal{L}_{\omega_1 \omega}$.

Definition 3.5 (Fragments). $\Delta \subseteq \mathcal{L}_{\omega_1 \omega}$ is a fragment if it contains every atom and is closed under subformulas and term substitution.

Typically, we work within a particular countable fragment of $\mathcal{L}_{\omega_1 \omega}$, and so elementary embeddings are taken with respect to the fragment, rather than all of $\mathcal{L}_{\omega_1 \omega}$. Since elementary embeddings are fragment-dependent, so too are the $\mathcal{L}_{\omega_1 \omega}$-analogues of some well-known results.

Proposition 3.6 (Vaught’s test). Suppose that $\Delta \subseteq \mathcal{L}_{\omega_1 \omega}$ is a countable fragment, $\mathcal{N}$ is a $\mathcal{L}_{\omega_1 \omega}$-structure and that $M \subseteq N$. $M \preceq N$ if for every $\phi(x_1, \ldots, x_n) \in \Delta$ (for all arities $n$), and all $\bar{a} \in M^{n-1}$, there is $b \in M$ for which $\inf_x \phi^N(x, \bar{a}) = \phi^N(b, \bar{a})$.

Proof. Proceed by induction on the complexity of $\phi$.

1. If $\phi(\bar{x})$ is atomic, then without loss of generality we can assume $\phi(\bar{x}) = R \circ (f_1, \ldots, f_m)(\bar{x})$. So for any $\bar{a} \in M^n$,

   $$\phi^N(\bar{a}) = R^N(f_1^N(\bar{a}), \ldots, f_m^N(\bar{a}))$$
   $$= R^N(f_1^M(\bar{a}), \ldots, f_m^M(\bar{a}))$$
   $$= R^M(f_1^M(\bar{a}), \ldots, f_m^M(\bar{a}))$$
   $$= \phi^M(\bar{a})$$

2. Suppose $f : [0,1]^m \rightarrow [0,1]$ is uniformly continuous and the result holds for $\phi_1(\bar{x}), \ldots, \phi_m(\bar{x})$. Then for all $\bar{a} \in M^n$,

   $$f^N(\phi_1(\bar{a}), \ldots, \phi_m(\bar{a})) = f(\phi_1^N(\bar{a}), \ldots, \phi_m^N(\bar{a}))$$
   $$= f(\phi_1^M(\bar{a}), \ldots, \phi_m^M(\bar{a}))$$
   $$= f^M(\phi_1(\bar{a}), \ldots, \phi_m(\bar{a}))$$

3. If the result holds for $\phi(\bar{x})$, then for all $\bar{a} \in M^{n-1}$:
\[
\inf_x \phi^N(x, \bar{a}) = \phi^N(b, \bar{a}) \text{ for some } b \in M \\
= \phi^M(b, \bar{a}) \text{ by hypothesis} \\
= \inf_x \phi^M(x, \bar{a}) \text{ as } M \subseteq N
\]

4. If the result holds for \((\phi_m(\bar{x})), m \in \mathbb{N}\), then for all \(\bar{a} \in M^n\):

\[
\bigwedge_{m \in \mathbb{N}} \phi^N_m(\bar{a}) = \sup_{m \in \mathbb{N}} \phi^N_m(\bar{a}) = \sup_{m \in \mathbb{N}} \phi^M_m(\bar{a}) = \bigwedge_{m \in \mathbb{N}} \phi^M_m(\bar{a})
\]

5. If the result holds for \(\phi(\bar{x})\), then for all \(\bar{a} \in M^n\):

\[
\rho^N(\bar{a}, Z^N \phi) = \inf \{d^N(\bar{a}, \bar{z}) : \bar{z} \in Z^N \phi\} \\
= \inf \{d^N(\bar{a}, \bar{z}) : \bar{z} \in Z^M \phi\} \text{ by hypothesis} \\
= \inf \{d^M(\bar{a}, \bar{z}) : \bar{z} \in Z^M \phi\} \text{ as } \bar{a}, \bar{z} \in M^n \\
= \rho^M(\bar{a}, Z^M \phi)
\]

\[\square\]

**Theorem 3.7** (Downward Löwenheim-Skolem). *Given a fragment \(\Delta\) of a metric language \(L\), an infinite cardinal \(\kappa \geq \chi(\Delta)\), a \(L\)-structure \(M\) and a subset \(A\) of \(\chi(A) \leq \kappa\), there exists a \(\Delta\)-elementary substructure \(N\) of \(M\) of density character \(\kappa\) having \(A\) as a subset.*

**Proof.** Let \(A_0\) be a dense subset of \(A\). By the axiom of choice, for each formula \(\phi(x_1, \ldots, x_{n+1})\) of in \((n + 1)\) variables there is a function \(f^0_\phi : M^n \to M\) with \(f_\phi(a_1, \ldots, a_n) = b\) for \(b \in M\) such that \(d^M(\phi^M(b, a_1, \ldots, a_n), \inf_y \phi^M(y, a_1, \ldots, a_n)) \leq \frac{1}{2}\). In general, we want the function \(f^k_\phi\) to be accurate within tolerance \(\frac{1}{2k+1}\).

