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On the discriminants of forms with Arf invariant one

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If A denotes the ring of integers in a dyadic local field E with trivial involution, then the Arf invariant of a quadratic form (with unimodular symmetric bilinearization) on a free A-module is determined by its discriminant in $E^\times/E^{\times 2}$. In this case either one of these invariants together with the rank classifies the form up to isometry [A], [O'M], § 93, [W 1], p. 66.

This fact is the basic for understanding the map between surgery obstruction groups [W 2]:

$$\Psi_i : L_i^K(\hat{\mathbb{Z}}_2\pi) \longrightarrow L_i^K(\hat{\mathbb{Q}}_2\pi) \tag{0. 1}$$

induced by inclusion of scalars $\hat{\mathbb{Z}}_2 \subseteq \hat{\mathbb{Q}}_2$, where $\hat{\mathbb{Z}}_2\pi$ is the group ring of the finite group π over the 2-adic integers and $\hat{\mathbb{Q}}_2$ denotes the 2-adic completion of the rational numbers. The map Ψ_i is the key to systematic calculations of the surgery obstruction groups of $\mathbb{Z}\pi$ (compare [W 2], § 4. 3). In this paper we calculate (0. 1) for 2-hyperelementary groups and express the answer in terms of representation theory. The result is used in [HM 2] to tabulate $L_i^{\langle -1\rangle}(\mathbb{Z}\pi)$, and correct the calculations of $L_i^p(\mathbb{Z}\pi)$ given in [BK], [K 1], and [K 2].

A 2-hyperelementary group is a semi-direct product $\pi = \mathbb{Z}/m \rtimes \sigma$, where $\mathbb{Z}/m$ is the cyclic group of odd order m, σ is a 2-group and the action of σ on $\mathbb{Z}/d$ is via a homomorphism $t : \sigma \to (\mathbb{Z}/d)^\times$. To define hermitian and quadratic forms, the group ring $\mathbb{Z}\pi$ must also be equipped with an involution. For example, the standard involution induced by

$$g \longrightarrow g^{-1}, \quad \text{for} \quad g \in \pi$$

arises from surgery on oriented manifolds. Our main result is Theorem 1. 16, where the answer is given for an arbitrary geometric anti-structure (see (1. 5)). This result covers all the involutions usually encountered in surgery theory.

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The L-theory and the map (0. 1) have a natural direct sum splitting indexed by the divisors of m [HM 1], § 6, and for $d|m$ the d-component $L_i^K(\hat{\mathbb{Z}}_2\pi)(d)$ is isomorphic to the L-group $L_i^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]^t\sigma)$ of the twisted group ring. In particular it is enough to consider the d-component of $\mathbb{Z}/d \rtimes \sigma$. After reducing modulo the radical, we see that the domain of $\Psi_i(d)$ is just a direct sum of $\mathbb{Z}/2$'s, (detected by the Arf invariant if i is even) one for each factor of $\hat{\mathbb{Q}}_2 \otimes \mathbb{Z}[\zeta_d]^t\bar{\sigma}$ with trivial involution on its centre ($\bar{\sigma} = t(\sigma)$). The range of Ψ_i is the direct sum of L-groups of complete local dyadic fields (the centres of the involution-invariant irreducible rational representations of π which are faithful on $\mathbb{Z}/d$). These are determined [W 2] by their discriminants and Hasse invariants.

The irreducible complex characters of $\mathbb{Z}/d \rtimes \sigma$ which are faithful on $\mathbb{Z}/d$ are induced up from $\chi \otimes \xi$ on $\mathbb{Z}/d \times \sigma_1$, where χ is a linear faithful character on $\mathbb{Z}/d$ and ξ is an irreducible character of $\sigma_1 = \ker t$. This is a 1-1 correspondence on the orbits of the conjugation action of σ/σ_1 [S 1]. A character is *type* I if the involution induced on the centre field of the associated simple algebra is trivial, otherwise *type* II. The simple involution-invariant summand of $\hat{\mathbb{Q}}_2 \otimes \mathbb{Q}[\zeta_d]^t\sigma$ containing the induced character $(\chi \otimes \xi)^*$ is further classified by a sub-type (O, Sp, GL or U). These depend also on the action of σ/σ_1 and the anti-structure. Our main result shows that the map $\Psi_i(d)$ is either injective or zero and that the types of the simple summands of $\hat{\mathbb{Q}}_2\pi$ corresponding to type I linear characters of σ_1 are enough to decide this. The projection of the image of $\Psi_{2n}(d)$ however, is non-trivial also at certain type II linear characters of σ_1. The precise result is stated in Theorem 1. 16.

The calculation of (0. 1) is given in [W 2], 4. 3. 2, assuming σ abelian (including the non-oriented involutions), and is implicit in [LM] for the case when σ_1 is abelian. Compare also [C], § 4, for an overlapping result assuming the standard involution. For $d=1$ and any geometric anti-structure the map (0. 1) was computed in [HTW 1]. An incorrect assertion about this map is contained in [K 1], 4. 23, 4. 24, [K 2], 3. 5, [BK], 3. 4. The simplest counter-example is $\pi = \mathbb{Z}/3 \rtimes \mathbb{Z}/4$ with $\ker t = \mathbb{Z}/2$, where we prove that $\ker \Psi_2 \cong \mathbb{Z}/2$ *not* $(\mathbb{Z}/2)^2$. The source of the discrepancy seems to be an error in [Bak], Cor. 4: for $\lambda = -1$, the maximal form parameter on

$$\left(M_2(\hat{\mathbb{Z}}_2),\, \beta: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right)$$

does not reduce to the maximal form parameter on $M_2(\mathbb{F}_2)$. This affects the contribution of the type Sp factors to the calculation.

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§ 1. Recollections and notations

In this paper we follow the definitions and conventions for quadratic forms found in [W 2], § 1. 1. Thus $\mathcal{Q}(A, \alpha_0, a_0)$ denotes the category of non-singular quadratic forms on free finitely generated right A-modules, associated to the anti-structure (A, α_0, a_0), and $\alpha_0^2(x) = a_0 x a_0^{-1}$, $\alpha_0(a_0) = a_0^{-1}$.

Hermitian bimodules give functors between quadratic categories. We recall this process, referring to [Fr-McE], [HRT], § 5, for alternatives and more details. For a given anti-structure and left (resp. right) A-modules N, we define the transposed right (resp. left) A-module N^t by

$$a \cdot m = m\alpha_0(a), \text{ (resp. } m \cdot a = \alpha_0(a)m).$$

Then $M \cong M^{tt}$ via translation by a_0.

A *hermitian* (B, β_0, b_0)-(A, α_0, a_0) *bimodule* is a pair $h = (W, h)$ consisting of an B-A bimodule $W = {}_BW_A$, and a bimodule isomorphism

$$h: W \longrightarrow \mathrm{Hom}_A(W, A)^t$$

with

$$(1.1) \qquad \alpha_0 h(w_1, w_2) = h(w_2, b_0 w_1 a_0^{-1}).$$

Here $h(w_1, w_2) = h(w_1)(w_2)$, and $(\ )^t$ transposes simultaneously the A- and B-structure (so $h(w_1, w_2 a) = h(w_1, w_2)a$, $h(w_1 a, w_2) = \alpha_0(a)\, h(w_1, w_2)$ and $h(bw_1, w_2) = h(w_1, \beta_0(b) w_2)$).

If $(M, q) \in \mathcal{Q}(B, \beta_0, b_0)$ and W is finitely generated A-free we define

$$(M \otimes_B W, q \otimes h) \in \mathcal{Q}(A, \alpha_0, a_0)$$

by the formula

$$(1.2) \qquad (q \otimes h)(m_1 \otimes w_1, m_2 \otimes w_2) = h(w_1, q(m_1, m_2) w_2).$$

This gives a functor from $\mathcal{Q}(B, \beta_0, b_0)$ to $\mathcal{Q}(A, \alpha_0, a_0)$ and hence a homomorphism

$$h_{\#}: L_n^K(B, \beta_0, b_0) \longrightarrow L_n^K(A, \alpha_0, a_0).$$

We make this more explicit in some cases relevant to our calculations:

(i) Let $f: (A, \alpha_0, a_0) \to (B, \beta_0, b_0)$ be a map. Set $W = {}_AB_B$ with

$$h(b_1, b_2) = \beta_0(b_1) b_2$$

(and reverse the roles of A and B in (1.1)). The induced $h_{\#}$ is the usual covariant f_*. Of particular interest to us is the case $B = A/I$ where I is a 2-sided ideal and A is complete in the I-adic topology. In this case f_* is an isomorphism [W 1], § 2.

(ii) Suppose $i: A \to B$ is a map of rings (not necessarily preserving the anti-structures) with B finitely generated A-free. Suppose $\mathrm{Tr}: B \to A$ is a right A-module homomorphism with

$$(1.3) \qquad \alpha_0 \mathrm{Tr}(b) = \mathrm{Tr}(\beta_0(b) b_0 a_0^{-1}).$$

Set $W = {}_BB_A$ and consider the trace form

$$h(b_1, b_2) = \mathrm{Tr}(\beta_0(b_1) b_2).$$

This satisfies (1. 1), so if non-singular, induces a map f^* from $L_n^K(B, \beta_0, b_0)$ to $L_n^K(A, \alpha_0, a_0)$, depending on Tr.

A simple special case occurs for a Galois extension of (commutative) rings, e.g. an unramified extension of a complete local 2-ring, [AG]. Given an involution β_0 on B which commutes with the Galois action, let $\alpha_0 = \beta_0 | A$ and $b_0 = a_0$. Then the usual trace satisfies (1. 3), and $\operatorname{Tr}(\beta_0(b_1) b_2)$ is non-singular.

Another special case occurs for group rings $B = \Lambda G$ and $A = \Lambda H$, where Λ is a commutative ring and $H \subset G$ is a subgroup. If $b_0 \in A$ and $\beta_0(A) = A$, we can let $\alpha_0 = \beta_0 | A$ and $a_0 = b_0$. Then the Λ-linear map

$$\operatorname{Tr}(g) = g \text{ if } g \in H, \quad \operatorname{Tr}(g) = 0 \text{ if } g \notin H$$

satisfies the condition (1. 3) and induces the usual restriction map

$$i^* : L_n^K(B, \beta_0, b_0) \longrightarrow L_n^K(A, \alpha_0, a_0).$$

(iii) Let (Λ, c) be a commutative ring with involution c, and $B = \Lambda^\gamma G$ the twisted group ring of a finite group G over Λ, associated with a homomorphism $\gamma : G \to \operatorname{Aut}_c(\Lambda)$ commuting with the action of c. Choose an automorphism θ of G with θ^2 inner, say $\theta^2(g) = \hat{b} g \hat{b}^{-1}$ for some $\hat{b} \in G$, and let $w : G \to \{\pm 1\}$ be a homomorphism. Suppose

$$(1.4) \qquad w \circ \theta = w, \quad \gamma \circ \theta = \gamma, \quad \theta(\hat{b}) = \hat{b}, \quad w(\hat{b}) = 1, \quad \gamma(\hat{b}) = \mathrm{id}.$$

Then a *geometric* anti-structure on B is defined by (the Λ-linear extension of):

$$(1.5) \qquad \beta(\lambda g) = w(g)\, \theta(g^{-1})\, c(\lambda); \; b = u \hat{b}$$

where $g \in G$, $\lambda \in \Lambda$ and $u \in \{\pm 1\}$.

