Minimizers for Axisymmetric Membranes

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Differential Geometry
- Surfaces
- Curvature

Biological Membranes
- Examples
- Models

Calculus
- Variations

Results
- Calculations

Example

References
Motivation

- Why are cells shaped the way they are?
- Possible models to explain behaviour?
- Geometry behind membrane structures?
- Characterizing energy of cells?

\[ W(\mathcal{M}) = \int_{\mathcal{M}} [H^2 - K] dS \]
How are distances measured on a surface? (Metric)

**Definition (First Fundamental Form)**

The first fundamental form is the inner product on the tangent space of a surface. Which can be conveniently written as a metric tensor $g$, in which the components are calculated using

$$g_{ij} = \vec{r}_i \cdot \vec{r}_j$$
What else can we say about surfaces? (Tangent Space)

**Definition (Second Fundamental Form)**

The second fundamental form is a quadratic form on the tangent plane of a smooth surface. It can be conveniently written as a tensor $\mathbb{II}$, in which the components are calculated using

$$\mathbb{II}_{ij} \equiv \vec{r}_{ij} \cdot \vec{N}$$
Our Main Tool For Curvature

Definition (Shape Operator)

The differential \((df)\) of the Gauss map \((f)\) defines the extrinsic curvature of a surface. It can be defined as a linear operator on the tangent space by the inner product \((S_\mathbf{x} \mathbf{v}, \mathbf{w}) = (df(\mathbf{v}), \mathbf{w})\) for tangent vectors \(\mathbf{v}, \mathbf{w}\) and \(\mathbf{x}\) is the point on the surface. It can be more conveniently written as

\[
S = g^{-1} \mathbf{II}
\]
What exactly is curvature?
Curvature in 3D

planes of principal curvatures

normal vector

tangent plane

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Notation for curvature (Importance of the Shape Operator)

- Principal Curvatures
  \[ \lambda_1 \text{ & } \lambda_2 \]

- Mean Curvature
  \[ H = \frac{\lambda_1 + \lambda_2}{2} = \frac{1}{2} \text{Trace}(S) \]

- Gaussian Curvature
  \[ K = \lambda_1 \lambda_2 = Det(S) = \frac{Det(\mathbb{II})}{Det(g)} \]
Gauss-Bonnet Theorem

Theorem

Suppose $\mathcal{M}$ is a compact two-dimensional Riemannian manifold without boundary. Let $K$ be the Gaussian curvature of $\mathcal{M}$. Then

$$\int_{\mathcal{M}} KdS = 2\pi \chi(\mathcal{M})$$

where $dS$ is the area element of the surface, and $\chi(\mathcal{M})$ is the Euler characteristic of $\mathcal{M}$ (Note that the Euler characteristic is defined as $\chi = V - E + F$).
Examples of Surfaces (General Cells)

Prokaryotic cell

- Plasma membrane
- Cytoplasm
- DNA
- Nucleoid region
- Nucleus
- Ribosomes

0.1-10 μm

Eukaryotic cell

- Plasma membrane
- Cytoplasm
- DNA
- Nucleus
- Ribosomes

10-100 μm
Biological Membranes

- Liposome
- Micelle
- Bilayer sheet
A closer look at the bilayer

Saturated lipids only

Mixed saturated and unsaturated
Why this isn’t a happy ending
Phases of membranes
The Willmore and Canham-Helfrich energy functionals

\[ W(M) = \int_M [H^2 - K]dS \quad - \text{Willmore} \]

\[ F(M) = \int_M \left[ \frac{\kappa_H}{2} (2H - H_0)^2 + \kappa_G K \right]dS \quad - \text{Canham-Helfrich} \]

- \( \mathcal{M} \) - The surface (membrane)
- \( \kappa_H \) & \( \kappa_G \) - Bending and Gaussian Rigidity
Generalizing the Canham-Helfrich functional

\[ F(M, \phi) = \int_M \left[ \frac{\kappa_H(\phi)}{2} (2H - H_0(\phi))^2 + \kappa_G(\phi)K + \eta(\phi) \right] dS \]

- Incorporating phase

\[ \phi : [0, 1] \times [0, 2\pi] \rightarrow (1, \infty) \]

- Entropy element

\[ \eta(\phi) \equiv \phi \log \phi \]
How to mathematically work with these models?