Let \(A = F^0(A)\), and define

\[
F^{k+1}(A) = \left\{ b \in M : b = f^k_\phi(a_1, \ldots, a_n) \mid \phi \in \Delta, a_1, \ldots, a_n \in F^k(A) \right\} \\
\text{and } d^M\left(\phi^M(b, a_1, \ldots, a_n), \inf_y \phi^M(y, a_1, \ldots, a_n)\right) \leq \frac{1}{2k + 1}\]
Notice that if we take $\phi(x,y)$ to be the formula $x = y$, then for all $a \in F^k(A)$ we have $f_\phi(a) = a$. So for all $k \in \mathbb{N}$, $F^k(A) \subseteq F^{k+1}(A)$. Let $F^\omega(A)$ denote the limit of this operation. Suppose $b \in F^{\omega+1}(A)$. Then there is a formula $\phi(x_1, ..., x_{n+1})$ and elements $a_1, ..., a_n \in F^\omega(A)$ so that $b = f^\omega_\phi(a_1, ..., a_n)$. For each $i \leq n$, there is $k_i \in \mathbb{N}$ such that $a_i \in F^{k_i}(A)$, so letting $k = \max \{k_i : i \leq n\}$, we see that $f^\omega_\phi(a_1, ..., a_n) = f^k_\phi(a_1, ..., a_n)$, so $b \in F^{k+1}(A) \subset F^\omega(A)$. To turn $\tilde{N} = F^\omega(A)$ into a metric structure, we need to ensure it is complete, so we close $\tilde{N}$ in $\mathcal{M}$ to form a structure $N$. Completion along with the closeness requirement ensures that we can realize all infima over $F^k(A)$ for all natural $k$, and so in particular we can do so over $F^0(A) = A$.

So for all $n \in \mathbb{N}$, every $\phi(x_1, ..., x_{n+1}) \in \Delta$ and for all $a_1, ..., a_n \in N$, if there is $b \in M$ for which

$$\phi^M(b, a_1, ..., a_n) \leq \frac{1}{2^m}$$

for all $m \in \mathbb{N}$, then for every $k \in \mathbb{N}$ there is $a_k \in N$ such that

$$\phi^N(a_k, a_1, ..., a_n) \leq \frac{1}{2^m} + \frac{1}{2^k}$$

So every infimum in $M$ can be witnessed in $N$. By the Vaught test, $N \preceq M$.

Notice that $\chi(F(A)) \leq \chi(\Delta) \cdot \chi(A) = \kappa + \chi(\Delta) \cdot \kappa = \kappa$, since (i) $F(A)$ contains at most one element for every formula in $\Delta$ and element of $A$, as well as containing every element of $A$ itself and (ii) $\chi(\Delta) \leq \kappa$. Similarly, $\chi(F^{k+1}(A)) \leq \chi(F^k(A)) + \chi(\Delta) \cdot \chi(F^k(A)) = \kappa$. So $\chi(\tilde{N}) = \chi(F^\omega(A)) \leq \kappa$. $N$ is the completion of $F^\omega(A)$, so $\chi(N) = \aleph_0 \cdot \chi(F^\omega(A)) \leq \aleph_0 \cdot \kappa \leq \kappa$. 

4 Scott isomorphism theorem

**Definition 4.1.** Let $L$ be a metric language and $\phi$ a sentence of $L_{\omega_1 \omega}(\rho)$. For a cardinal $\kappa$, $I(\phi, \kappa)$ denotes the number of models of $\phi$ of density character $\kappa$ up to isomorphism.

The main theorem of this section is:

**Theorem 4.2** (Scott isomorphism theorem). For every separable structure in a metric language $L$, there is a sentence $\phi$ of $L_{\omega_1 \omega}$ for which $I(\phi, \aleph_0) = 1$. 

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To prove it, we’ll need a few definitions. Let $\mathcal{M} \in \mathcal{L}_{\omega_1\omega}(\rho)$ be separable. Choose and well-order a dense subset $A = (a_n)_{n \in \mathbb{N}}$ of $M$. Fix $\bar{a} \in A^n$ for some $n$.

Let’s define a notion of $\alpha$-equivalence for each ordinal $\alpha$. If $\bar{a}, \bar{b} \in A^n$, we say $\bar{a} \sim_0 \bar{b}$ if they satisfy the same formulas. Then $\bar{a} \sim_{\alpha+1} \bar{b}$ if for every one point extension $a$ of $\bar{a}$ there is $b$ such that $\bar{a}a \sim_\alpha \bar{b}b$ and for every one point extension $b$ of $\bar{b}$ there is $a$ such that $\bar{b}b \sim_\alpha \bar{a}a$. If $\alpha$ is a limit ordinal, then $\bar{a} \sim_\alpha \bar{b}$ if $\bar{a} \sim_\beta \bar{b}$ for all $\beta < \alpha$. Let’s flesh this out.

**Definition 4.3 (Types).** A type $p$ in variables $\bar{x}$ is a set of conditions on formulas in the free variables $\bar{x}$. A type is said to be realized by an element $\bar{a}$ of a metric structure $\mathcal{M}$ if $\bar{a}$ satisfies every condition in $p$. If no such $\bar{a}$ exists, then $\mathcal{M}$ is said to omit $p$. The notation $tp^\mathcal{M}(\bar{a})$ refers to the finitary type of $\bar{a}$ in $\mathcal{M}$.