(iv) Let (B, β_0, b_0) be the geometric anti-structure from (iii) and consider the subring $A = \Lambda G_1$, where $G_1 = \ker(\gamma : G \to \operatorname{Aut}(\Lambda))$. Note that θ induces an automorphism of G_1, and assume $\gamma(b) = 1$ so that $b_0 \in A$. Then (A, α_0, a_0) is an anti-structure where $\alpha_0 = \beta_0 | A$, $a_0 = b_0$ and the transfer map given in (ii) induces

$$i^* : L_n^K(B, \beta_0, b_0) \longrightarrow L_n^K(A, \alpha_0, a_0).$$

(v) Let Λ / A be a finite Galois extension of commutative rings with Galois group G, [AG], and c an involution of Λ commuting with G. Give $B = \Lambda^\gamma G$ the anti-structure from (1. 5) with $\theta = $ identity and $b_0 = a_0 \in A$. There are two hermitian bimodules, both supported on Λ, namely

$$W = {}_B\Lambda_A, \quad h(\lambda_1, \lambda_2) = \operatorname{Tr}(c(\lambda_1)\lambda_2),$$
$$V = W^t, \qquad h^t(\lambda_1, \lambda_2) = c(\lambda_1)\lambda_2.$$

They give inverse isomorphisms (Morita invariance)

$$h_\# : L_n^K(B, \beta_0, b_0) \longrightarrow L_n^K(A, \alpha_0, a_0),$$
$$h_\#^t : L_n^K(A, \alpha_0, a_0) \longrightarrow L_n^K(B, \beta_0, b_0).$$

Indeed the two compositions, induced from the hermitian bimodules $W \otimes_A V \cong {}_B B_B$ and $V \otimes_B W \cong {}_A A_A$ with their standard hermitian structure, are the identity.

(vi) For any $u \in B^\times$, scaling a quadratic form (M, q) by u replaces $q(m_1, m_2)$ by $u \cdot q(m_1, m_2)$. This defines an isomorphism

$$L_n^K(B, \beta_0, b_0) \xrightarrow{\cong} L_n^K(B, \beta_1, b_1)$$

where $\beta_1(b) = u\beta_0(b)u^{-1}$ and $b_1 = u\beta_0(u^{-1})b_0$.

For the rest of the paper, we fix a 2-hyperelementary group:

(1. 6) $$\pi = \mathbb{Z}/d \rtimes \sigma, \quad t : \pi \longrightarrow (\mathbb{Z}/d)^\times.$$

Here σ is a fixed 2-Sylow subgroup, d is odd, and t is the twisting homomorphism defined by $sgs^{-1} = t(s)g$ for $s \in \sigma$ and $g \in \mathbb{Z}/d$. We set

$$\sigma_1 = \ker(t|\sigma), \quad \pi_1 = \ker t = \mathbb{Z}/d \times \sigma_1.$$

Let $(\mathbb{Z}\pi, \theta, b, w)$ define a geometric anti-structure, as in (iii) with $\Lambda = \mathbb{Z}$. Since $t \circ \theta = t$, σ_1 is θ-invariant but σ may not be. However, $\theta(\sigma)$ is another 2-Sylow subgroup of π, so $\theta(\sigma) = x^{-1}\sigma x$ for some $x \in \mathbb{Z}/d$. By scaling the anti-structure (vi) using x, we have

Assumption 1. 7. *The 2-Sylow subgroup σ of π is θ-invariant.*

Notice that the scaled anti-structure and the original one agree on σ_1, and that under the assumption 1. 7, the quotient ring $\mathbb{Z}[Sd]^t\bar{\sigma}$ inherits a geometric anti-structure from $(\mathbb{Z}\pi, \theta, b, w)$ in the sense of (iii) with $\Lambda = \mathbb{Z}[\zeta_d]$ if $b \in \sigma_1$. Furthermore, the automorphism induced by θ on $\bar{\sigma} = \sigma/\sigma_1$ is the identity.

We are interested in the map

$$\varrho_* : L_n^K(\hat{\mathbb{Z}}_2\pi, \beta, b) \longrightarrow L_n^K(\hat{\mathbb{Q}}_2\pi, \beta, b),$$

induced from the inclusion $\hat{\mathbb{Z}}_2 \subseteq \hat{\mathbb{Q}}_2$. Both groups are finite 2-groups, and by (i) and (iv), modules over the 2-local Burnside ring $A(\pi) \otimes \mathbb{Z}_{(2)}$, as described in [HM 1], § 6. This ring decomposes into a product of rings, indexed by the subgroups of $\mathbb{Z}/d$, and ϱ_* decomposes accordingly:

$$\varrho_* : \prod_{m|d} L_n^K(\hat{\mathbb{Z}}_2\pi, \beta, b)(m) \longrightarrow \prod_{m|d} L_n^K(\hat{\mathbb{Q}}_2\pi, \beta, b)(m).$$

Moreover, for each divisor m of d the inclusion induces an isomorphism

$$i_* : L_n^K(\hat{\mathbb{Z}}_2[\mathbb{Z}/m \rtimes \sigma], \beta, b)(m) \xrightarrow{\cong} L_n^K(\hat{\mathbb{Z}}_2\pi, \beta, b)(m)$$

and similarly with $\hat{\mathbb{Q}}_2$ scalars. Hence, it suffices to calculate the *top component*, corresponding to $m = d$.

Following the notation from [W 2], § 4. 1, let

$$(1.8)\qquad \begin{aligned} &R(d) = \mathbb{Z}[\zeta_d]^t\sigma, && \hat{R}_p(d) = \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} R(d),\ p \text{ prime},\\ &S(d) = \mathbb{Q} \otimes_{\mathbb{Z}} R(d), && \hat{S}_p(d) = \hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} S(d),\ p \text{ prime}. \end{aligned}$$

By [W 2], § 4. 1, or [HM 1], § 7, the natural projections of group rings induce isomorphisms for $p \nmid d$

$$(1.9)\qquad \begin{aligned} &L_n^K(\hat{\mathbb{Z}}_p\pi, \beta, b)(d) \cong L_n^K(\hat{R}_p(d), \beta, d),\\ &L_n^K(\hat{\mathbb{Q}}_p\pi, \beta, b)(d) \cong L_n^K(\hat{S}_p(d), \beta, b). \end{aligned}$$

This reduces the study of ϱ_* to

$$\varrho_* : L_n^K(\hat{R}_2(d), \beta, b) \longrightarrow L_n^K(\hat{S}_2(d), \beta, b).$$

By Wedderburn theory, the rings $\hat{S}_p(d)$ and $S(d)$ are products of matrix rings over division algebras. The decomposition is controlled by the irreducible (complex) representations of π.

Let $\mathrm{irr}_{\mathbb{C}}(\pi)$ denote the irreducible $\mathbb{C}\pi$-modules and $\mathrm{irr}_{\mathbb{C}}(\pi)(d)$ the subset of modules which are faithful when restricted to $\mathbb{Z}/d$. The $\mathbb{Z}$-span of $\mathrm{irr}_{\mathbb{C}}(\pi)$ is the representation ring $R(\pi)$. It contains the $\mathbb{Z}$-span $R(\pi)(d)$ of $\mathrm{irr}_{\mathbb{C}}(\pi)(d)$.

Each element $\psi \in \mathrm{irr}_{\mathbb{C}}(\pi)$ induces a simple summand $S(\psi)$ of $S(d)$, and its p-adic completion is a simple summand of $\hat{S}_p(d)$.

Let $\bar{\mathbb{Q}}/\mathbb{Q}$ be the extension which contains all roots of one, and let Ω be its Galois group. Each $\mathbb{C}\pi$-module has the form $V \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$ for a (unique up to isomorphism) $\bar{\mathbb{Q}}\pi$-module V. Thus Ω acts on $\mathrm{irr}_{\mathbb{C}}(\pi)$, and

$$(1.10)\qquad S(\psi) = S(\psi') \Leftrightarrow \psi' \in \Omega \cdot \psi.$$

The center of $S(\psi)$ is the field $\mathbb{Q}(\psi)$ generated by all character values of ψ,

$$\mathbb{Q}(\psi) = \mathrm{Span}_{\mathbb{Q}}\{\psi(g) \mid g \in \pi\}.$$

Here, as is common practice, $\psi(g)$ denotes the value of the character of ψ at g. If Ω_ψ is the stabilizer of ψ, then Ω/Ω_ψ is the Galois group of $\mathbb{Q}(\psi)/\mathbb{Q}$.

Actually, we are more interested in the p-local situation. So let $\Omega_p \subseteq \Omega$ be the (local) Galois group of $\bar{\mathbb{Q}}_p/\hat{\mathbb{Q}}_p$. Concretely, there are identifications

$$(1.11)\qquad \Omega = \hat{\mathbb{Z}}^\times, \quad \Omega_p = \langle p \rangle \times \hat{\mathbb{Z}}_p^\times$$

where $\langle p \rangle \subset \hat{\mathbb{Z}}^\times/\hat{\mathbb{Z}}_p^\times$ is the multiplicative subgroup of $\hat{\mathbb{Z}}^\times/\hat{\mathbb{Z}}_p^\times = \prod_{q \neq p} \hat{\mathbb{Z}}_p^\times$ generated by p.

The simple summands of $\hat{S}_p(d)$ are in one-to-one correspondence with the orbits $\mathrm{irr}_{\mathbb{C}}(\pi)(d)/\Omega_p$. The summand corresponding to ψ is the completion of $S(\psi)$ with center $\hat{\mathbb{Q}}_p(\psi)$, and

$$\hat{S}_p(\psi) = \hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} S(\psi) = \prod_{l|p} S(\psi)_l.$$

The number of factors is the index of $\Omega_p/(\Omega_p)_\psi$ in Ω/Ω_ψ.

The geometric anti-involution β acts on $\mathrm{irr}_{\mathbb{C}}(\pi)(d)$ by

$$\beta(\psi)(g) = \psi(\theta(g^{-1}))\, w(g).$$

Note that $\psi \otimes w$ is irreducible if ψ is because $w^2 = 1$, and that β commutes with the action of Ω_p (resp. Ω). One compares the two actions by introducing the concept of *p-types* (resp. *types*) as follows:

$$\text{(1. 12)} \qquad \begin{array}{lll} \psi \text{ has } p\text{-type GL (type GL)} & \text{if} & \beta(\psi) \notin \Omega_p \cdot \psi \quad (\beta(\psi) \notin \Omega \cdot \psi), \\ \psi \text{ has } p\text{-type U (type U)} & \text{if} & \beta(\psi) = \omega \cdot \psi,\ \omega \notin (\Omega_p)_\psi \quad (\omega \notin \Omega_\psi), \\ \psi \text{ has type I} & \text{if} & \beta(\psi) = \psi. \end{array}$$

Note that for characters of a 2-group, the 2-type equals the type. For type I one has the subtypes

$$\text{(1. 13)} \qquad \begin{array}{lll} \psi \text{ has type O} & \text{if} & \sum w(g)\,\psi(g\theta(g)b) > 0, \\ \psi \text{ has type Sp} & \text{if} & \sum w(g)\,\psi(g\theta(g)b) < 0. \end{array}$$

Write $\mathrm{irr}^0_{\mathbb{C}}(\pi) \subset \mathrm{irr}_{\mathbb{C}}(\pi)$ for the subset of characters *not* of type GL.

