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Finite Group Actions on $P^2(\mathbb{C})$

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Consider the question: which finite groups operate as symmetries of the complex projective plane $P^2(\mathbb{C})$? Any finite subgroup of $PGL_3(\mathbb{C})$ acts as a group of collineations and these give the linear models. The list of such groups is relatively short [MBD] but contains, for example, abelian groups of rank $\leqslant 2$, subgroups of $U(2)$, and the simple groups A_5, A_6, and $PSL(\mathbb{F}_7)$. It turns out that these linear groups are the only ones which can operate topologically on $P^2(\mathbb{C})$ with reasonable behavior near the singular set. An action is called "locally linear" if each singular point has an invariant neighborhood which is equivariantly homeomorphic to a neighborhood of 0 in a (real) representation space.

THEOREM. *Let G be a finite group with a locally linear action on $P^2(\mathbb{C})$. If G induces the identity on homology, then G is isomorphic to a subgroup of $PGL_3(\mathbb{C})$.*

Our proof shows that the result also holds when G acts as above on an integral homology $P^2(\mathbb{C})$. We will use this remark in a further paper, to study smooth actions on manifolds with definite intersection forms.

We are informed that a result similar to this has been obtained independently by D. Wilszynski (Ph.D. thesis, in preparation). It is also a pleasure to thank W. Feit for simplifying the proof of Theorem 3.1 and M. Pettet for a helpful letter.

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1. Finite Subgroups of $PGL_3(\mathbb{C})$

We briefly recall the classification of Blichfeldt [MBD] for the finite subgroups of $PGL_3(\mathbb{C})$ described by 3×3 matrices. There are two types of intransitive groups:

(A) abelian groups of rank $\leqslant 2$,

(B) subgroups of $U(2)$.

The imprimitive groups are all monomial. There are two types:

(C) a group generated by an abelian group of substitutions of the form

$$H\colon x_1' = \alpha x_1, \qquad x_2' = \beta x_2, \qquad x_3' = \gamma x_3$$

together with an order 3 permutation

$$T\colon x_1' = x_2, \qquad x_2' = x_3, \qquad x_3' = x_1,$$

(D) a group generated by H, T of (C) and a transformation

$$R\colon x_1' = a x_1, \qquad x'_2 = b x_3, \qquad x_3' = c x_2.$$

The groups of type (C) or (D) have either one or four invariant triangles. The second case is very special and occurs only for the particular group H generated by

$$S_1 = (1, \omega, \omega^2), \qquad S_2 = (\omega, \omega, \omega) \qquad (\omega^3 = 1)$$

the transformation T, and the particular transformation

$$x_1' = -x_1, \qquad x_2' = -x_3, \qquad x_3' = -x_2$$

for R. There are in addition three primitive groups having a normal imprimitive subgroup of the special type (D). The permutation induced on the set of four triangles is a subgroup of A_4 generated by the transformations

$$U\colon x_1' = \varepsilon x_1, \qquad x_2' = \varepsilon x_2, \qquad x_3' = \varepsilon\omega x_3 \qquad (\varepsilon^3 = \omega^2)$$

and

$$V\colon \begin{cases} x_1' = \rho(x_1 + x_2 + x_3) \\ x_2' = \rho(x_1 + \omega x_2 + \omega^2 x_3) \\ x_3' = \rho(x_1 + \omega^2 x_2 + \omega x_3) \end{cases} \qquad \rho = 1/(\omega - \omega^2).$$

The subgroups of this type are

(E) the group of order 36 generated by S_1, T, and V,

(F) the group of order 72 generated by S_1, T, V and UVU^{-1},

(G) the group of order 216 generated by S_1, T, V, and U.

Note that (E) and (F) are subgroups of the Hessian group (G) which acts on the triangles by the full alternating group A_4.

Finally there are three simple groups contained in $PGL_3(\mathbb{C})$:

(H) the alternating group A_5 of order 60,

(I) the alternating group A_6 or order 360,

(J) the group $PSL_2(\mathbb{F}_7)$ or order 168.