The complete type of an element $\bar{a}$ of a $\mathcal{L}$-structure $\mathcal{M}$ with respect to a fragment $\Delta \subseteq \mathcal{L}_{\omega_1\omega}$ is the set of conditions in $\Delta$ satisfied by $\bar{a}$, or formulas $\phi(\bar{x}) \in \Delta$ for which $\phi^\mathcal{M}(\bar{a}) = 0$. This is denoted $tp^\Delta(\bar{a})$.

Let $\bar{a} \in A^n$. By separability of $\mathcal{L}$, $tp^\mathcal{M}(\bar{a})$ is determined by a dense subset $\{\theta_n(\bar{v}) : n \in \mathbb{N}\}$. For $\alpha < \omega_1$, we construct the sequence of Scott formulas $\phi_{\mathcal{M},\bar{a}}^\alpha$ as follows:

1. For $\alpha = 0$,

   $$\phi_{\mathcal{M},\bar{a}}^0(\bar{v}) = \sum_{n \in \mathbb{N}} \theta_n(\bar{v}) 2^{-n}$$

2. For $\alpha$ a countable limit ordinal,

   $$\phi_{\mathcal{M},\bar{a}}^\alpha(\bar{v}) = \bigwedge_{\beta < \alpha} \phi_{\mathcal{M},\bar{a}}^\beta(\bar{v})$$

3. For $\alpha = \beta + 1$, $\phi_{\mathcal{M},\bar{a}}^\alpha(\bar{v})$ is the maximum of:

   (a) $\phi_{\mathcal{M},\bar{a}}^\beta(\bar{v})$

   (b) $\bigwedge_{a \in A} \inf_s \rho^N_s((\bar{v},w), Z^N_a \phi_{\mathcal{M},\bar{a}}^\beta)$

   (c) $\sup_u \bigvee_{a \in A} \rho^N_u((\bar{v},u), Z^N_a \phi_{\mathcal{M},\bar{a}}^\beta)$
The interpretation of $\phi^{\alpha}_{M,\bar{a}}$ in a metric structure $N$ is denoted in the usual manner: $\phi^{\alpha,N}_{M,\bar{a}}$.

Notice that we used a countable conjunction in the case where $\alpha$ is a limit ordinal, but did not use it in the case $\alpha = 0$. In the latter case, the moduli of uniform continuity of the formulas $\theta_n \in p$ could be anything as long as they exist, so for a given $\epsilon > 0$ we may have $\lim_{n \to \infty} \delta_{\theta_n,x}(\epsilon) = 0$. Thus the countable conjunction would not be guaranteed a modulus of uniform continuity, and would therefore not necessarily be an admissible formula. In the case where $\alpha$ is a limit ordinal, however, we may take a countable conjunction. This is possible because the modulus of continuity of the $(\alpha+1)^{th}$ Scott formula is no worse than that of the $\alpha^{th}$, and so for every $\epsilon > 0$ we have $\inf_{\beta < \alpha} \delta_{\phi^\beta_{M,x}}(\epsilon) > 0$ in every variable $x$.

**Lemma 4.4.** Let $\mathcal{L}$ be a separable metric language. For every separable $M \in \text{Mod}(\mathcal{L})$, there is a countable ordinal $\alpha$ such that for all arities $k$ and $\bar{a}, \bar{b} \in A^k$, $\phi^\alpha_{M,\bar{a}}(\bar{b}) \leq \phi^\beta_{M,\bar{a}}(\bar{b})$ for all $\beta \geq \alpha$.

**Proof.** Let $M$ be arbitrary. Write $A$ for a dense subset of $M$. For any $\bar{a}, \bar{b} \in A$, it is clear that $\phi^\beta_{M,\bar{a}}(\bar{b}) \leq \phi^\alpha_{M,\bar{a}}(\bar{b})$ for all $\beta \leq \alpha$.

For the other direction, suppose not. So there is a separable structure $M$ with dense subset $A$ and some $k$-tuples $\bar{a}, \bar{b} \in A^k$ such that for every $\alpha < \omega_1$ there is $\beta > \alpha$ for which $\phi^\alpha_{M,\bar{a}}(\bar{a}) < \phi^\beta_{M,\bar{a}}(\bar{b})$. But then $\{\phi^\beta_{M,\bar{a}}(\bar{b}) : \beta < \omega_1\}$ forms a strictly increasing sequence of length $\omega_1$ in $[0, 1]$, an impossibility.

So the previous lemma guarantees a countable ordinal $\alpha(k, \bar{a}, \bar{b})$ for all $k \in \mathbb{N}$ and all $\bar{a}, \bar{b} \in A^k$. This allows us to define the sentence we’re looking for.

**Definition 4.5.**

1. The Scott height of $M$ is the countable ordinal

\[
\alpha_M = \text{sup} \text{ sup} \text{ sup} \alpha(k, \bar{a}, \bar{b})
\]

2. The Scott sentence of $M$ is the $\mathcal{L}_{\omega_1\omega}(\rho)$-sentence

\[
\Phi_M := \max \left\{ \phi^\alpha_{M,\bar{a}}(\bar{b}) \bigg| \text{sup} \text{ sup} \left\{ \phi^\alpha_{M,\bar{a}}(\bar{x}) - \phi^{\alpha+1}_{M,\bar{a}}(\bar{x}) : \ell(\bar{x}) = n \right\} \right\}
\]
Note 4.6. The construction of $\Phi_M$ depends on the choice of dense subset $A$, unlike the discrete case where we simply enumerate the structure whose Scott sentence we would like to compute.