We are now ready to formulate our main result. First we give the setting:

Let $(\mathbb{Z}\pi, \beta, b)$ be a geometric anti-structure as in (iii) with $\Lambda = \mathbb{Z}$, where π is the 2-hyperelementary group from (1. 6), and suppose $b \in \pi$.

Let $\vartheta \in \mathbb{Z}/d^\times$ be such that $\beta(T) = T^\vartheta$ for $T \in \mathbb{Z}/d$. Assuming there exists

$$\text{(1. 14)} \qquad g_0 \in \sigma \quad \text{with} \quad t(g_0) = -\vartheta^{-1},$$

define a scaled anti-structure on $\hat{\mathbb{Q}}_2\sigma_1$ by

$$\text{(1. 15)} \qquad \beta_0(a) = g_0\,\beta(a)\,g_0^{-1}, \quad b_0 = g_0\,\beta(g_0^{-1})\,b\,w(g_0).$$

Call $\xi \in \mathrm{irr}_{\mathbb{C}}(\sigma_1)$ linear, if it is 1-dimensional, $\xi : \sigma_1 \to \mathbb{C}^\times$. Its order is the order of the cyclic subgroup $\xi(\sigma_1)$ of $\mathbb{C}^\times$.

Let $\chi : \mathbb{Z}/d \to \mathbb{C}^\times$ be any faithful linear character of $\mathbb{Z}/d$. For $\xi \in \mathrm{irr}^0_{\mathbb{C}}(\sigma_1)$, $\chi \otimes \xi \in \mathrm{irr}_{\mathbb{C}}(\pi_1)$ and we can consider the induced character $\mathrm{Ind}(\chi \otimes \xi)$ of π. Write $\hat{S}_2(d, \xi)$ for the summand of $\hat{S}_2(d)$ associated with $\mathrm{Ind}(\chi \otimes \xi)$.

Theorem 1. 16. *If there is no element $g_0 \in \sigma$ satisfying* (1. 14), $L_i^K(\hat{\mathbb{Z}}_2\pi, \beta, b)(d) = 0$. *If g_0 exists, set $m = i + (1 - w(g_0))$. For each $\xi \in \mathrm{irr}_{\mathbb{C}}^0(\sigma_1)$ the composite*

$$L_i^K(\hat{\mathbb{Z}}_2\pi, \beta, b)(d) \xrightarrow{\Psi_i(d)} L_i^K(\hat{S}_2(d), \beta, b) \longrightarrow L_i^K(\hat{S}_2(d, \xi), \beta, b)$$

is injective or zero. It is injective, if and only if the character ξ is:

(a) *linear type* O (*and* $m \equiv 0$ *or* 1 (mod 4)),

(b) *linear type* Sp (*and* $m \equiv 2$ *or* 3 (mod 4)),

(c) *linear type* U (*and m even*), *order 2^l and* $\xi(b_0^{2^{l-1}}) = -1$.

Here types refer to the anti-structure $(\hat{\mathbb{Q}}_2\sigma_1, \beta_0, b_0)$ *of* (1. 15).

Remarks 1. 17. (i) Note that for a type I linear character $\xi(b_0) = 1 (= -1)$ if and only if ξ has type O (type Sp). Since types (and the condition that a linear character ξ has order 2^l and $\xi(b_0^{2^{l-1}}) = -1$) are preserved by scaling the conclusions above are independent of choice of g_0.

(ii) If σ_1 has a linear character ξ of type 1. 16(c), then (by projecting onto the $\mathbb{Z}/2$ quotient of $\xi(\sigma_1)$) it also has linear characters of type O and Sp. Therefore the map $\psi_i(d)$ is injective if and only if σ_1 has a linear character of type O ($m \equiv 0, 1$ (mod 4)), or type Sp ($m = 2, 3$ (mod 4)). For $d = 1$, the case of a 2-group, we recover the result of [HTW 1], AI 2. 1.

§ 2. Discriminant calculations

This section evaluates the "discriminant"

(2. 1) $$d_m^K : L_m^K(\hat{\mathbb{Z}}_2\pi_1, \beta, b) \longrightarrow \hat{H}^m(K_1(\hat{\mathbb{Q}}_2\pi_1), \beta)$$

for 2-elementary groups π_1. The range in (2. 1) denotes Tate cohomology of $\mathbb{Z}/2$ with coefficient in $K_1(\hat{\mathbb{Q}}_2\pi_1)$, equipped with the usual involution (β-conjugate, transposition of matrices). Our calculations will use the character homomorphism description of $K_1(\hat{\mathbb{Q}}_2\pi_1)$, which we recall below.

One has isomorphisms of Ω-modules

$$J_2(\bar{\mathbb{Q}}) = (\hat{\mathbb{Q}}_2 \otimes \bar{\mathbb{Q}})^\times \cong \mathrm{Hom}_{\Omega_2}(\Omega, \bar{\mathbb{Q}}_2^\times)$$

and, following [F], an isomorphism

(2. 2) $$K_1(\hat{\mathbb{Q}}_2\pi) \cong \mathrm{Hom}_\Omega(R(\pi), J_2(\bar{\mathbb{Q}})) \cong \mathrm{Hom}_{\Omega_2}(R(\pi), \bar{\mathbb{Q}}_2^\times).$$

This is natural with respect to both the covariant and contravariant structure of the involved functors. Thus if $\pi' \subseteq \pi$ then

$$\text{(2. 3)} \qquad \begin{array}{ccc} K_1(\hat{\mathbb{Q}}_2\pi) & \cong & \mathrm{Hom}_\Omega(R(\pi), J_2(\bar{\mathbb{Q}})) \\ {\scriptstyle i^*}\downarrow\uparrow{\scriptstyle i_*} & & {\scriptstyle \mathrm{Ind}^*}\downarrow\uparrow{\scriptstyle \mathrm{Res}^*} \\ K_1(\hat{\mathbb{Q}}_2\pi') & \cong & \mathrm{Hom}_\Omega(R(\pi'), J_2(\bar{\mathbb{Q}})) \end{array}$$

give two commutative diagrams.

The isomorphism from left to right in (2. 3) can be described as follows. Let $\varrho : \pi \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ an irreducible module, and let $S_\varrho \subseteq \mathrm{End}_{\mathbb{C}}(V)$ be the image of $\mathbb{Q}\pi$. It is a simple algebra whose centre is the field $\mathbb{Q}(\varrho)$ of character values. The composition

$$\varrho_{\#} : K_1(\hat{\mathbb{Q}}_2\pi) \xrightarrow{\varrho} K_1(\hat{\mathbb{Q}}_2 \otimes_{\mathbb{Q}} S_\varrho) \xrightarrow{\mathrm{Nrd}} J_2(\bar{\mathbb{Q}})$$

(Nrd = reduced norm) is adjoint to (2. 3): for $\chi \in K_1(\hat{\mathbb{Q}}_2\pi)$, $\chi(\varrho) = \varrho_{\#}(\chi)$.

We now specialize to π from (1. 6) and a geometric anti-structure $(\hat{\mathbb{Z}}_2\pi, \beta, b)$ defined by a pair (θ, b, w) as in § 1 (iii), satisfying 1. 7. We are interested only in the top component $L_n^K(\hat{\mathbb{Z}}_2\pi, \beta, \mathrm{b})$ (d), or equivalent by (1. 9), in the groups $L_n^K(\hat{R}_2(d), \beta, b)$.

The automorphism θ of π restricts to an automorphism of $\mathbb{Z}/d$,

$$\text{(2. 4)} \qquad \theta(g) = g^{\vartheta}, \quad \text{for} \quad g \in \mathbb{Z}/d$$

for some $\vartheta \in (\mathbb{Z}/\mathrm{d})^\times$. We will assume from now on

Assumption 2. 5. *The unit b is a group element rather than an element in* $\{\pm\pi\}$.

The change of unit by -1 simply shifts the calculation from L_m *to* L_{m+2} so this is just a normalization.

Assumption 2. 6. *There exists* $g_0 \in \sigma$ *with* $t(g_0) = -\vartheta^{-1}$.

In fact, $L_n^K(\hat{R}_2(d), \beta, b) \neq 0$ if and only if 2. 6 is satisfied. To see this, for $t(\sigma) \cong \bar{\sigma} = \sigma/\sigma_1$ we have the isomorphism:

$$L_n^K(\hat{R}_2(d), \beta, b) \cong L_n^K(\mathbb{F}_2 \otimes \mathbb{Z}[\zeta_d]^t\bar{\sigma}, \bar{\beta}, \bar{b})$$

from § 1, (i), and the involution $\bar{\beta}$ on the centre $\mathbb{F}_2 \otimes \mathbb{Z}[\zeta_d]^{\bar{\sigma}}$ corresponds to $-\vartheta$ in the Galois group $(\mathbb{Z}/d)^\times/t(\sigma)$. If 2. 6 is not satisfied, then $\bar{\beta}$ is non-trivial on the centre, so all the summands on the right-hand side have type U or GL. These anti-structures have vanishing L_n^K ([W 2], § 1. 2). If 2. 6 is satisfied, then we check in 2. 9 below that the right-hand side is isomorphic to

$$L_n^K(\mathbb{F}_2 \otimes \mathbb{Z}[\zeta_d]^{\bar{\sigma}}, 1, 1) \cong g_2^\sigma(d) \cdot (\mathbb{Z}/2),$$

a direct sum of $g_2^\sigma(d) = |(\mathbb{Z}/d)^\times : t(\sigma) \cdot \langle 2 \rangle|$ copies of $\mathbb{Z}/2$.

We now scale the anti-structure (§ 1, (vi)) by $g_0 \in \sigma$ to eliminate the action on $\mathbb{Z}/d \subseteqq \pi$:

$$L_m^K(\hat{R}_2(d), \beta, b) \cong L_m^K(\hat{R}_2(d), \beta_0, b_0 w(g_0))$$

$$(2.7) \qquad \beta_0(g) = g_0 \beta(g) g_0^{-1}, \quad b_0 = g_0 \beta(g_0)^{-1} b w(g_0).$$

In this formula we have arranged that b_0 is still a group element, and $\beta_0(\sigma) = \sigma$. Notice also that if $g \in \mathbb{Z}/d$, then $\beta(g) = \theta(g^{-1})$ implies $\beta_0(g) = g$. Since $\beta_0^2(g) = b_0 g b_0^{-1} = t(b_0)(g)$, we see that $t(b_0) = 1$. Since d is odd, and $\beta_0(b_0) = b_0^{-1}$, it follows that $b_0 \in \sigma_1 \subseteqq \mathbb{Z}/d \times \sigma_1$.

We will use the inclusions

$$(2.8) \qquad \begin{aligned} i_0^K &: (\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_0, \beta_0, b_0) \longrightarrow (\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_1, \beta_0, b_0), \\ i_1^K &: (\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_1, \beta_0, b_0) \longrightarrow (\hat{R}_2(d), \beta_0, b_0) \end{aligned}$$

where $\sigma_0 = \langle b_0 \rangle \subseteqq \sigma_1$ is the cyclic group generated by b_0. The composite inclusion $i_1^K \circ i_0^K$ will be denoted i^K.

Lemma 2.9. $L_n^K(\hat{R}_2(d), \beta_0, b_0) \cong L_n^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]^\sigma, 1, 1)$.