The groups of type (B) can be described more explicitly (see [MBD, TS, Wo]). For our purposes it is sufficient to list the possibilities: (i) G is conjugate to a subgroup of $U(1)\times U(1)$, (ii) G is the product of a dihedral group by a cyclic group of odd order, or (iii) G has a unique central element of order 2 (this case includes for example

$$G \cong C_a \times_{\{\pm 1\}} H,$$

where $H \subseteq SU(2)$ is a binary dihedral or a binary polyhedral group). Note that if $G \subseteq U(2)$ has odd order then G is abelian, and $G \subseteq U(2)$ is abelian then rank $G \leqslant 2$.

2. Actions of Solvable Groups

In this section we prove the main result assuming that G is solvable. First note that if G has a locally linear actin on $P^2(\mathbb{C})$, then the Lefschetz fixed-point theorem gives

$$\chi(\mathrm{Fix}(g)) = L(g) = 3$$

for any $g \in G$. The following result shows that there are only two possibilities for $\mathrm{Fix}(g)$: a point and a disjoint 2-sphere (type I), or 3 isolated points (type II). To emphasize the resemblance between our topological actions and the linear models we will refer to topologically embedded 2-spheres in $P^2(\mathbb{C})$ as lines.

Proposition 2.1 [B, pp. 378, 382; Su]. *Let G be a cyclic group of prime order p with a locally linear action on $P^2(\mathbb{C})$. If $p=2$ and G acts trivially on homology, then* Fix(G) *is type* I; *when $p \neq 2$,* Fix(G) *is either type* I *or* II. *The 2-sphere in a type* I *fixed set represents the generator of $H_2(P^2(\mathbb{C}); \mathbb{F}_p)$.*

Note that this result is proved by homological methods only. By using the Atiyah–Singer Index Theorem one can show that the local representations for these C_p actions are the same as those for the linear models. In this paper it will be convenient to have available the two easiest cases (whose proofs can be left to the reader):

(i) the action has a type I fixed set;

(ii) the action extends to a dihedral group action.

In (i) the 2-sphere in the type I fixed set represents the generator of $H_2(P^2(\mathbb{C}); \mathbb{Z})$. Case (ii) is used in the proof of (2.5) to show that the local representations for dihedral group actions are complex linear.

One consequence of (2.1) which will be used repeatedly below is

PROPOSITION 2.2. *Let G be an elementary abelian group $C_p \times C_p$ with a locally linear action on $P^2(\mathbb{C})$ inducing the identity on homology.*

(i) *If $p \neq 3$ there exists a basis S, T for G so that the fixed-point sets of S, T, and ST are type* I. *The singular set is a triangle with sides the three distinct lines in these fixed sets and vertices the three points (forming* Fix(G)). *These vertices are also the type* II *fixed set of the elements ST^a for* $2 \leqslant a \leqslant p-1$.

(ii) *If $p = 3$ the singular set is either the pattern above or that of the special linear model* (C). *In this case there are four subgroups of order* 3 *with disjoint type* II *fixed sets and* Fix(G) *is empty.*

Proof. If $p \neq 3$ there is a fixed-point x_3 for G. By considering the local representation for G at x_3 we can find a basis S, T for G such that Fix(S) and Fix(T) contain lines l_1 and l_2 intersecting at x_3. If x_1 and x_2 denote the isolated points fixed by S and T, respectively, it follows that $x_1 \in l_2$, $x_2 \in l_1$, and $\{x_1, x_2, x_3\} \subseteq$ Fix(G). Since T acts semifreely on l_1, Fix(G) consists of these three distinct points. Now consider the local representation at x_2: there must be an element $g \in G$ independent of S with a type I fixed set. By renaming T^a as T (for some a, $2 \leqslant a \leqslant p-1$) if necessary we may assume that $g = ST$. Now Fix(ST) consists of a line l_3 passing through x_1 and x_2 and an isolated point. Since this point is fixed by G and distinct from x_1 and x_2 it must be x_3.

When $p = 3$ the argument is the same if G has a fixed point. If not, each $g \in G$ has a type II fixed set consisting of three points permuted cyclically by $G/\langle g \rangle \cong C_3$.