The main theorem can now be rephrased in more transparent terms using our newly-expanded vocabulary:

**Theorem 4.7 (Scott isomorphism theorem v.2).** Suppose $\mathcal{M}, \mathcal{N}$ are separable structures in $\mathcal{L}_{\omega_1\omega}(\rho)$. If $\mathcal{N} \models \Phi \mathcal{M}$ then $\mathcal{N} \cong \mathcal{M}$.

**Proof.** Let $A = (a_n)_{n \in \mathbb{N}}$ be a countable dense subset of $\mathcal{M}$, and let $C = (c_n)_{n \in \mathbb{N}}$ be a countable dense subset of $\mathcal{N}$. Let $D$ be the disjoint union of countably many copies of $C$, so every element of $C$ appears countably many times in $D$. Clearly $D$ is a countable dense subset of $\mathcal{N}$. We proceed back-and-forth:

1. **Stage 1.** By assumption, $\phi_{M,\emptyset}^{\alpha_N} = (\phi_{M,\emptyset}^{\alpha_M} - \phi_{M,\emptyset}^{\alpha_M+1})^N = 0$, so $\phi_{M,\emptyset}^{\alpha+1_N} = 0$. Therefore $\inf_w \rho_N(w, Z^N \phi_{M,a}^{\alpha_M}) = 0$ for all $a \in A$, so $\inf_w \rho_N(w, Z^N \phi_{M,a_1}^{\alpha_M}) = 0$. Since the quantification is over $N$, we are not guaranteed an exact witness. So witness the infimum with $b_1^1 \in C \subseteq D$ such that $\rho_N(b_1^1, Z^N \phi_{M,a_1}^{\alpha_M}) \leq \frac{1}{2}$.

2. **Stage 2n.** At this stage, we have already chosen a subset of $A$, say $A_{2n} = \{a_1, a_{(2)}, ..., a_{2n-1}\}$. Since we choose from $A$ at odd stages and from $D$ at even stages and cannot be guaranteed that the order from $D$ can be mirrored exactly in $A$, we write $i(n)$ for the index corresponding to the $n^{th}$ choice from $D$. For notational convenience, define the function taking naturals to naturals $j(k) = \begin{cases} i(k) & \text{if } k \in 2\mathbb{N} \\ k & \text{else} \end{cases}$

In addition to the elements from $A$, we also have $k$-tuples $(b_1^k, ..., b_k^k) \in C^k$ (for all $1 \leq k \leq 2n-1$) such that

$$\rho_N \left( (b_1^k, ..., b_k^k), Z^N \phi_{M,a_1 \cdots a_{2n-1}}^{\alpha_M} \right) \leq \frac{1}{2^k}$$

Since $\Phi_{M,\emptyset}^{\mathcal{N}} = 0$, $\sup \left( \phi_{M,a_1 \cdots a_{2n-1}}^{\alpha_M}(\bar{v}) - \phi_{M,a_1 \cdots a_{2n-1}}^{\alpha_M+1}(\bar{v}) \right)^N = 0$, so $Z^N \phi_{M,a_1 \cdots a_{2n-1}}^{\alpha_M+1} \subseteq Z^N \phi_{M,a_1 \cdots a_{2n-1}}^{\alpha_M+1}$. Therefore,

$$\sup_u \bigvee_{a \in A} \rho_N \left( (b_1^{2n-1}, ..., b_{2n-1}^{2n-1}, u), Z^N \phi_{M,a_1 \cdots a_{2n-1}}^{\alpha_M} \right) = 0$$
Choose $c_{2n}$ least in $D \setminus \{c_m : m < 2n\}$. We can find $a_{i(2n)} \in A$ and $(b_{2n}^1, b_{2n}^2) \in C^{2n}$ such that

$$d^N((b_{2n-1}^1, \ldots, b_{2n-1}^2), (b_{2n}^1, \ldots, b_{2n}^2)) \leq \rho^N((b_{2n}^1, \ldots, b_{2n}^2), Z^N_{\phi_{M,a_1\cdots a_{i(2n)}}}) \leq \frac{1}{2^{2n}}$$

3. Stage $2n + 1$. At this stage, we have elements $a_1, \ldots, a_{i(2n)} \in A$ and $k$-tuples $(b_j^1, \ldots, b_j^k) \in C^k$ (for all $1 \leq k \leq 2n$) such that

$$\rho^N((b_j^1, \ldots, b_j^k), Z^N_{\phi_{M,a_1 \cdots a_{i(k)}}}) \leq \frac{1}{2^k}$$

By the same argument (using the Scott sentence) as the previous case, we see that

$$\bigwedge_{a \in A} \inf_x \rho^N((b_{2n}^1, \ldots, b_{2n}^2, x), Z^N_{\phi_{M,a_1 \cdots a_{i(2n)n}}}) = 0$$