Proof. Suppose first that the orientation character $w(g) \equiv 1$. Then $\beta_0(g) = \theta_0(g^{-1})$ where $\theta_0(\ ) = g_0 \theta(\ ) g_0^{-1}$ and $t \circ \theta_0 = t$. Therefore there is a projection

$$(\hat{R}_2(d), \beta_0, b_0) \longrightarrow (\bar{R}_2(d), \bar{\beta}_0, \bar{b}_0)$$

with $\bar{R}_2(d) = \hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]^t \bar{\sigma}$, $\bar{\theta}_0 = \mathrm{id}$ on $\bar{\sigma}$, and $\bar{b}_0 = 1$. Since $\hat{R}_2(d)$ and $\bar{R}_2(d)$ have the same simple quotient (upon dividing out the radical) the surjection induces isomorphism on L_n^K.

From Morita invariance (§ 1, (v)):

$$\mu_* : L_m^K(\bar{R}_2(d), \bar{\beta}_0, \bar{b}_0) \cong L_m^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]^\sigma, 1, 1).$$

This proves our claim when $w(g) \equiv 1$.

In general,

$$L_n^K(\hat{R}_2(d), \beta_0, b_0) \cong L_n^K(\hat{R}_2(d), w \cdot \beta_0, b_0)$$

for any homomorphism $w : \sigma \to \langle \pm 1 \rangle$, again by reducing modulo the radical. ■

Let $g_2(d)$ denote the number of dyadic primes in $\mathbb{Q}(\zeta_d)$ and $g_2^\sigma(d)$ the number of dyadic primes in $\mathbb{Q}(\zeta_d)^\sigma$. Specifically,

$$g_2(d) = |\mathbb{Z}/d^\times : \langle 2 \rangle|, \quad g_2^\sigma(d) = |\mathbb{Z}/d^\times : \langle 2 \rangle \cdot t(\sigma)|$$

and $\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]$ is $g_2(d)$ copies of $\hat{\mathbb{Z}}_2[\zeta_d]$.

Proposition 2.10. *The inclusion i^K in* (2.8) *induces a split surjection*

$$i_*^K : L_{2n}^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_0, \beta_0, b_0) \longrightarrow L_{2n}^K(\hat{R}_2(d), \beta_0, b_0).$$

In fact, i_*^K is naturally identified with the projection

$$\mathbb{Z}/2[\mathbb{Z}/d^\times/\langle 2\rangle] \longrightarrow \mathbb{Z}/2[\mathbb{Z}/d^\times/\langle 2\rangle \cdot \bar{\sigma}].$$

Proof. By 2. 9 it is equivalent to study

$$j^*: L_{2n}^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d], 1, 1) \longrightarrow L_{2n}^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]^\sigma, 1, 1).$$

Indeed, using § 1, (i), (ii) and (v), one has a commutative diagram

$$\begin{array}{ccc} L_n^K(\hat{R}_2(d), \beta_0, b_0) & \xrightarrow{\quad q_* \quad} & L_m^K(\bar{R}_2(d), \bar{\beta}_0, \bar{b}_0) \\ \uparrow i_* & & \downarrow \mu_* \\ L_m^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d], 1, 1) & \xrightarrow{\quad j^* \quad} & L_m^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]^\sigma, 1, 1), \end{array}$$

and $\mu_* \circ q_*$ is the isomorphism constructed in the proof of 2. 9. The simple summands of the rings in question are $\hat{\mathbb{Z}}_2[\zeta_d]$ and $\hat{\mathbb{Z}}_2[\zeta_d]^{\bar{\sigma} \cap \langle 2\rangle}$, so it is enough to conclude that

$$j^*: L_{2n}^K(\hat{\mathbb{Z}}_2[\zeta_d], 1, 1) \longrightarrow L_{2n}^K(\hat{\mathbb{Z}}_2[\zeta_d]^{\bar{\sigma} \cap \langle 2\rangle}, 1, 1)$$

is an isomorphism. This is the case if for each pair of finite fields $E_0 \subset E$ of char $= 2$,

$$j^*: L_{2n}^K(E, 1, 1) \longrightarrow L_{2n}^K(E_0, 1, 1)$$

is bijective. Both groups have order 2 and the non-trivial element is represented by a quadratic plane $(E \oplus E, Q)$ with Arf invariant 1, i.e.

$$Q = \begin{pmatrix} 1 & 1 \\ 0 & \delta \end{pmatrix}, \quad \mathrm{Tr}_{\mathbb{F}_2}^E(\delta) = 1.$$

Suppose $|E : E_0| = 2$. Then $j^*(E \oplus E, Q)$ is represented by $(E \oplus E, Q_0)$ with $Q_0 = \mathrm{Tr}_{E_0}^E \circ Q$, of dim 4 over E_0. Let $\{e_1, f_1\}$ be the (symplectic) basis for $(E \oplus E, Q)$. Let $w = \delta/\delta_0$ where $\delta_0 = \mathrm{Tr}_{E_0}^E(\delta)$. Then $\mathrm{Tr}_{E_0}^E(w) = 1$ and $\{e_1, f_1 w^{-1}, e_1 w, f_1\}$ is a symplectic basis for Q_0. It has non-vanishing Arf invariant. Indeed,

$$\mathrm{Tr}_{\mathbb{F}_2}^{E_0}(Q_0(e_1, e_1) \cdot Q_0(f_1 w^{-1}, f_1 w^{-1}) + Q_0(e_1 w, e_1 w)\, Q_0(f_1, f_1)) = \mathrm{Tr}_{\mathbb{F}_2}^{E_0} \mathrm{Tr}_{E_0}^E(\delta)$$

is non-zero. ■

We can now calculate the discriminant

(2. 11) $$d_{2n}^K: L_{2n}^K(\hat{\mathbb{Z}}_2 \pi_0, \beta_0, b_0) \longrightarrow \hat{H}^0(\mathrm{Hom}_{\Omega_2}(R(\pi_0)(d), \bar{\mathbb{Q}}_2^\times), \beta_0),$$

where $\pi_0 = \mathbb{Z}/d \times \sigma_0$ and $R(\pi_0)(d) = \mathbb{Z}[\mathrm{irr}_{\mathbb{C}}(\pi_0)(d)]$ are the characters which act faithfully on $\mathbb{Z}/d$. Recall that $\sigma_0 = \langle b_0\rangle$, the cyclic subgroup of π_1 generated by the element $b_0 \in \sigma_1$ from (2. 7), and $\beta_0(b_0) = b_0^{-1}$.

Choose faithful characters $\xi:\sigma_0 \to \mathbb{C}^\times$ and $\chi:\mathbb{Z}/d \to \mathbb{C}^\times$. The set $\operatorname{irr}_{\mathbb{C}}(\pi_0)(d)$ is given by $\chi^j \otimes \xi_0^i$ with $(j, i) \in \mathbb{Z}/d^\times \times \mathbb{Z}/|\sigma_0|$. The 2-local Galois group Ω_2 acts with orbits

$$\operatorname{irr}_{\mathbb{C}}(\pi_0)(d)/\Omega_2 = \{\chi^j \otimes \xi_0^{2^i} \mid j \in \mathbb{Z}/d^\times/\langle 2\rangle,\ 2^i \in \sigma_0/\sigma_0^\times\}.$$

Let $R(\pi_0)(d)$ be the $\mathbb{Z}$-span of $\operatorname{irr}_{\mathbb{C}}(\pi_0)(d)$. By (2. 2)

$$K_1(\hat{\mathbb{Q}}_2\pi_0)(d) = \operatorname{Hom}_{\Omega_2}(R(\pi_0)(d), \bar{\mathbb{Q}}_2^\times).$$

Concretely, it is the direct product of the groups

$$\hat{\mathbb{Q}}_2(\chi^j \otimes \xi_0^{2^i})^\times \cong \hat{\mathbb{Q}}_2(\zeta_d, \zeta_0^{2^i})^\times, \qquad 2^i \in \sigma_0/\sigma_0^\times$$

where ζ_0 is a primitive $|\sigma_0|$'th root of 1. There are $|(\mathbb{Z}/d)^\times : \langle 2\rangle|\,(k+1)$ factors in $K_1(\hat{\mathbb{Q}}_2\pi)$ when $|\sigma_0| = 2^k$.

Fix a 2-local integer δ in $\hat{\mathbb{Z}}_2[\zeta_d]$ whose reduction to the residue field has non-zero trace in $\mathbb{F}_2$. Define a Galois homomorphism

$$A_{2n}^r : R(\pi_0)(d) \longrightarrow \bar{\mathbb{Q}}_2^\times$$

by

(2. 12)
$$A_{2n}^r(\chi^j \otimes \xi_0^{2^i}) = 1 - (-1)^n\,\delta\{1 + (-1)^n \xi_0^{2^i}(b_0)\}^2/\xi_0^{2^i}(b_0), \text{ for } r = j \text{ in } \mathbb{Z}/d^\times/\langle 2\rangle$$

$$A_{2n}^r(\chi^j \otimes \xi_0^{2^i}) = 1, \text{ for } r \neq j \text{ in } \mathbb{Z}/d^\times/\langle 2\rangle, \text{ with } i = 1, 2, \ldots, k.$$

The involution on

$$K_1(\hat{\mathbb{Q}}_2\pi_0)(d) = \prod \hat{\mathbb{Q}}_2(\chi^j \otimes \xi_0^{2^i})^\times$$

induced from β_0 fixes χ^j and maps $\xi_0^{2^i}$ to its complex conjugate. We are interested in the Tate cohomology class

$$\hat{A}_{2n}^r(\chi^j \otimes \xi_0^{2^i}) \in \hat{H}^0(\hat{\mathbb{Q}}_2(\chi^j \otimes \xi_0^{2^i})^\times, \beta_0).$$

Lemma 2. 13. *For $i = 1, 2, \ldots, k-2$ the element $\hat{A}_{2n}^j(\chi^j \otimes \xi_0^{2^i})$ is non-trivial. For $i = k-1$ (resp. $i = k$) it is non-trivial if and only if n is odd (resp. n is even).*

Proof. We will do the case n odd, and leave the similar argument for n even to the reader. If $i = k-1$ the character field is fixed by β_0, and $A_{2n}^j(\chi^j \otimes \xi_0^{2^{k-1}}) = 1 - 4\delta$ is a non-square if $\hat{\mathbb{Q}}_2(\zeta_d)^\times$, [S2], XIV, Prop. 9. If $i = k - v$ with $v \geqq 2$, β_0 acts non-trivially on the character field $E = \hat{\mathbb{Q}}_2(\zeta_d, \zeta_{2^v})$ with fixed field $E_0 = \hat{\mathbb{Q}}_2(\zeta_d, \zeta_{2^v} + \zeta_{2^v}^{-1})$. The element

$$A_{2n}^j(\chi^j \otimes \xi_0^i) = 1 - (-1)^n\,[(1 + (-1)^n \zeta_{2^v})^2/\zeta_{2^v}]\,\delta$$

is not a norm from E [S2], XV. ■

The discriminant

$$d_{2n}^K : L_{2n}^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_0, \beta_0, b_0) \longrightarrow \hat{H}^0(\mathrm{Hom}_{\Omega_2}(R(\pi_0)(d), \bar{\mathbb{Q}}_2^\times), \beta_0)$$

maps the $|\mathbb{Z}/d^\times : \langle 2\rangle|$ copies of $\mathbb{Z}/2$ into the $|\mathbb{Z}/d^\times : \langle 2\rangle|$ cohomology classes $\hat{A}_{2n}^r$ calculated in 2. 13. Indeed, the Arf invariant plane has quadratic form $Q = \begin{pmatrix} 1 & 1 \\ 0 & \delta \end{pmatrix}$ with bilinearization $Q + (-1)^n b_0 Q^t$ and discriminant $1 - (-1)^n [(1 + (-1)^n b_0)^2 / b_0]\delta$.