COROLLARY (2.3). *If a finite group G acts as above on $P^2(\mathbb{C})$, then* Fix(H) *has even dimension for all $H \subseteq G$.*

Proof. It is enough to consider $H=G$. Since $\mathrm{Fix}(g)$ has dimension $\leqslant 2$ for all $g \in G$, if $\mathrm{Fix}(G)$ has odd dimension then its dimension is 1 and $G \subseteq SO(3)$. Clearly this cannot happen for G cyclic, and the other cases except for dihedral groups are ruled out by (2.2), since a tetrahedral group must act with no fixed points. If G is dihedral, and $\mathrm{Fix}(G)$ is 1-dimensional, then the action must contain an invariant line ($\approx S^2$) on which G acts by reversing the orientation. This is impossible by (2.1) since our actions induce the identity on homology (if G is a 2-group, use (2.2) or the remark following (2.1) instead).

Remark 2.4. A similar elementary argument can be used to show that $G \cong C_2 \times C_2 \times C_2$ does not act as above on $P^2(\mathbb{C})$, indicating that the local representations for our actions respect some complex structure.

PROPOSITION 2.5. *Let G act locally–linearly on $P^2(\mathbb{C})$ inducing the identity on homology. If* $\mathrm{Fix}(G)$ *is non-empty then G is conjugate in $SO(4)$ to a subgroup of $U(2)$.*

Proof. Let $x \in \mathrm{Fix}(G)$, then $G \subseteq SO(4)$ since the local representation ρ around x is faithful. Since the universal covering group is $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$, it follows that $SO(4)$ is isomorphic to

$$SU(2) \times_{\{\pm I\}} SU(2)$$

and there is a projection

$$j\colon SO(4) \to SO(3) \times SO(3).$$

If $j(G) \cong G$, then G is the pullback of subgroups $\Gamma_1, \Gamma_2 \subseteq SO(3)$ via an isomorphism

$$\mu\colon \Gamma_1/\gamma_1 \xrightarrow{\approx} \Gamma_2/\gamma_2$$

for some $\gamma_i \lhd \Gamma_i$; that is $G = \{(x, y) \in \Gamma_1 \oplus \Gamma_2 \mid \mu(\bar{x}) = \bar{y}\}$. There exists an isomorphic lifting of such a subgroup if and only if γ_1 and γ_2 are odd order cyclic. By checking the list in [TS] with the methods above, one finds that the only such groups that can act on $P^2(\mathbb{C})$ satisfying (2.3) with $\mathrm{Fix}(G)$ non-empty are already conjugate to subgroups of $U(2)$.

If $j(G)$ is not isomorphic to G, then G contains a central element t or order 2. Then $\mathrm{Fix}(t)$ consists of a point p and a line l left invariant by G. Let K denote the subgroup of G fixing l and consider the extension

$$1 \to K \to G \to \Gamma \to 1, \qquad (*)$$

where $\Gamma \subseteq SO(3)$ since it operates effectively on S^2 preserving the orientation. Since any cyclic group acting effectively on S^2 has two fixed points, it follows that the inverse image in G of any cyclic subgroup of Γ is abelian and so K is a central cyclic subgroup of G. Using this partial description of the singular set, we can easily eliminate the remaining subgroups of $S(O(2)\times O(2))$ which are not in $U(2)$.

We may now assume that the inclusion $\rho\colon G\subseteq SO(4)$ is an absolutely irreducible real representation. Indeed if ρ is reducible, then G is conjugate to a subgroup of $U(2)$ since local representations in $S(O(1)\,xO(3))$ must satisfy (2.3). It follows that $K\cong C_2$, the quotient Γ is non-cyclic, and $(*)$ is non-split. If Γ is dihedral, then G has an abelian subgroup of index 2 and so all irreducible complex characters have degree $\leqslant 2$. The same conclusion holds if Γ is tetrahedral, so it remains to consider the cases when Γ is octahedral or icosahedral. In these cases, the action of Γ on l may be identified with the standard action on S^2. From the solution of the spherical space form problem in [Wo, p. 224], we see that the irreducible free representations of G are not absolutely irreducible, and hence in the present situation, G does not act freely in a neighborhood of p. Then there is a conjugacy class of elements $g\in G$ of order 3 or 5 which have lines in their type I fixed sets intersecting p and the singular set of Γ in l. By looking at the action of an element $h\in G$ of order 4 such that $h^2=t$ on this configuration it follows that h fixes l. Since this is impossible for $h\notin K$, we have a contradiction.