Choose $a_{2n+1}$ least in $A \setminus \{a_{j(m)} : m \leq n\}$. We can find $(b_{2n+1}^1, \ldots, b_{2n+1}^2) \in C^{2n+1}$ such that

$$d^N((b_{2n}^1, \ldots, b_{2n}^2), (b_{2n+1}^1, \ldots, b_{2n+1}^2)) \leq \rho^N((b_{2n+1}^1, \ldots, b_{2n+1}^2), Z^N_{\phi_{M,a_1 \cdots a_{2n+1}}}) \leq \frac{1}{2^{2n+1}}$$

**Lemma 4.8.** For each $i \in \mathbb{N}$, the sequence $(b_j^i)_{j \in \mathbb{N}}$ is Cauchy.

**Proof.** Consider the $i^{th}$ sequence, and let $\epsilon > 0$ be given. Choose $k$ least such that $2^{-(k+1)} \leq \epsilon$. Then

$$d^N(b_i^1, b_i^{k+1}) \leq \max_{j \leq k} d^N(b_j^k, b_j^{k+1})$$

$$= \inf_x d^N((b_1^k, \ldots, b_i^k, x), (b_i^{k+1}, \ldots, b_i^{k+1})) \leq \frac{1}{2^{k+1}} \leq \epsilon$$

Applying this argument for any $j \geq (k+1)$ gives the result, as $k \geq j$ implies $2^{-(j+1)} \leq 2^{-(k+1)}$. \qed
As $\mathcal{N}$ is complete, every such sequence converges to some $b_i$. Define

$$B = \bigcup_{i \in \mathbb{N}} b_i$$

For all natural $n$, $tp^M(a_1, ..., a_n)$ can be approximated in $\mathcal{N}$ by elements of the set of $\langle b_1^{i_1}, ..., b_n^{i_n} \rangle$, where $m \geq n$ and the $(m + 1)^{th}$ such is within $1/2$ of the $m^{th}$. In the limit, $tp^M(a_1, ..., a_n) = tp^N(b_1, ..., b_n)$ and so $A$ and $B$ satisfy the same atomic formulas.

$B$ is dense in $\mathcal{N}$ since it is as dense as $C$; for $K$ large enough we have:

$$\left| \psi^N(d_1, ..., d_m) - \psi^N(b_{i(1)}, ..., b_{i(K)}) \right| \leq \left| \psi^N(d_1, ..., d_m) - \psi^N(b_{i(1)}, ..., b_{i(m)}) \right| + \left| \psi^N(b_{i(1)}, ..., b_{i(m)}) - \psi^N(b_{i(1)}, ..., b_{i(m)}) \right|$$

$$\leq 2 \cdot \frac{\epsilon}{2} = \epsilon$$

So there is a bijection between $A$ and $B$ which preserves all atomic formulas; by uniform continuity, this extends to an isomorphism $M \cong N$.

\[ \square \]

5 Infinitary stability

**Definition 5.1 (Amalgamation).** Let $\kappa$ be an infinite cardinal and $\mathcal{L}$ a metric language. A subset $S$ of the class of all $\mathcal{L}$-structures of density character $\kappa$ admits $\kappa$-amalgamation if for all $M_0, M_1, M_2 \in S$ for which there exist elementary embeddings $f_i : M_0 \hookrightarrow M_i$ ($i = 1, 2$) there is $N \in S$ and elementary maps $g_i : M_i \hookrightarrow N$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

In $\mathcal{L}_{\omega_1\omega}(\rho)$, we work within a countable fragment $\Delta$ and take all embeddings to be $\Delta$-elementary.

The following is an epsilonized version of lemma 3.4 in [11].

**Theorem 5.2.** Assume $V = L$. Let $\mathcal{L}$ be a separable metric language and let $\phi \in \mathcal{L}_{\omega_1\omega}(\rho)$. Suppose that $I(\phi, \aleph_0) = 1$ and that $I(\phi, \aleph_1) \neq 0$. If $\text{Mod}(\phi)$ does not admit $\aleph_0$-amalgamation, then $I(\phi, \aleph_1) = 2^{\aleph_1}$.

**Proof.** Since there are at most $2^{\aleph_1}$ many $\mathcal{L}$-structures of density character $\aleph_1$, it suffices to show that $I(\phi, \aleph_1) \geq 2^{\aleph_1}$.

For any subset $X \subseteq \omega_1$, construct a $\omega_1$-indexed sequence of separable models $M_\lambda^X \models \phi$ such that $M_\lambda^X$ is the (metric) completion of $\lambda < \omega_1$. The
successor case is the only one we will pay particular attention to in this construction. We can find a model $\mathcal{M}_0^X$, and for $\lambda$ a limit ordinal we simply define $\mathcal{M}_\lambda^X = \bigcup_{\gamma < \lambda} \mathcal{M}_\gamma^X$.

Partition $\omega_1$ into $\aleph_1$ disjoint stationary sets $S_\alpha$.