We can now study the map in (2. 1) for $m = 2n$ via the diagram

$$\text{(2. 14)}\qquad \begin{array}{ccc} L_{2n}^K(R_2(d), \beta_0, b_0) & \longrightarrow & \hat{H}^0(\mathrm{Hom}_{\Omega_2}(R(\pi)(d), \bar{\mathbb{Q}}_2^\times), \beta_0) \\ \uparrow{\scriptstyle i_*^K} & & \uparrow{\scriptstyle \hat{H}^0(\mathrm{Res}_*)} \\ L_{2n}^K(\mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_0, \beta_0, b_0) & \longrightarrow & \hat{H}^0(\mathrm{Hom}_{\Omega_2}(R(\pi_0)(d), \bar{\mathbb{Q}}_2^\times), \beta_0). \end{array}$$

The lower horizontal map was calculated above and the i_*^K was calculated in 2. 10. Thus we have left to study the right-hand vertical map. This will be done in two steps using the intermediate group

$$\pi_0 \subseteqq \pi_1 \subseteqq \pi.$$

The representations of 2-type GL (1. 12) do not contribute to the Tate cohomology groups in (2. 14). Write

$$\mathrm{irr}_{\mathbb{C}}^0(\pi)(d) \subseteqq \mathrm{irr}_{\mathbb{C}}(\pi)(d)$$

for the complementary subset of 2-type U and type I characters (with respect to the scaled anti-involution β_0), and $R^0(\pi)(d)$ for its $\mathbb{Z}$-span. Then

$$\hat{H}^*(\mathrm{Hom}_{\Omega_2}(R(\pi)(d), \bar{\mathbb{Q}}_2^\times), \beta_0) = \hat{H}^*(\mathrm{Hom}_{\Omega_2}(R^0(\pi)(d), \bar{\mathbb{Q}}_2^\times), \beta_0).$$

Note for π_1 that

$$\mathrm{irr}_{\mathbb{C}}^0(\pi_1)(d) = \{\chi^j \otimes \xi \mid j \in (\mathbb{Z}/d)^\times,\ \xi \in \mathrm{irr}_{\mathbb{C}}^0(\sigma_1)\}.$$

We collect a few standard facts about subfields of $\bar{\mathbb{Q}}_2$. The reader is referred to [S2], V, XIII, for proofs. Let $L \subset \bar{\mathbb{Q}}_2$ be a subfield and β_0 any Galois involution on L. If $\beta_0 \neq 1$, $\hat{H}^0(L^\times, \beta_0) = \mathbb{Z}/2$; otherwise $\hat{H}^0(L^\times, \beta_0) = L^\times / L^{\times 2}$. In the case of trivial β_0 however, we are mostly interested in the order 2 subgroup

$$U_L^{2e} / U_L^{2e+1} \subset L^\times / L^{\times 2}, \quad e = e(L/\hat{\mathbb{Q}}_2)$$

the total ramification index. This subgroup is generated by $1 - 4\delta$ where $\delta \in U_L$ is a unit whose residue class has non-zero trace in $\mathbb{F}_2$ (U_L^i is the subgroup of integral elements x with valuation $v_L(x-1) \geqq i$).

Let $K \subset L$. Then β_0 induces an involution of K which can be trivial or not. Consider the norm $N: L \to K$ and the inclusion $I: K \to L$. For their induced maps

$$N^*: \hat{H}^0(L^\times, \beta_0) \longrightarrow \hat{H}^0(K^\times, \beta_0), \quad I_*: \hat{H}^0(K^\times, \beta_0) \longrightarrow \hat{H}^0(L^\times, \beta_0)$$

we have

(C1) If $\beta_0|K \neq 1$, N^* is bijective and, if $[L:K]$ is even, $I_* = 0$ ([S2], XIII, § 4).

(C2) If $\beta_0|K = 1$ but $\beta_0 \neq 1$, $N^* = 0$.

(C3) $N: U_L^{2e}/U_L^{2e+1} \to U_K^{2e}/U_K^{2e+1}$ is zero if and only if L/K is ramified.

(C4) $I: U_K^{2e}/U_K^{2e+1} \to U_L^{2e}/U_L^{2e+1}$ is bijective if and only if L/K is totally ramified.

We can give a calculation of the discriminant map for the group π_1 in terms of certain character sums. By (2. 14) with π replaced by π_1, this amounts to calculating

$$\hat{H}^0(\mathrm{Res}_*^1): \hat{H}^*(\mathrm{Hom}_{\Omega_2}(R^0(\pi_0)(d), \bar{\mathbb{Q}}_2^\times), \beta_0) \longrightarrow \hat{H}^*(\mathrm{Hom}_{\Omega_2}(R^0(\pi_1)(d), \bar{\mathbb{Q}}_2^\times), \beta_0).$$

We shall need a certain function

$$\mu: \mathrm{irr}_{\mathbb{C}}(\sigma_1) \times \sigma_1 \longrightarrow \mathbb{Z}.$$

For $s \in \sigma_1$, let ξ_s be a faithful character of $\langle s \rangle$, so that $\xi_s(s) = e^{2\pi i/|s|}$. Then

$$\mu(\xi, s) = \sum_{r=1}^{|s|} \langle \mathrm{Res}_{\langle s \rangle}^{\sigma_1}(\xi), \xi_s^r \rangle \mid \hat{\mathbb{Q}}_2(\xi_s^r) : \hat{\mathbb{Q}}_2(\xi) |^{-1}. \tag{2. 15}$$

Here we interpret the inverse index $|\hat{\mathbb{Q}}_2(\xi_s^r) : \hat{\mathbb{Q}}_2(\xi)|^{-1}$ to be zero whenever $\hat{\mathbb{Q}}_2(\xi)$ is not contained in $\hat{\mathbb{Q}}_2(\xi_s^r)$ (i.e. when the conductor $f_\xi > |s^r|$).

The formula (2. 15) defines an integer, because if we write

$$\mathrm{Res}_{\langle s \rangle}^{\sigma_1}(\xi) = \sum_{v=0}^{k} \sum_{v_2(j)=k-v} m_j \xi_s^j, \quad k = \log_2 |s|,$$

then each subsum with fixed v is invariant under $(\Omega_2)_\xi$, and hence by the Galois group Γ_v of $\hat{\mathbb{Q}}_2(\zeta_{2^v})/\hat{\mathbb{Q}}_2(\zeta_{2^v}) \cap \hat{\mathbb{Q}}_2(\xi)$. For $v_2(j) = k - v$ write $j = j' \cdot 2^{k-v}$, $j' \in (\mathbb{Z}/2^v)^\times$. The function $m_j = m_{j'}$ only depends on j' in $(\mathbb{Z}/2^v)^\times/\Gamma_v$. Hence $\sum_{v_2(j)=k-v} m_j$ is divisible by $|\Gamma_v|$. This implies integrality in (2. 15).

Recall from (1. 12) that $\xi \in \mathrm{irr}_{\mathbb{C}}^0(\sigma_1)$ has type I if $\beta_0(\xi) = \xi$. Otherwise we say ξ has type II ($=$2-type U).

Proposition 2. 16. *Let $\xi \in \mathrm{irr}_C^0(\sigma_1)$. For each $r \in (\mathbb{Z}/d)^\times/\langle 2\rangle$,*

$$\hat{A}^r_{2n}(\mathrm{Res}^{\pi_1}_{\pi_0}(\chi^j \otimes \xi)) \neq 0 \quad in \quad \hat{H}^0(\hat{\mathbb{Q}}_2(\chi^j \otimes \xi)^\times, \beta_0)$$

if only if $j = r$ and one of the following two conditions is satisfied:

(I) *ξ has type* I *and*

(a) *for n even,* $|b_0|^{-1} \sum_{i=1}^{|b_0|} \xi(b_0^i) \equiv 1 \pmod 2$,

(b) *for n odd,* $|b_0|^{-1} \sum_{i=1}^{|b_0|} (-1)^i \xi(b_0^i) \equiv 1 \pmod 2$;

(II) *ξ has type* II *and* $\mu(\xi, b_0) \equiv 1 \pmod 2$.

Proof. The condition $r = j$ comes from definition (2. 12). We have

$$\mathrm{Res}^{\sigma_1}_{\sigma_0}(\xi) = \sum_{v=0}^{k} \sum_{v_2(j)=k-v} m_j(v)\, \xi_0^j.$$

We saw above that the v'th sum is invariant under Γ_v, the Galois group of $\hat{\mathbb{Q}}_2(\zeta_{2^v})/\hat{\mathbb{Q}}_2(\xi) \cap \hat{\mathbb{Q}}_2(\zeta_{2^v})$. Thus

$$A^r_{2n}(v) := A^r_{2n}(\chi^r \otimes \sum_{v_2(j)=k-v} m_j(v)\, \xi_0^j) = \prod_{j \in (\mathbb{Z}/2^v)^\times/\Gamma_v} N[A^r_{2n}(\chi^r \otimes \xi_0^{2^{k-v}})]^{m_j(v)}$$

where $N : \hat{\mathbb{Q}}_2(\zeta_d, \zeta_{2^v}) \to \hat{\mathbb{Q}}_2(\zeta_d, \xi) \cap \hat{\mathbb{Q}}_2(\zeta_d, \zeta_{2^v})$ is the norm.

In case (I), β_0 acts trivially on $\hat{\mathbb{Q}}_2(\zeta_d, \xi) \cap \hat{\mathbb{Q}}_2(\zeta_d, \zeta_{2^v})$ so $A^r_{2n}(v) = 1$ for $v \geqq 2$ by (C 2). Hence in case (b)

$$A^r_2(\mathrm{Res}^{\pi_1}_{\pi_0}(\chi^r \otimes \xi)) = A^r_2(\chi^r \otimes \xi_0^{2^{k-1}})^{m(1)} = (1 - 4\delta)^{m(1)}$$

in U_L^2/U_L^3 with $L = \hat{\mathbb{Q}}_2(\zeta_d)$. But

$$m(1) = \langle \mathrm{Res}^{\sigma_1}_{\sigma_0}(\xi), \xi_0^{2^{k-1}} \rangle = |b_0|^{-1} \sum_{i=1}^{|b_0|} (-1)^i \xi(b_0^i),$$

and we can use (C 4) to complete the proof in case (I) (b). Case (a) is similar but easier. From (2. 13) the answer is non-trivial if and only if $\mathrm{Res}^{\sigma_1}_{\sigma_0}(\xi)$ contains an odd multiple of the trivial character.