Corollary 2.6. *If G is an abelian group acting as above on $P^2(\mathbb{C})$ then rank $G\leqslant 2$.*

Proof. Let G be an elementary abelian p-group of rank 3. If $1\neq g\in G$ the subgroup H fixing the singular set of g has index $\leqslant 3$. When $H\neq G$ it follows from (2.2) that there is an $h\in H$ with a type I fixed set. Then in either case Fix(G) is non-empty so $G\subseteq U(2)$ and rank $G\leqslant 2$.

Corollary 2.7. *If G acting on $P^2(\mathbb{C})$ as above contains a subgroup $H\cong C_3\times C_3$ with singular set pattern* (2.2)(ii) *then H is a maximal abelian subgroup of G.*

Proof. We may suppose that G is an odd-order abelian group of rank 2. Let A, B generate G so that $A^m=S$ and $B^n=T$ generate H. Since T permutes the three points in Fix(S), $\langle B\rangle\subseteq \mathrm{Aut}(\mathrm{Fix}(S))\cong\Sigma_3$ and so B has order 3. Similarly A has order 3 and $H=G$.

Suppose now that G is a solvable group acting as above on $P^2(\mathbb{C})$. We will use several times the result of (2.5), that when Fix(G) is non-empty then $G\subseteq U(2)$. Since a minimal normal subgroup of a solvable group is elementary abelian we can distinguish cases depending on the rank of such a subgroup. From (2.6) this rank is $\leqslant 2$ and if H is normal in G, $H\cong C_p$, let K be the subgroup of G fixing the singular set of H. If $K=G$ then $G \subseteq U(2)$ as above; if not there is an exact sequence

$$1 \to K \to G \to \Sigma_3 \tag{2.8}$$

where Σ_3 is the symmetric group on three letters. Since $\mathrm{Fix}(K)$ is non-empty, $\rho: K \subseteq U(2)$ is faithful and so is either K is abelian (of rank $\leqslant 2$) or ρ is irreducible. In the second case, either K has a unique central element k or order 2 (implying that k is central in G and so $G \subseteq U(2)$), or the centre of K has odd order. This can only happen if $K \cong D_{2n} \times Z(K)$ for some odd integer n. Since K is supposed to operate trivially on $\mathrm{Fix}(H)$ this possibility can easily be eliminated.

We conclude that for this case it is enough to study extensions (2.8) with $K \neq G$ albelian and $\mathrm{Fix}(H)$ of type II.

The second case occurs when G has no normal cyclic subgroups but a normal subgroup $H \cong C_p \times C_p$. If the singular set for H is described by (2.2)(i) then $\mathrm{Fix}(H)$ is non-empty and the argument above reduces us to the extensions (2.8) of rank $\leqslant 2$ abelian groups again where $\mathrm{Fix}(H) = \mathrm{Fix}(K)$. The remaining possibility is for $H \cong C_3 \times C_3$ with the special type (C) singular set (2.2)(ii). In the linear model for this action there exist trinagles invariant under each of the subgroups of order 3. It will be convenient to adopt this terminology to refer to the type II fixed sets of elements of H (even though the pattern contains only the vertices).

First consider the actions with four invariant triangles.

LEMMA 2.9. *Let $H \cong C_3 \times C_3$ be normal in G and* $\mathrm{Fix}(H)$ *empty. If G leaves invariant the triangles in the singular set of H then H has index at most* 2 *in G. If $H \neq G$ then G is a special type* (D) *group.*

Proof. By (2.6) H is a maximal abelian subgroup of G so the map

$$j: G/H \to \mathrm{Aut}(H) \cong GL_2(\mathbb{F}_3)$$

is injective. But the action of $GL_2(\mathbb{F}_3)$ on the four subgroups of H has kernel $\{\pm I\}$ and since the triangles in $\mathrm{Sing}(H)$ are the type II fixed sets of these four subgroups we obtain $[G: H] \leqslant 2$. If $G \neq H$ then G/H acts as $-I$ on $C_3 \times C_3$. From the presentation given in Section 1 we see that G is a special type (D) group.