Since $V = L$ implies $\diamondsuit_{\aleph_1}(S_\alpha)$ for all $\alpha$, we may associate to each $\lambda \in \omega_1$ countable models $\mathcal{M}_\lambda^0, \mathcal{M}_\lambda^1$ with domain $\lambda$, along with a function $f_\lambda : \mathcal{M}_\lambda^0 \to \mathcal{M}_\lambda^1$ such that for every pair of $\mathcal{L}$-structures $\mathcal{M}^0, \mathcal{M}^1$ of density character $\aleph_1$ and for every function $g : \omega_1 \to \omega_1$, the set

$$\{ \lambda \in S_\alpha : \mathcal{M}^0|\lambda = \mathcal{M}_\lambda^0, \mathcal{M}^1|\lambda = \mathcal{M}_\lambda^1, g|\lambda = f_\lambda \}$$

is stationary in $\aleph_1$. By $\diamondsuit_{\aleph_1}(S_\alpha)$, for stationarily many $\lambda$, one of $\mathcal{M}^0, \mathcal{M}^1$ is dense in the restriction of any model $\mathcal{M}$ of density $\aleph_1$ to $\lambda$. So one of $\mathcal{M}_\lambda^0$, $\mathcal{M}_\lambda^1$ is dense in $\mathcal{M}_\lambda^X$ stationarily often. Let $l \in \{0, 1\}$. If $\beta$ is limit and $\mathcal{M}_\beta^l$ is dense in $\mathcal{M}_\beta^X$ for all $\lambda < \beta$, then $\bigcup \mathcal{M}_\lambda^l$ is dense in $\mathcal{M}_\lambda^X$. So the result holds on a closed set. Since $V = L$ implies $\diamondsuit_{\aleph_1}(S_\alpha)$ for all $\alpha$ (whose union covers $\omega_1$), the result also holds on an unbounded, and hence club set. So $\mathcal{M}_\lambda^l$ is dense in $\mathcal{M}_\lambda^X$ club often.

Suppose $\mathcal{M}_\lambda^X$ is defined. Witness the failure of $\aleph_0$-amalgamation with a separable model of $\phi$, $\mathcal{N}_0$, and elementary extensions $\mathcal{N}_1, \mathcal{N}_2$. Since $I(\phi, \aleph_0) = 1$, we can choose an isomorphism $g : \mathcal{N}_0 \to \mathcal{M}_\lambda^0$.

Suppose that $\lambda$ is part of the club set for which $\mathcal{M}_\lambda^l$ is dense in $\mathcal{M}_\lambda^X$. Since $\{ S_\alpha \}$ is a disjoint cover of $\omega_1, \lambda \in S_\alpha$ for exactly one $\alpha < \aleph_1$. Suppose further that we have chosen $\mathcal{M}_\lambda^X$ so that $\alpha \in X$ if and only if $l = 0$. Since $\aleph_0$-amalgamation fails for the triple $(\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)$, for each such $\lambda < \omega_1$ there are two choices for $\mathcal{M}_{\lambda+1}^X$.

Since $X$ is an arbitrary subset of $\omega_1$ and $\alpha \in X$ if and only if $l = 0$, we can make different choices of $X$ render different $X$-indexed sequences of models $\mathcal{M}_\lambda^X$. Choose a separable structure $\mathcal{M}_{\lambda+1}^X \succ \mathcal{M}_\lambda^X$ such that:

1. If $\alpha \in X$, then extend $g$ to an isomorphism $\mathcal{N}_1 \cong \mathcal{M}_{\lambda+1}^X$.
2. If $\alpha \notin X$, then extend $g$ to an isomorphism $\mathcal{N}_2 \cong \mathcal{M}_{\lambda+1}^X$.

Let $\mathcal{M}^X = \bigcup \mathcal{M}_\lambda^X$. Since $\text{Mod}(\phi)$ is closed under unions of chains and $\mathcal{M}^X$ is of density character $\aleph_1$, it suffices to see that distinct choices of $X$ result in distinct models $\mathcal{M}^X$. Suppose that $\mathcal{M}^X \cong \mathcal{M}^Y$ for $X \neq Y \in \mathcal{P}(\omega_1)$ by some isomorphism $h$.

For stationarily many $\lambda < \omega_1$, $\mathcal{M}_\lambda^0$ is dense in $\mathcal{M}_\lambda^X$, $\mathcal{M}_\lambda^1$ is dense in $\mathcal{M}_\lambda^X$, and $h : \mathcal{M}^X \to \mathcal{M}^Y$ restricts to an isomorphism $f_\lambda : \mathcal{M}_\lambda^X \to \mathcal{M}_\lambda^Y$. This,
along with the fact that \( g : N_0 \to \mathcal{M}_X^X \) is an isomorphism, implies that \( f_\lambda \circ g : N_0 \to \mathcal{M}_X^X \) is an isomorphism. We assumed \( N_1, N_2 \) cannot be amalgamated over \( N_0 \cong \mathcal{M}_X^X \cong \mathcal{M}_Y^Y \), but either of \( \mathcal{M}_X^X, \mathcal{M}_Y^Y \) is such as \( h \) is an isomorphism, a contradiction. 

So categoricity and no amalgamation in \( \aleph_0 \) give us as many models as possible in character density \( \aleph_1 \). Suppose that \( \phi \) is such that \( I(\phi, \aleph_0) = 1 \). Then if there are fewer than \( 2^{\aleph_1} \) models of density character \( \aleph_1 \), Mod(\( \phi \)) must admit \( \aleph_0 \)-amalgamation. This implies the following theorem, whose proof follows the same general lines as the similar result in [11]:

**Theorem 5.3.** Assume \( V = L \) and let \( L \) be a separable metric language. Suppose \( \phi \) is a \( L_{\omega_1\omega}(\rho) \)-sentence such that \( I(\phi, \aleph_0) = I(\phi, \aleph_1) = 1 \). Then \( I(\phi, \aleph_2) > 0 \).