In case (II), $A^r_2(v) = 1$ by (C 1) unless $\hat{\mathbb{Q}}_2(\zeta_d, \zeta_{2^v}) \supset \hat{\mathbb{Q}}_2(\zeta_d, \xi)$. When this is the case, then, again by (C 1),

$$N[A^r_2(\chi^r \otimes \xi_0^{2^{k-v}})] \neq 0$$

in $\hat{H}^0(\hat{\mathbb{Q}}_2(\zeta_d, \xi)^\times, \beta_0) = \mathbb{Z}/2$. Thus $A^r_2(v) = \sum m_j(v)$, $j \in (\mathbb{Z}/2^v)^\times/\Gamma_v$ with

$$\Gamma_v = G(\hat{\mathbb{Q}}_2(\zeta_{2^v})/\hat{\mathbb{Q}}_2(\xi)).$$

Equivalently

$$A_2^r(v) = \sum m_j(v)\, |\hat{\mathbb{Q}}_2(\zeta_{2^v}) : \hat{\mathbb{Q}}_2(\xi)|^{-1}, \qquad j \in (\mathbb{Z}/2^v)^\times$$

when $f_\xi \leqq 2^v$ and zero otherwise. Sum over v to complete the proof. ■

We now consider (2. 1) with m odd. In analogy with 2. 10 we have

Proposition 2. 17. *The inclusion i_1^K induces a split injection*

$$(i_1^K)^* : L_{2n+1}^K(\hat{R}_2(d), \beta_0, b_0) \longrightarrow L_{2n+1}^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_1, \beta_0, b_0)$$

which can be identified with the natural injection

$$\mathrm{Map}(\mathbb{Z}/d^\times/\langle 2\rangle \bar{\sigma}, \mathbb{Z}/2) \longrightarrow \mathrm{Map}(\mathbb{Z}/d^\times/\langle 2\rangle, \mathbb{Z}/2).$$

Proof. The proof is similar to that of 2. 10 and reduces to showing that, for an extension $F \subset E$ of finite fields of characteristic 2,

$$i_* : L_{2n+1}^K(F, 1, 1) \longrightarrow L_{2n+1}^K(E, 1, 1)$$

is non-trivial (both groups are equal to $\mathbb{Z}/2$). But this is clear as the non-trivial element is represented by the automorphism $\tau = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$ of the hyperbolic plane. ■

Consider the diagram

(2. 18)

$$\begin{array}{ccc}
L_{2n+1}^K(R_2(d), \beta_0, b_0) & \xrightarrow{\ d\ } & \hat{H}^1(\mathrm{Hom}_{\Omega_2}(R(\pi)(d), \bar{\mathbb{Q}}_2^\times), \beta_0) \\
\Big\downarrow{\scriptstyle i_1^*} & & \Big\downarrow{\scriptstyle \hat{H}^1(\mathrm{Ind}_1^*)} \\
L_{2n+1}^K(\mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_1, \beta_0, b_0) & \xrightarrow{\ d^1\ } & \hat{H}^1(\mathrm{Hom}_{\Omega_2}(R(\pi)(d), \bar{\mathbb{Q}}_2^\times), \beta_0) \\
\cong \Big\uparrow{\scriptstyle i_{0*}} & & \Big\uparrow{\scriptstyle \hat{H}^1(\mathrm{Res}_*^1)} \\
L_{2n+1}^K(\mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_0, \beta_0, b_0) & \xrightarrow{\ d^0\ } & \hat{H}^1(\mathrm{Hom}_{\Omega_2}(R(\pi_0)(d), \bar{\mathbb{Q}}_2^\times), \beta_0).
\end{array}$$

Here we need only the lower square, but the upper square is used in § 4. The Galois homomorphisms in (2. 12) are replaced by

(2. 19) $$A_{2n+1}^r(\chi^j \otimes \xi_0^i) = \begin{cases} (-1)^{n+1}\, \xi_0^i(b_0) & \text{if } i = 2^k \text{ or } 2^{k-1} \text{ and } r = j \text{ in } (\mathbb{Z}/d)^\times/\langle 2\rangle, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 2. 20. *The map*

$$d_{2n+1}^K : L_{2n+1}^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_0, \beta_0, b_0) \longrightarrow \hat{H}^1(\mathrm{Hom}_{\Omega_2}(R(\pi_0)(d), \bar{\mathbb{Q}}_2^\times), \beta_0)$$

maps the $|\mathbb{Z}/d^\times : \langle 2\rangle|$ copies of $\mathbb{Z}/2$ into the homomorphisms in (2. 19). ■

Theorem 2. 21. *Let $\xi \in \mathrm{irr}^0_{\mathbb{C}}(\sigma_1)(d)$ and $r \in (\mathbb{Z}/d)^\times/\langle 2 \rangle$. Then*

$$\hat{A}^r_{2n+1}(\mathrm{Res}^{\pi_1}_{\pi_0}(\chi^j \otimes \xi)) \neq 0$$

if and only if $r = j$ and one of the following holds:

(a) *n is odd, ξ has type* I *and*

$$|b_0|^{-1} \sum_{i=1}^{|b_0|} (-1)^i \xi(b_0^i) \equiv 1 \pmod 2.$$

(b) *n is even, ξ has type* I *and*

$$|b_0|^{-1} \sum_{i=1}^{|b_0|} \xi(b_0^i) \equiv 1 \pmod 2.$$

Proof. The proof is similar to, but easier than that of 2. 16. One uses the diagram (2. 18) along with 2. 17 and (2. 19). The condition for n odd is simply that some odd multiple of the character $\xi_0^{2^{k-1}}$ extends to an irreducible character of σ_1. ■

§ 3. Final results for 2-elementary groups

We now finish the calculation of

$$\Psi_m : L^K_m(\hat{\mathbb{Z}}_2 \pi_1, \beta_0, b_0) \longrightarrow L^K_m(\hat{\mathbb{Q}}_2 \pi_1, \beta_0, b_0)$$

where $\pi_1 = \mathbb{Z}/d \times \sigma_1$ (i.e. for 2-elementary groups) using a more detailed study of the character theory for finite 2-groups. We show that the results of § 2 can be improved, so that non-linear characters can be neglected and handle the cases where the range of Ψ_m is not detected by the discriminant.

Recall that β_0 has the form $\beta_0(g) = w(g)\,\theta_0(g^{-1})$ with $\beta_0(g) = g$ for $g \in \mathbb{Z}/d$, and that $b_0 \in \sigma_1^+ = \ker(w : \sigma_1 \to \{\pm 1\})$.

If $f : (A, \alpha_0, a_0) \to (B, \beta_0, b_0)$ is a map of rings with anti-structure, we have the transfer map I^* (§ 1, (ii)). In addition, for any $v \in B^\times$ with $vAv^{-1} = A$, there is a generalized transfer, defined by the composite

$$I^*_v : L^K_m(B, \beta_0, b_0) \xrightarrow{\text{"scale by } v\text{"}} L^K_m(B, \beta^v_0, b^v_0) \xrightarrow{\quad I^* \quad} L^K_m(A, \alpha^v_0, a^v_0).$$

The first ingredient is

Lemma 3. 1. *Let $\sigma_2 \subsetneqq \sigma_1$ be a proper subgroup such that $\beta_0(\sigma_2) = \sigma_2$ and $b_0 \in \sigma_2$. Then for any $v \in \sigma_1$ such that $v\sigma_2 v^{-1} = \sigma_2$, the generalized transfer map*

$$L^K_m(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_1, \beta_0, b_0) \longrightarrow L^K_m(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_2, \beta^v_0, b^v_0)$$

is zero.

Proof. From § 1 (i) and (ii), it is easy to see that the following diagram commutes:

$$\begin{array}{ccc} L_m^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_1, \beta_0^v, b_0^v) & \xrightarrow{I_v^*} & L_m^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_2, \beta_0^v, b_0^v) \\ \uparrow \cong & & \downarrow \cong \\ L_m^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d], 1, 1) & \xrightarrow{[\sigma_1:\sigma_2]} & L_m^K(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d], 1, 1) \end{array}$$

where the vertical maps are induced by the inclusion and reduction modulo a radical ideal respectively. ■

The second ingredient is a special case of the Detection Theorem of [HTW 2], 5. 6. For detecting maps into $\hat{H}^*(K_1(\mathbb{Q}\pi))$, this involves a variant of the Witt-Roquette character theory and some attention to the geometric anti-structure. For the benefit of the reader, we will give a proof adapted to our special case and just refer to [HTW 2] for a group-theoretic result (3. 4).

Definition 3. 2. If G is a finite 2-group and θ is an automorphism of G, then G is called *θ-basic* if G contains no normal θ-invariant $\mathbb{Z}/2 \times \mathbb{Z}/2$ subgroups K.

If θ is an inner automorphism, then G is θ-basic if and only if G is basic in the classical sense. By Roquette's Theorem [Ro], the basic groups are

(3. 3) $\qquad G = \mathbb{Z}/2^k, \quad Q2^k\ (k \geqq 3), \quad D2^k\ (k \geqq 4), \quad \text{or} \quad SD2^k\ (k \geqq 4).$

Theorem 3. 4 ([HTW 2], 5. 4). *If G is a θ-basic 2-group and θ has even order in* $\mathrm{Out}(G)$, *then G is basic or $G \cong D8$ and θ represents the non-trivial element in* $\mathrm{Out}(D8)$.

We now return to the evaluation of the Arf-classes

(3. 5) $$\hat{A}_m^r \in \hat{H}^m(\mathrm{Hom}_{\Omega_2}(R(\pi_1)(d), \bar{\mathbb{Q}}_2^\times), \beta_0)$$

for the 2-elementary group $\pi_1 = \mathbb{Z}/d \times \sigma_1$, explicating the character formulas (2. 16), (2. 21). To shorten the notation, we will use $\hat{A}_m^r(\chi^j \otimes \xi)$ instead of the more precise $\hat{A}_m^r(\mathrm{Res}_{\pi_0}^{\pi_1}(\chi^j \otimes \xi))$ from § 2.

Here is the key result:

Proposition 3. 6. *Let $\mathbb{Z}/d \times G$ be a 2-elementary group with geometric anti-structure (θ, w, b) such that $\beta(g) = g$ for $g \in \mathbb{Z}/d$, $\beta(G) = G$ and $b \in G$. Suppose that G is not θ-basic and that $\xi \in \mathrm{irr}_{\mathbb{C}}^0(G)$ is faithful. Then $\hat{A}_m^r(\chi^j \otimes \xi) = 0$ for all $r, j \in (\mathbb{Z}/d)^\times/\langle 2 \rangle$.*

Proof. It is enough to do the case $r = j = 1$. Since G is not θ-basic, it contains a θ-invariant normal subgroup $K \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Let K_0, K_1 and K_2 denote the distinct $\mathbb{Z}/2$ subgroups of K. Since ξ is faithful, K is normal but non-central, and intersects the centre of G, say in K_0. Let $g \in G$ be an element with $gK_1g^{-1} = K_2$, and G_0 be the centralizer of K in G. Let V be the representation space of ξ. Since ξ is irreducible $V^{K_0} = 0$ so $\mathrm{Res}_G^K(V) = V^{K_1} \oplus V^{K_2}$, interchanged by g. Then $\mathrm{Res}_G^{G_0}(V) = V^{K_1} \oplus V^{K_2}$ and

$$V = \mathrm{Ind}_{G_0}^G(V^{K_1}) = \mathrm{Ind}_{G_0}^G(V^{K_2}),$$

without change of centre or Schur index.