In general G will not leave the triangles invariant so let K denote the subgroup of G with this property. Then G is an extension

$$1 \to K \to G \to \Sigma_4, \tag{2.10}$$

where K is by (2.9) a special type (C) or (D) group normal in G and G/K permutes the triangles in $\mathrm{Sing}(H)$. Again $H \cong C_3 \times C_3$ is a maximal abelian normal subgroup in G and G/H injects into $GL_2(\mathbb{F}_3)$. Note that the diagram

$$\begin{array}{ccc} G/H & \longrightarrow & GL_2(\mathbb{F}_3) \\ \downarrow & & \downarrow \\ G/K & \longrightarrow & \Sigma_4 \end{array}$$

commutes.

LEMMA 2.11. *If $G \neq K$ in the description* (2.10) *above then G is type* (E), (F) *or* (G).

Proof. It is enough to show that the composite

$$G/H \to GL_2(\mathbb{F}_3) \to \Sigma_4 \xrightarrow{\omega} C_2$$

is trivial, where $\omega(\sigma) = \operatorname{sgn}(\sigma)$. If not, for a suitable choice of basis for H, there is an element $g \in G$ acting as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ on H. (The possibility that an element of G acts as an order 4 cyclic permutation of the triangles may be ruled out by considering the orbit space $(P^2(\mathbb{C})\text{-Sing}(H))/K$, which would inherit a semi-free C_4 action.) Then the extension

$$1 \to H \to G_0 = \langle H, g \rangle \to C_2 \to 1$$

is split and $G_0 \cong \sum_3 \times C_3$. But the action of Σ_3 has a unique fixed point and so $\operatorname{Fix}(G_0)$ is non-empty contradicting the assumption on H.

Since the above composite is trivial G is an extension

$$1 \to K \to G \to A_4,$$

where K is the special type (D) group of order 18 and A_4 acts faithfully on the singular set of K by permuting the triangles.

This completes the proof for the case $H \cong C_3 \times C_3$ and $\operatorname{Sing}(H)$ containing four invariant triangles. To complete the argument for solvable groups in general, it remains to discuss the particular extensions (2.8) of rank $\leqslant 2$ abelian groups K by a subgroup of Σ_3, where $\operatorname{Fix}(K) = \operatorname{Fix}(H)$. We will show that all such extensions lead to groups in $PGL_3(\mathbb{C})$. As above $H \lhd G$, $H \subseteq K$, and H is elementary abelian. Notice that if rank $H = 1$ and rank $K = 2$ then $\operatorname{Fix}(H)$ of type I implies that G has a fixed point, and $\operatorname{Fix}(H)$ of type II implies that K contains $H' \cong C_q \times C_q$ normal in G with the same fixed set. We replace H by H' and consider these extensions in Case 2 below.

Case 1. Rank $K = 1$ and $H \subseteq K$, $H \cong C_p$ with a type II fixed set; G/K operates faithfully on $\operatorname{Fix}((H)$.

Case 2. Rank $K = 2$, G has no normal cyclic subgroups with type I fixed sets, and $H \cong C_p \times C_p$ with $\mathrm{Fix}(H)$ non-empty; G/K operates faithfully on $\mathrm{Fix}(H)$.

First we consider Case 1 and make a number of remarks to show that these groups are type (B), (C), or (D).

(i) The K-action is semifree with 3 isolated fixed points and K has odd order, otherwise we could find an element $k \in K$ with a type I fixed set implying $G \subseteq U(2)$. If $|\mathrm{im}\, j| = 2$ then $G \subseteq U(2)$.

(ii) If $\mathrm{im}(j\colon G \to \Sigma_3)$ contains C_3, let G_0 be the subextension

$$1 \to K \to G_0 \to C_3 \to 1.$$

This extension splits (choose $g_0 \in G_0$ with $j(g_0) \neq 1$ then g_0 permutes $\mathrm{Fix}(K) = \{p_1, p_2, p_3\}$ so $\mathrm{Fix}(g_0) \subseteq P^2(\mathbb{C}) - \{p_1, p_2, p_3\}$. But if $1 \neq g_0^3 \in K$ then $\mathrm{Fix}(g_0^3) \subseteq \{p, p_2, p_3\}$, a contradiction).