- Define our shape as some parametrization

\[ r(u, v) = (f(u, v), g(u, v), \xi(u, v)) \subset \mathbb{R}^3 \]

- Define a functional that inputs our shape

**Definition**

A functional \( (\mathcal{F}) \) is defined as:

\[ \mathcal{F} : V \rightarrow \hat{\mathbb{R}} \]

where \( V \) is some function space and \( \hat{\mathbb{R}} \) is the one-point compactification of the real line (i.e. \( \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \)).
Why do we care?

- We can apply concepts from calculus (such as minimizers)
- Things commonly seen around us are minimizers

**Definition (Minimizers)**

A minimizer of $\mathcal{F}$ is some function $\mathcal{M}_0 \in V$, such that for all $\mathcal{M} \in U \subset V$ we have that:

$$\mathcal{F}(\mathcal{M}_0) \leq \mathcal{F}(\mathcal{M})$$
The First & Second Variations

Definition
If $\mathcal{M}$ is a stable minimizer of some functional $\mathcal{F}$ and $h$ is a sufficiently small and smooth function then:

$$\delta^{(1)}\mathcal{F}(\mathcal{M}) = \lim_{\epsilon \to 0} \frac{\mathcal{F}(\mathcal{M} + \epsilon h) - \mathcal{F}(\mathcal{M})}{\epsilon} = \frac{d}{d\epsilon} \mathcal{F}(\mathcal{M} + \epsilon h)\bigg|_{\epsilon=0} = 0$$

&

$$\delta^{(2)}\mathcal{F}(\mathcal{M}) = \frac{d^2}{d\epsilon^2} \mathcal{F}(\mathcal{M} + \epsilon h)\bigg|_{\epsilon=0} > 0$$
Assumptions

- $\mathcal{M}$'s parameterization
  $$r(t, \theta) = (\gamma_1(t) \cos \theta, \gamma_1(t) \sin \theta, \gamma_2(t))$$

- Spaces for $\gamma_1$ & $\gamma_2$
  $$\gamma_1, \gamma_2 \in W^{2,2}(\{\gamma_1(t) > 0\}; \mathbb{R})$$

- No kinks around rotation axis
Consequences

- Principal Curvatures:

\[ \lambda_1 = \frac{\ddot{\gamma}_2 \dot{\gamma}_1 - \ddot{\gamma}_1 \dot{\gamma}_2}{|\dot{\gamma}|^3} \quad \& \quad \lambda_2 = \frac{\dot{\gamma}_2}{\gamma_1 |\dot{\gamma}|} \]

- Area Form

\[ A(M) = 2\pi \int_0^1 \gamma_1 |\dot{\gamma}| dt \]

- Volume Form

\[ V(M) = \pi \int_0^1 \gamma_1^2 \dot{\gamma}_2 dt \]
Let’s poke it with a stick! (Variations)

- How to poke our system?
- Constrain area and volume forms.
- Add a sufficiently small variation $h$ with a scaling factor $\epsilon$.

\[
 r_{\text{new}}(t, \theta) = r_{\text{old}}(t, \theta) + \epsilon h(t, \theta) N
\]

- It’s a whole different ball game now!
The Shape Equation

\[ \mathcal{F}(t, \theta) = \int_0^1 \int_0^{2\pi} \left[ \frac{\kappa_H(\phi)}{2} (2H-H_0(\phi))^2 + \kappa_G(\phi) K + \eta(\phi) \right] \gamma_1 |\dot{\gamma}| d\theta dt \]

- Approximating \( \kappa_G(\phi) = \text{Const} \) almost everywhere, allows Gaussian term to drop

\[ \kappa_H(\phi)(2H-H_0(\phi))(2H^2-2K-H_0(\phi)H)+2\Delta(\kappa_H(\phi)H)-2H\eta(\phi) = 0 \]
The Torus (Red Blood Cell)
References


4. Timon Idema: Structure, shape and dynamics of biological membranes (2009)

Thanks for listening

The End! Questions?