### 6 Appendices

#### 6.1 Ultrafilters, ultralimits, and ultraproducts

**Definition 6.1.** Given a set \( X, F \subseteq \mathcal{P}(X) \) is a filter if:

- \( \emptyset \notin F \),
- if \( A, B \in F \) then \( A \cap B \in F \), and
- if \( A \in F \) and \( A \subseteq B \subseteq X \) then \( B \in F \).

**Lemma 6.2.** \( G \subseteq \mathcal{P}(X) \) is contained in a filter iff \( G \) has the finite intersection property i.e. for every finite \( G_0 \subseteq G \), \( \bigcap G_0 \neq \emptyset \).

**Proof.** Suppose \( G \subseteq F \) for a filter \( F \) and consider some finite subcollection \( G_0 \subseteq G \). Since every \( x \in G_0 \) is in \( F \), \( \bigcap G_0 \in F \) as \( F \) is a filter. So \( G \) has the finite intersection property.

Conversely, if \( G \) has the finite intersection property, then every finite subset \( G_0 = (x_1, ..., x_n) \) has \( \bigcap G_0 \neq \emptyset \). In particular, note that \( G \) almost has two of the three properties required for it to be a filter already, as \( G \) can easily closed under finite intersections and \( \emptyset \notin G \). So that \( G \) actually has two of the three filter properties, close \( G \) under finite intersections. Since \( \emptyset \notin G \) and \( G \) has the f.i.p., closing \( G \) under finite intersections will give a collection \( G' \) which has the f.i.p. For the last property, close \( G' \) upwards under inclusion to get a filter \( G'' \). To do this, partially order \( \mathcal{P}(X) \) by
inclusion. For each $x \in G'$, add to $G''$ every $y \in \mathcal{P}(X)$ such that $y \geq x$. Note that $G''$ has the f.i.p., since every element of $G''$ is either in $G'$ or a superset of an element of $G'$, and $G'$ has the f.i.p. from above. So $G''$ is a filter, and so $G$ is included in a filter.

**Definition 6.3.** For $F$ a filter on $X$, if it holds that either $A \in F$ or $X \setminus A \in F$ for all $A \in \mathcal{P}(X)$, then $F$ is called an **ultrafilter**.

**Lemma 6.4.**

1. A filter is an ultrafilter if and only if it is maximal.
2. Any filter on $X$ can be extended to an ultrafilter on $X$.

**Proof of (1).** Suppose $F$ is an ultrafilter, and suppose there exists some $F'$ strictly containing $F$. So there exists some $A \in (F' \cap F^c)$, which implies that $A^c \in F$. Since $A \cap A^c = \emptyset$ and we supposed that $F'$ strictly contained $F$, $A^c \in F'$, and so $F'$ is not a filter as $A \cap A^c = \emptyset$. So $F$ is a maximal filter.

Now suppose $F$ is a maximal filter and choose any $A \in \mathcal{P}(X)$. For any $B, C \in \mathcal{P}(X)$ for which $A \cap B = \emptyset = A^c \cap C$, it follows that $B \cap C = (A \cap B) \cap (A^c \cap C) = \emptyset$, and so at least one of $B, C \in F^c$. Since $F$ is maximal, this says that at least one of $A \in F$ or $A^c \in F$. Since $\emptyset \notin F$ and $F$ is closed under finite intersections, exactly one of $A, A^c$ is in $F$ for any $A \in \mathcal{P}(X)$ and so $F$ is an ultrafilter.

**Proof of (2).** The set of filters on $X$ can be partially ordered by $\subseteq$. Let $(F_i, \subseteq)_{i \in I}$ be a chain of filters on $X$ indexed by $I$. Then $\cup_{i \in I} F_i$ contains every $F_i$. Furthermore, $\cup F_i$ is a filter since each $F_i$ is a filter. Thus by Zorn’s lemma, every chain has a maximal element, and so every filter $F$ is contained in a maximal filter, and hence in an ultrafilter by (1) of this lemma.

**Example 6.5.** Suppose that $X$ is a set.

- If $a \in X$ then $U = \{A \in \mathcal{P}(X) : a \in A\}$ is an ultrafilter; ultrafilters of this kind are called principal.

- If $X$ is infinite, the set of cofinite subsets of $X$ is a filter called the Frechet filter on $X$; it is contained in all non-principal ultrafilters on $X$.

- Let $Y = \mathcal{P}_{fin}(X)$ be the set of finite subsets of $X$. For any finite subset $A$ of $X$, let $O_A = \{B \in Y : A \subseteq B\}$. The set $F = \{O_A : A \in Y\}$ has the finite intersection property and is not contained in a principal ultrafilter.
Definition 6.6. Let $I$ be any set, $\mathcal{U}$ be a nonprincipal ultrafilter on $I$ and $\vec{r} = (r_i)_{i \in I}$ a sequence of elements of $\mathbb{R}$. Then

$$\lim_{\mathcal{U}} r_i = r \text{ iff for every } \epsilon > 0, \{i \in I : |r - r_i| < \epsilon\} \in \mathcal{U}$$

Then $r \in \mathbb{R}$ is called the **ultralimit** of $\vec{r}$ along $\mathcal{U}$.