(iii) In the extension for G_0, either $C_3 \subseteq \mathrm{Aut}(K)$, and G_0 is type (C), or C_3 commutes with K, $K \cong C_3$, and G_0 is a special type (C) group. Indeed if $C_3 = \langle g_0 \rangle$ commutes with K, then $\mathrm{Fix}(g_0) = \{q_1, q_2, q_3\}$ is a K-invariant subset of $P^2(\mathbb{C}) - \mathrm{Fix}(K)$ so $|K| = 3$.

(iv) The remaining case is

$$1 \to G_0 \to G \to C_2 \to 1,$$

where G_0 is type (C). If G_0 is not special type (C) group $C_3 \times C_3$ then G/G_0 preserves the unique triangle of singular points for G_0 and the extension splits. It follows that G is type (D) by comparing if with the presentation of Section 1.

(v) If G_0 is the special type (C) group then G is the special type (D) group of order 18.

We turn now to Case 2 where G has no normal cyclic subgroups.

(i) Let $K \cong C_a \times C_b$ generated by A, B such that $A^m = S$ and $B^n = T$ gives the basis (2.3)(i) for $H \cong C_p \times C_p$. Since $\mathrm{Fix}(K) = \mathrm{Fix}(H)$, C_a and C_b are the subgroups of K which act faithfully on the lines in $\mathrm{Fix}(T)$ and $\mathrm{Fix}(S)$, respectively. As above we may asume $\mathrm{im}(j\colon G \to \Sigma_3) \supseteq C_3$ otherwise $G \subset U(2)$. Since C_3 permutes the lines in $\mathrm{Sing}\,(H)$ it follows that $a = b$.

(ii) The subextension $1 \to K \to G_0 \to C_3 \to 1$ splits as in Case 1. If $(|K|, 3) = 1$ the group G_0 is type (C) and, since the further extension $1 \to G_0 \to G \to C_2$ also splits, one can easily check that G is type (D).

(iii) If G_0 contains a subgroup $(C_{3^k} \times C_{3^k}) \rtimes C_3$ the fact that G_0/K permutes the isotropy groups of the lines in $\mathrm{Sing}\,(H)$ implies that $C_{3^k} \times C_{3^k}$ is a cyclic $\mathbb{Z}/3^k[C_3]$ module.

LEMMA 2.10. *Let $M \cong C_{3^k} \times C_{3^k}$ be a cyclic $\mathbb{Z}/3^k[C_3]$ module. Then there are generators A, B for M and T for C_3 such that $T(A) = B$ and $T(B) = -A - B$.*

Proof. We can use the classification of [S] to show that this is the only cyclic indecomposable module of rank 2.

(iv) The Lemma implies that G_0 is a type (C) group and it then follows that G is type (D), since the extension $1 \to G_0 \to G \to C_2$ is again split.

3. ACTIONS OF SIMPLE GROUPS

We now consider actions of simple groups on complex projective space $P^2(\mathbb{C})$. Our object is to prove the following:

THEOREM 3.1. *If G is a nonabelian simple group operating on $P^2(\mathbb{C})$, then G can only be one of the three simple groups A_5, A_6, $PSL_2(7)$.*

First, from the discussion in the previous section, it is clear that any 2-subgroup of G must have a fixed point and so such a subgroup must also be a subgroup of $U(2)$. In particular, the 2-rank of such a simple group (the maximum rank of abelian 2-subgroups) is at most 2. From the classification theory of finite simple groups, we obtain (see p. 72 of [G]):

THEOREM 3.2. *If G is a simple group of 2-rank at most 2, then G is isomorphic to one of the following:*

(i) $PSL_2(q)$, *q odd*, $q \geq 5$,

(ii) $PSL_3(q)$, *q odd*,

(iii) $PSU_3(q)$, *q odd*,

(iv) $U_3(4)$,

(v) A_7,

(vi) M_{11}.