The automorphism θ preserves K_0 and hence $\theta(K_1)=K_1$ or $\theta(K_1)=K_2$. In both cases θ^2 is the identity on K so $b \in G_0$, and $\theta(G_0)=G_0$. Then (β, b) restricts to an anti-structure on G_0. If the character ξ_1 of V^{K_1} is not β-invariant, we scale by v $(=e$ or $g)$ and find that ξ_1 is β^v-invariant. Then since

$$(\hat{\mathbb{Q}}_2(\chi \otimes \xi_1)^\times, \beta^v) = (\hat{\mathbb{Q}}_2(\chi \otimes \xi)^\times, \beta),$$

the summand given by ξ of $\hat{H}^m(K_1(\hat{\mathbb{Q}}_2 G), \beta)$ is mapped isomorphically under Ind* to the summand given by ξ_1 of $\hat{H}^m(K_1(\hat{\mathbb{Q}}_2 G_0), \beta^v)$.

Consider the following commutative diagram:

$$\begin{array}{ccc}
L^K_m(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]G, \beta, b) & \xrightarrow{I^*} & L^K_m(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]G_0, \beta^v, b^v) \\
\downarrow{\scriptstyle d^K_m} & & \downarrow{\scriptstyle d^K_m} \\
\hat{H}^m(\mathrm{Hom}(R(\mathbb{Z}/d \times G)(d), \bar{\mathbb{Q}}_2^\times), \beta) & \xrightarrow{\mathrm{Ind}^*} & \hat{H}^m(\mathrm{Hom}(R(\mathbb{Z}/d \times G_0)(d), \bar{\mathbb{Q}}_2^\times), \beta^v) \\
\downarrow{\scriptstyle e(\xi)_*} & & \downarrow{\scriptstyle e(\xi_1)_*} \\
\hat{H}^m(\hat{\mathbb{Q}}_2(\chi \otimes \xi)^\times, \beta) & = & \hat{H}^m(\hat{\mathbb{Q}}_2(\chi \otimes \xi_1)^\times, \beta^v),
\end{array}$$

where $e(\xi)$ denotes the evaluation map of a character homomorphism on the character $\chi \otimes \xi$. Since G_0 is a proper subgroup of G, $I^*=0$ by (3. 1) and hence $\mathrm{Ind}^* \hat{A}^r = \{0\}$. However,

$$(\mathrm{Ind}^* \hat{A}^r_m)(\chi \otimes \xi_1) = \hat{A}^r_m(\chi^j \otimes \xi)$$

and the result follows. ■

It follows from 3. 4 that irreducible representations of θ-basic groups (with $\theta^2=1$) have degrees 1 or 2.

Lemma 3. 7. *Suppose $\pi_1 = \mathbb{Z}/d \times \sigma_1$ as in (3. 5). If σ_1 is a θ-basic group and ξ is a faithful irreducible character of degree two, then $\hat{A}^r_m(\chi^r \otimes \xi)=0$.*

Proof. If σ_1 has a faithful irreducible character ξ of degree 2, then σ_1 is non-cyclic. Let $\sigma_2 \subset \sigma_1$ be a cyclic subgroup of index 2. Then $b_0 \in \sigma_2$. Indeed, suppose if possible that $b_0 \notin \sigma_2$. Then conjugation with b_0 is a non-trivial automorphism of σ_2 and it has a square root $\theta_0 \in \mathrm{Aut}(\sigma_1)$. One checks for $\sigma_1 = Q2^k$, $D2^k$ and $SD2^k$ that this cannot happen.

The restriction of ξ to $\sigma_0 = \langle b_0 \rangle$ is the sum $\xi_0 + \xi_0^w$ of a faithful linear character and a Galois conjugate. We can now use (2. 16) or (2. 21). Both character sums $\mu(\xi, b_0)$ and $|b_0|^{-1} \sum (-1)^i \xi(b_0^i)$ are zero (mod 2), so regardless of the type of ξ (w.r.t. β_0) the Arf-class vanishes. ■

Now we can eliminate the non-linear characters.

Proposition 3. 8. *For* $\xi \in \operatorname{irr}^0_{\mathbb{C}}(\sigma_1)$, $\hat{A}^r_m(\chi^r \otimes \xi) = 0$ *unless* ξ *is a linear character.*

Proof. If ξ is faithful then we are done by (3. 6) or 3. 7. If ξ is non-faithful but $w(\ker \xi) = 1$, then the projection map

$$\mathbb{Z}/d \times \sigma_1 \longrightarrow \mathbb{Z}/d \times (\sigma_1/\ker \xi)$$

induces a map of rings with anti-structure. Since the summand corresponding to ξ in $K_1(\hat{\mathbb{Q}}_2(\pi_1))$ is mapped isomorphically by this projection, we are reduced to the previous case.

Finally, if $w(\ker \xi) \neq 1$ then let $\sigma_1^+ = \ker w$ and note that $\xi^+ = \operatorname{Res}(\xi)$ is irreducible on σ_1^+. Now we finish by considering the commutative diagram

$$\begin{array}{ccc} L^K_m(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_1^+, \beta_0, b_0) & \xrightarrow{I_*} & L^K_m(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_1, \beta_0, b_0) \\ \downarrow d^K_m & & \downarrow d^K_m \\ \hat{H}^m(\operatorname{Hom}(R(\pi_1^+)(d), \bar{\mathbb{Q}}_2^\times), \beta_0) & \xrightarrow{\operatorname{Res}_*} & \hat{H}^m(\operatorname{Hom}(R(\pi_1)(d), \bar{\mathbb{Q}}_2^\times), \beta_0), \end{array}$$

where the map I_* is an isomorphism by reduction (§ 1, (i)). ■

We would also like to express the answer for the linear characters in terms of the *sub-types* O, U or Sp introduced earlier. This will allow us to state the main result 1. 16 in an invariant way.

Given a linear character $\xi : \sigma_1 \to \mathbb{C}^\times$. We say ξ has *order* 2^l if $\xi(\sigma_1)$ is cyclic of order 2^l. Choose $g_1 \in \sigma_1$ such that $\xi(g_1)$ generates $\xi(\sigma_1)$. Then ξ has type I when $\xi(\theta_0(g_1^{-1})) = w(g_1)\,\xi(g_1)$.

Theorem 3. 9. *Let* $\xi \in \operatorname{irr}^0_{\mathbb{C}}(\sigma_1)$. *For each* $r \in (\mathbb{Z}/d)^\times$,

$$\hat{A}^r_{2n}(\operatorname{Res}^{\pi_1}_{\pi_0}(\chi^j \otimes \xi)) \neq 0 \in \hat{H}^0(\hat{\mathbb{Q}}_2(\chi^r \otimes \xi)^\times, \beta_0)$$

if and only if $j = r$, *and the character* ξ *is linear and has*: *type* I *and* $\xi(b_0) = (-1)^n$; *or type* U *of order* 2^l *and* $\xi(b_0^{2^{l-1}}) = -1$.

Proof. If $\hat{A}^r_{2n}(\chi^j \otimes \xi) \neq 0$ then $j = r$ (cf. Section 2) and ξ is linear by 3. 8. Suppose ξ has order 2^l and choose v so that $\operatorname{Res}^{\sigma_1}_{\sigma_0}(\xi)$ is Galois conjugate to $\xi_0^{2^{k-v}}$ with the notation of 2. 13. Note that $v \leqq l$ since $\xi(b_0)^{2^l} = 1$.

If ξ has type I then

$$|b_0|^{-1} \sum (-1)^i\, \xi(b_0^i) = \langle \operatorname{Res}^{\sigma_1}_{\sigma_0}(\xi), \xi_0^{2^{k-1}} \rangle = 1$$

precisely when $v = 1$, i.e. when $\xi(b_0) = -1$.

If ξ has type U then

$$\mu(\xi, b_0) = \begin{cases} 1 & \text{if } \; v = l, \\ 0 & \text{otherwise,} \end{cases}$$

and case $v = l$ is equivalent to $\xi(b_0^{2^{l-1}}) = -1$. Apply 2. 16 to complete the proof. ■

The odd Arf-classes

$$\hat{A}^r_{2n+1} \in \hat{H}^1(\mathrm{Hom}_{\Omega_2}(R(\pi_1)(d), \bar{\mathbb{Q}}_2^\times), \beta_0)$$

can be calculated in a similar fashion, using 2. 21 instead of 2. 16. We leave for the reader to prove

Proposition 3. 10. *Let $\xi \in \mathrm{irr}^0_{\mathbb{C}}(\sigma_1)$. Then $\hat{A}^r_{2n+1}(\mathrm{Res}^{\pi_1}_{\pi_0}(\chi^j \otimes \xi)) \neq 0$ if and only if $j = r$, ξ is linear of type* I *and $\xi(b_0) = (-1)^n$.* ■

Remark 3. 11. Note as in 1. 17 that our conclusions 3. 9 and 3. 10 are independent of the choice of scaling elements in 1. 7 and 2. 6. ■

We conclude by observing that for 2-elementary groups the map

$$\Psi_n(d) : L^K_n(\hat{R}_2(d), \beta_0, b_0) \longrightarrow L^K_n(\hat{S}_2(d), \beta_0, b_0)$$

is detected by the discriminant. This is clear for n odd (even in the 2-hyperelementary case) since

$$d^K_n : L^K_n(\hat{S}_2(d), \beta_0, b_0) \longrightarrow \hat{H}^1(K_1(\hat{S}_2(d)), \beta_0)$$

is injective. But d^K_{2m} is not injective in general.

Let $\xi \in \mathrm{irr}^0_{\mathbb{C}}(\sigma_1)$. The irreducible representation $\Phi = \chi^r \otimes \xi$ gives a direct, β_0-invariant simple summand $S_2(\Phi)$ of $\hat{S}_2(d)$ and d^K_n decomposes into the corresponding sum of $d^K_n(\Phi)$.

If $(S_2(\Phi), \beta_0, b)$ has type O and $S_2(\Phi)$ has trivial Schur index (i.e. type OK) then $d^K_0(\psi)$ has kernel $\mathbb{Z}/2$, detected by the Hasse-invariant, cf. [W 1]. In all other cases $d^K_0(\Phi)$ is injective. Similar for $d^K_2(\psi)$ where the kernel $\mathbb{Z}/2$ appears for type SpK.

Lemma 3. 12. *For Φ of type* OK, *$\Psi_0(d)$ and $d^K_0(\Phi) \circ \Psi_0(d)$ have isomorphic images.*

Proof. Kolster [K 1], 4. 11, has shown that in type OK there is a β_0-invariant maximal order $\mathcal{M}_2(\Phi) \subset S_2(\Phi)$. Hence the question reduces to the study of

$$\begin{array}{ccc} L^K_0(\hat{A}_2, 1, 1) & \longrightarrow & L^K_0(\hat{E}_2, 1, 1) \\ \downarrow d^K_0 & & \downarrow d^K_0 \\ \hat{A}_2^\times / \hat{A}_2^{\times 2} & \longrightarrow & \hat{E}_2^\times / \hat{E}_2^{\times 2}, \end{array}$$

where $\hat{E}_2$ is a 2-local field with integers $\hat{A}_2$. The left hand vertical map is an isomorphism, [W 2], and the lower map is an injection. ■

Lemma 3. 12 and its counterpart for type SpK representation give

Corollary 3. 13. *The maps*

$$\Psi_m(d): L_m^K(\hat{R}_2(d), \beta_0, b_0) \longrightarrow L_m^K(\hat{S}_2(d), \beta_0, b_0),$$
$$d_m^K \circ \Psi_m(d): L_m^K(R_2(d), \beta_0, b_0) \longrightarrow \hat{H}^m(K_1(\hat{S}_2(d)), \beta_0)$$

have the same kernels.