**Lemma 6.7.** If $\vec{r}$ is bounded then

1. $\lim_{\mathcal{U}} r_i$ exists and is unique;

2. $\lim_{\mathcal{U}} r_i = \inf_{\mathcal{U}} \{B : \{i \in I : r_i \leq B\} \in U\} = \sup_{\mathcal{U}} \{B : \{i \in I : r_i \geq B\} \in U\}$

**Proof of (1).** Since $\vec{r}$ is a bounded sequence of real numbers, there is $B \in \mathbb{N}$ such that for all $i \in I$, $r_i \in [-B, B]$. As $\mathcal{U}$ is an ultrafilter, there are either $\mathcal{U}$-many $r_i$ in $[-B, 0]$ or $\mathcal{U}$-many $r_i$ in $[0, B]$. Suppose without loss of generality that there are $\mathcal{U}$-many $r_i$ in $[0, B]$. Then, again there are $\mathcal{U}$-many $r_i$ in either $[0, \frac{B}{2}]$ or in $[\frac{B}{2}, B]$. Supposing that we choose the lower-half of the interval at each step, we are presented with a nested sequence of compact intervals $\{0, \frac{B}{2^n}\}$, and so the intersection $\bigcap_{n \in \mathbb{N}} [0, \frac{B}{2^n}] \neq \emptyset$. In general, the fact that $\mathcal{U}$ is an ultrafilter allows us to generate a $\mathbb{N}$-sequence of nested compact intervals $(C_n)$, which is nonempty by Cantor’s intersection theorem on nested sets. So the ultralimit always exists.

Now suppose that the ultralimit is not unique; i.e. that for some ultrafilter $\mathcal{U}$ on $I$, the sequence $\vec{r} = (r_i)$ has two distinct ultralimits $r_0, r_1 \in \mathbb{R}$. Since $\mathbb{R}$ is a metric space and $r_0 \neq r_1$, there exist open neighbourhoods $V_0, V_1$ of $r_0, r_1$ respectively such that $V_0 \cap V_1 = \emptyset$. In other words, there are some $\epsilon_0, \epsilon_1 > 0$ such that $B_{\epsilon_0}(r_0) \cap B_{\epsilon_1}(r_1) = \emptyset$. But by assumption $r_0, r_1$ are ultralimits of $\vec{r}$, so for $\mathcal{U}$-many $i \in I$, $r_i \in B_{\epsilon_0}(r_0)$ and for $\mathcal{U}$-many $j \in I$, $r_j \in B_{\epsilon_1}(r_1)$. Call the former subset of $I$ $I_0$ and the latter $I_1$. Since both are in $\mathcal{U}$, $I_0 \cap I_1 \in \mathcal{U}$. However, it is true for no $i \in I$ that $r_i \in B_{\epsilon_0}(r_0) \cap B_{\epsilon_1}(r_1)$, so $I_0 \cap I_1 = \emptyset$, contradicting the hypothesis that $\mathcal{U}$ is a filter. 

**Proof of (2).** From the definition, $\lim_{\mathcal{U}} r_i$ is the point $r \in \mathbb{R}$ such that for every $\epsilon > 0$, $\{i \in I : |r - r_i| < \epsilon\} \in \mathcal{U}$, so $\{i : r_i - \epsilon < r < r_i + \epsilon\} \in \mathcal{U}$. Since $r \geq r_i$ for $\mathcal{U}$-many $i$, from the inequality $r_i - \epsilon < r$, it follows that $r = \inf \{B : \{i \in I : r_i \leq B\} \in U\}$. Similarly, from the inequality $r < r_i + \epsilon$, it follows that $r = \sup \{B : \{i \in I : r_i \geq B\} \in U\}$.
6.2 Jensen’s diamond principle

**Definition 6.8 (Club set).** Let $\alpha$ be a limit ordinal. $\beta \subseteq \alpha$ is closed in $\alpha$ iff for all $\gamma < \alpha$, $\sup(\beta \cap \gamma) = \gamma$ implies that $\gamma \in \beta$. $\beta \subseteq \alpha$ is unbounded in $\alpha$ iff for all $\gamma < \alpha$ there is $\mu \in \beta$ such that $\gamma < \mu$. A set which is both closed and bounded is called club.

Intuitively, club sets can be thought of as sets of measure one. The corresponding notion for sets of positive measure follows.

**Definition 6.9 (Stationary set).** Let $\kappa$ be a cardinal with $\text{cf}(\kappa) > \aleph_0$. $S \subseteq \kappa$ is stationary if $S$ intersects every club set in $\kappa$.

With this, we can define an interesting set-theoretic property known as Jensen’s diamond principle. It features prominently in the proof of theorem 5.2.

**Definition 6.10 ($\Diamond_\kappa(S)$).** Let $\kappa$ be a regular cardinal and $S \subseteq \kappa$ a stationary set. Then for each $\alpha \in S$ there is $A_\alpha \subseteq \alpha$ such that for all $A \subseteq \kappa$, $\{ \alpha \in S : A \cap \alpha = A_\alpha \}$ is stationary in $\kappa$.

We state the following set-theoretic facts about $\Diamond$ without proof.

1. $V = L$ implies $\Diamond_\kappa(S)$ for all regular $\kappa$ and stationary subsets $S$.
2. $\Diamond_\kappa(S)$ implies $\kappa^+ = 2^\kappa$.

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