Note that the three simple groups appearing in $PGL_3(\mathbb{C})$ are $PSL_2(5) = A_5$, $PSL_2(7)$, and $PSL_2(9) = A_6$. Therefore to prove our theorem, we only need to show that all the groups in (3.2)(i–vi) except the above three groups are impossible.

Case 3.3(i): *Simple groups of Lie type*

$$PSL_2(q), \quad q \text{ odd}, \quad q \geq 5; \qquad PSL_3(q), \quad q \text{ odd}, \quad q > 3;$$
$$PSU_3(q), \quad q \text{ odd}, \quad q > 3.$$

Suppose G is one of the above groups and suppose G operates on $P^2(\mathbb{C})$. Let P be a Sylow p-subgroup and N be the normalizer of P in G. Then by our analysis in Section 2, N is a subgroup of $PGL_3(\mathbb{C})$. In particular there exists an epimorphism $\tilde{N} \to N$ such that $\tilde{N}$ has a faithful representation of degree 3.

Let $\tilde{P}$ be a p-Sylow subgroup of $\tilde{N}$. If $\tilde{P}$ is non-abelian then the degree of a faithful representation is at least p and so $p=3$. Suppose that $G=PSL_3(q)$ or $PSU_3(q)$, with $3 \mid q$. If $q>3$, then N contains an elementary abelian 3-group of rank $\geqslant 3$ contrary to (2.6). The situation for $q=3$ requires a separate argument in (3.4). If $G=PSL_2(q)$ and $q>9$, then again N contains a rank 3 abelian subgroup and so is ruled out.

Suppose that P is abelian. Then G can only be $PSL_2(q)$ and

$$|N: C(\tilde{P})| \geqslant |N: C(P)| = (q-1)/2.$$

By a well-known Theorem of Clifford in representation theory, the degree of a faithful representation of $\tilde{N}$ is at least $(q-1)/2$ because the restriction to $\tilde{P}$ has at least $(q-1)/2$ distinct linear constitutents. Hence $(q-1)/2 \leqslant 3$ and $q \leqslant 7$.

Case 3.4(ii): *$PSL_3(3)$ or $PSU_3(3)$*. In the case of $PSL_3(3)$, we consider the maximal parabolic subgroup consisting of matrices

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1/\Delta \end{pmatrix} \qquad \Delta = ad - bc.$$

This group is (abstractly) isomorphic to the semidirect product of the abelian group $C_3 \times C_3$ and the symmetric group Σ_4 on four letters. This is ruled out in (2.11).

The argument for $PSU_3(3)$ is similar. Again we consider the maximal parabolic subgroup $SU_3(3)$ which can be written as a group extension

$$1 \to C_3 \times C_3 \to (\text{parabolic subgroup}) \to U_2(3) \to 1$$

and in $PSU_3(3)$ this gives rise to a solvable group whose order is bigger than the Hessian group. Thus it is impossible for this group to act on $P^2(\mathbb{C})$.

The Remaining Cases 3.5: *$U_3(4)$, A_7, and M_{11}*. First of all, the Sylow 2-subgroup S of $U_3(4)$ has the following properties:

(i) $|S| = 64$;

(ii) centre of S is isomorphic to $C_2 \times C_2$; and

(iii) S/centre is an elementary group of order 16.

From these properties, it is not difficult to show that there is no faithful representation of S to $U(2)$. Since S does not lie in $U(2)$, it follows that $U_3(4)$ cannot operate on $P^2(\mathbb{C})$.

As for the alternating group A_7 on seven letters, it contains a subgroup $T = A_4 \times C_3$ of permutations which preserves the first four letters. In other words, T is a subgroup of index 2 in the product $\Sigma_4 \times \Sigma_3$ of symmetric groups

$$1 \to T \to \Sigma_4 \times \Sigma_3 \to C_2 \to 1.$$

Since $|T| = 8 \cdot 3^2$, our analysis of (2.6) shows that it cannot be a subgroup of the Hessian group. On the other hand, it is also easy to see that T does not belong to one of the solvable groups (A)–(D).

Finally, it is well known that the Mathieu group M_{11} contains the projective special linear group $PSL_2(11)$. Since we have ruled out this last group in (3.3), it is impossible to have M_{11} operating on $P^2(\mathbb{C})$.