§ 4. The 2-hyperelementary case

In this section we state and prove our main theorem 1. 16 which calculates the map

$$(4.1) \qquad \Psi_m(d): L_m^K(\hat{\mathbb{Z}}_2\pi, \beta, b)(d) \longrightarrow L_m^K(\hat{\mathbb{Q}}_2\pi, \beta, b)(d)$$

for a 2-hyperelementary group $\pi = \mathbb{Z}/d \rtimes \sigma$. It turns out Ψ_m is injective for π if and only if it is injective for the 2-elementary subgroup π_1. By 2. 10 and 3. 13 it is equivalent to show that the discriminant at a character $\mathrm{Ind}(\chi \otimes \xi)$ for π is non-trivial if and only if the discriminant at $\chi \otimes \xi$ for π_1 is non-trivial.

Before considering the passage from π_1 to π, we recall a few general facts about the relationship between representations of π and of π_1. Suppose more generally that $A \lhd G$ and let $\psi \in \mathrm{irr}_{\mathbb{C}}(A)$. The group G/A acts on ψ by conjugation ($\psi^g(a) = \psi(gag^{-1})$). We also have the Galois action of Ω_2 on ψ. To relate them, define

$$G_2(\psi) = \{g \in G/A \mid \psi^g = \omega_g \psi \text{ for some } \omega_g \in \Omega_2\}.$$

Note that $g \longrightarrow \omega_g$ defines a homomorphism of $G_2(\psi)$ into $\Omega_2/(\Omega_2)_\psi$, the Galois group of $\hat{\mathbb{Q}}_2(\psi)/\hat{\mathbb{Q}}_2$.

Lemma 4. 2. *If the induced character* $\psi^* = \mathrm{Ind}_A^G(\psi)$ *is irreducible, then*

$$\hat{\mathbb{Q}}_2(\psi^*) = \hat{\mathbb{Q}}_2(\psi)^{G_2(\psi)}.$$

Proof. Since $\psi^*(a) = \sum \psi^g(a)$, $\hat{\mathbb{Q}}_2(\psi^*) \subseteq \mathrm{Tr}(\hat{\mathbb{Q}}_2(\psi)) = \hat{\mathbb{Q}}_2(\psi)^{G_2(\psi)}$. If

$$\omega \in \mathrm{Gal}(\hat{\mathbb{Q}}_2(\psi)/\hat{\mathbb{Q}}_2(\psi^*))$$

then $(\omega\psi)^* = \psi^*$ so $\mathrm{Res}_A((\omega\psi)^*) = \mathrm{Res}_A(\psi^*)$, i.e. $\sum (\omega\psi)^* = \sum \psi^g$. Since ψ^* is irreducible, $\{\psi^g \mid g \in G/A\}$ is a set of distinct irreducible characters (calculate $\langle (\omega^g)^*, \psi^* \rangle$!), and ω acts on it by a permutation. Hence $\omega \cdot \psi = \psi^h$ for some $h \in G$, so $h \in G_2(\psi)$ and $\omega = \omega_h \in G(\psi)$. ■

Next recall from [S 1], § 9, that

$$\mathrm{Ind}_{\pi_1}^{\pi}: \mathrm{irr}_{\mathbb{C}}(\pi_1)(d) \longrightarrow \mathrm{irr}_{\mathbb{C}}(\pi)(d)$$

is surjective, and induces a bijection from the set of orbits under the free action of σ/σ_1 on $\mathrm{irr}_{\mathbb{C}}(\pi_1)(d)$.

Given $\psi^* \in \mathrm{irr}_C(\pi)(d)$, $\mathrm{Res}^{\pi}_{\pi_1} \psi^* = \sum \psi^g$ with $g \in \sigma/\sigma_1$; each ψ^g is irreducible and induces up to ψ^*. It can happen that $\psi^* \in \mathrm{irr}^0_C(\pi)(d)$ but $\psi \notin \mathrm{irr}^0_C(\pi_1)(d)$. However,

$$\psi \in \mathrm{irr}^0_C(\pi_1)(d) \Leftrightarrow \psi^g \in \mathrm{irr}^0_C(\pi_1)(d),$$

and ψ has type I or II if and only if the same is true for each ψ^g.

Lemma 4. 3. *The extension $\hat{\mathbb{Q}}_2(\psi)/\hat{\mathbb{Q}}_2(\psi^*)$ is unramified.*

Proof. Since $\Omega_2 = \langle 2 \rangle \times \hat{\mathbb{Z}}_2^{\times}$, $G_2(\psi)$ maps into the subgroup $\langle 2 \rangle \subset \mathbb{Z}/d^{\times}$ under the characteristic map $t: \sigma/\sigma_1 \to \mathbb{Z}/d^{\times}$. Thus the Galois group for the residue field extension is isomorphic to $G_2(\psi)$. ■

Lemma 4. 4. *Let $\psi \in \mathrm{irr}^0_C(\pi_1)(d)$ and let $\psi^* = \mathrm{Ind}^{\pi}_{\pi_1}(\psi)$. Then ψ has type* I *if and only if ψ^* has type* I.

Proof. Suppose $\beta_0(\psi^*) = \psi^*$. Restrict to π_1 to get

$$\beta_0\left(\sum \psi^g\right) = \sum \psi^g, \qquad g \in \sigma/\sigma_1$$

and hence $\beta_0(\psi) = \psi^h$ for some $h \in \sigma/\sigma_1$. Since β_0 acts trivially on $\mathbb{Z}/d$,

$$\beta_0(\mathrm{Res}^{\pi_1}_{\mathbb{Z}/d}(\psi)) = \mathrm{Res}^{\pi_1}_{\mathbb{Z}/d}(\psi).$$

But $\sigma/\sigma_1 \subseteq (\mathbb{Z}/d)^{\times}$ and $\mathrm{Res}^{\pi_1}_{\mathbb{Z}/d}(\psi)$ is faithful, so $\beta_0(\psi) = \psi$. The other implication is obvious. ■

Theorem 4. 5. *The "discriminant" from* (2. 1)

$$d^K_m : L^K_m(\hat{R}_2(d), \beta_0, b_0) \longrightarrow \hat{H}^m(\mathrm{Hom}_{\Omega_2}(R(\pi)(d), \bar{\mathbb{Q}}_2^{\times}), \beta_0)$$

is either injective or zero. It is injective for $m \equiv 0$ or 1 (mod 4). *For $m \equiv 2$ or* 3 (mod 4) *it is injective if and only if there exists $\xi \in \mathrm{irr}^0_C(\sigma_1)(d)$ which satisfies one of the conditions in* 3. 9 *or* 3. 10.

Proof. Consider the diagram

$$\begin{array}{ccc}
L^K_{2n}(R_2(d), \beta_0, b_0) & \xrightarrow{\ d\ } & \hat{H}^0(\mathrm{Hom}_{\Omega_2}(R(\pi)(d), \bar{\mathbb{Q}}_2^{\times}); \beta_0) \\
\uparrow{\scriptstyle i_{1*}} & & \uparrow{\scriptstyle \hat{H}^0(\mathrm{Res}_*)} \\
L^K_{2n}(\mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_1, \beta_0, b_0) & \xrightarrow{\ d^1\ } & \hat{H}^0(\mathrm{Hom}_{\Omega_2}(R(\pi_1)(d), \bar{\mathbb{Q}}_2^{\times}); \beta_0) \\
\uparrow{\scriptstyle i_{0*}} & & \uparrow{\scriptstyle \hat{H}^0(\mathrm{Res}_*)} \\
L^K_{2n}(\mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_0, \beta_0, b_0) & \xrightarrow{\ d^0\ } & \hat{H}^0(\mathrm{Hom}_{\Omega_2}(R(\pi_0)(d), \bar{\mathbb{Q}}_2^{\times}); \beta_0).
\end{array}$$

We know the left-hand vertical maps by 2. 10: i_{0*} is bijective and i_{1*} is split surjective. The $|(\mathbb{Z}/d)^\times : \langle 2\rangle|$ copies of $\mathbb{Z}/2$ in $L^K_{2n}(\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_d]\sigma_1, \beta_0, b_0)$ map into the homomorphisms

$$(\mathrm{Res}^{\pi_1}_{\pi_0})_* (\hat{A}^r_{2n}) : R(\pi_1)(d) \longrightarrow \hat{\bar{\mathbb{Q}}}{}_2^\times$$

where the $\hat{A}^r_{2n}$ are the basic Arf homomorphisms from (2. 12).

We first do the case n odd. The cohomology class $\hat{A}^r_2$ was given in 2. 16. Let $\xi \in \mathrm{irr}^0_{\mathbb{C}}(\sigma_1)(d)$, $\psi = \chi^r \otimes \xi$ and $\psi^* = \mathrm{Ind}^{\pi_1}_{\pi_1}(\psi)$. Then

$$\begin{aligned} A^r_2(\psi^*) &= A^r_2(\textstyle\sum \psi^g) && (g \in \sigma/\sigma_1) \\ &= A^r_2(\textstyle\sum_{\bar g} \sum_{\omega} \omega \cdot \psi^{\bar g}) && (\bar g \in (\sigma/\sigma_1)/G_2(\psi),\ \omega \in G_2(\psi)) \\ &= \textstyle\prod_{g} N_{\bar g}(A^r_2(\psi^{\bar g})), \end{aligned}$$

where $N_{\bar g} : \hat{\mathbb{Q}}_2(\psi^{\bar g}) \to \hat{\mathbb{Q}}_2(\psi^*)$ is the norm. Since $\psi^{\bar g} = \chi^{r\bar g} \otimes \xi^{\bar g}$ and $r\bar g \neq r$ in $(\sigma/\sigma_1)/\langle 2\rangle$, $A^r_2(\psi^{\bar g}) = 1$ except if $\bar g = 1$. This gives

$$\hat{A}^r_2(\psi^*) = N(\hat{A}^r_2(\psi)) \in \hat{H}^0(\hat{\mathbb{Q}}_2(\psi^*), \beta_0).$$

If ξ has type I then $\hat{A}^r_2(\psi) \in U^{2e}_L/U^{2e+1}_L$ is non-trivial with $L = \hat{\mathbb{Q}}_2(\psi)$. By 4. 3, 4. 4 and (C3), $\hat{A}^r_2(\psi^*) \neq 0$. If ξ has type II, then ψ^* has type II by 4. 4 and using (C1), $\hat{A}^r_2(\psi^*) \neq 0$. In all other cases $\hat{A}^r_2(\psi^*) = 0$.

For n even the situation is simpler. In (2. 12), $\hat{A}^r_0(\chi^r \otimes 1) = 1 - 4\delta$, so

$$A^r_0(\mathrm{Ind}^{\pi}_{\pi_1}(\chi^r \otimes 1)) = 1 - 4\delta \in \hat{\mathbb{Q}}_2(\zeta_d)^{\sigma/\sigma_1}.$$

Its cohomology class is non-trivial.

The argument for m odd is similar but easier. We use (2. 18) as the main diagram and note that $\hat{H}^1(\mathrm{Ind}^*_1)$ is injective by 4. 4. But i^*_1 is also injective by 2. 17, and so d is injective if and only if d_1 is injective. This completes the proof. ■

We have now proved our main result, Theorem 1. 16. Indeed, it was remarked after 2. 6 that $L^K_i(\hat{\mathbb{Z}}_2\pi, \beta, b)(d) = 0$ unless the element g_0 exists satisfying (1. 15). The rest of the theorem is contained in 3. 9, 3. 10 and 4. 5.

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