4. Proof of the Main Theorem

Suppose we are given a finte group G with a locally linear homologically trivial action on $P^2(\mathbb{C})$ as in the Introduction. Suppose in addition that it is not one of the groups in our list (A)–(J). We decompose G using a normal series of subgroups:

$$1 = G_{-1} \lhd G_0 \lhd G_1 \lhd \cdots \lhd G_n = G,$$

where G_i is normal in G_{i+1} and G_0 is simple or solvable. If G_0 is simple then by the previous section G_0 is A_5, A_6, or $PSL_2(7)$. Since G is different from these the factor group G_1/G_0 must be non-trivial. Thus to eliminate this situation we need the following:

Proposition 4.1. *Suppose G_1 is a group extension of a simple group G_0 by a cyclic group G_1/G_0*

$$1 \to G_0 \to G_1 \to G_1/G_0 \to 1,$$

where G_0 is one of the simple groups A_5, A_6, or $PSL_2(7)$. Then G_1 does not operate effectively on $P^2(\mathbb{C})$.

Proof. First there is a natural homomorphism $\varphi\colon G_1 \to \mathrm{Aut}(G_0)$. Since the centre of G_0 is trivial the restriction of this homomorphism to G_0 is an injection.

Suppose the homomorphism φ has a non-trivial kernel K in G_1. Then there exists a subgroup in G_1 isomorphic to the product $G_0 \times K$ and, since G_1/G_0 is cyclic, so is K. We can assume that $|K|$ is prime and from the discussion in Section 1 it follows that the fixed set $\mathrm{Fix}(K)$ consists of three isolated points or a point and a line. In either case $G_0 \times K$ acts with at least one fixed point. However, this is impossible because none of our simple groups can be embedded in $U(2)$. Thus K must be trivial and by means of φ our group G_1 is embedded as a subgroup of the automorphism group $\mathrm{Aut}(G_0)$.

From the work of Schreier and Van der Waerden in [SW], it is known that all automorphisms ψ of $PSL_2(q)$ can be expressed by

$$\psi(X) = B \cdot A \cdot X^{\sigma} \cdot A^{-1} \cdot B^{-1},$$

where σ is an automorphism of the field $\mathbb{F}_q$, B is an element in $PSL_2(q)$, $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ for some representative a of a coset in $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2$.

We now consider the situation when $q = 5$ and $G_0 = PSL_2(5) = A_5$. Since $\mathbb{F}_5$ is a prime field, there is no nontrivial field automorphism and the group $\mathrm{Aut}(PSL_2(5))$ can be identified with the projective general linear group $PGL_2(5)$. Note that the factor group $PGL_2(5)/PSL_2(5)$ contains only one non-trivial element. In order for G_1 to be strictly larger than G_0, it must coincide with $PGL_2(5)$.

The group $GL_2(5)$ contains a Borel subgroup H of matrices

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \qquad a, c \in \mathbb{F}_5^*, \qquad b \in \mathbb{F}_5.$$

In the projective linear group $PGL_2(5)$, the image $\bar{H}$ of this subgroup is a semidirect product of two cyclic groups:

$$1 \to C_5 \to \bar{H} \to C_4 \to 1.$$

It is not difficult to see that this semidirect product $\bar{H}$ does not belong to any of the solvable groups (A) – (G). Thus $\bar{H}$ does not operate effectively on $P^2(\mathbb{C})$ and neither does $PGL_2(5)$.

The same argument can be carried out if $G_0 \cong PSL_2(7)$. In this case G_1 can be identified with $PGL_2(7)$ and the image of the Borel subgroup is a semidirect product of cyclic groups C_7 and C_6. Once again this group does not belong to one of the solvable groups in the list so $PGL_2(7)$ does not operate effectively.

As for the group $PSL_2(9) = A_6$, the Galois group $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ is cyclic of order 2 generated by the field automorphism

$$\sigma\colon \mathbb{F}_9 \to \mathbb{F}_9, \qquad \sigma(x) = x^3.$$

Therefore the automorphism group $\mathrm{Aut}(PSL_2(9))$ is a semidirect product of $PGL_2(9)$ and this cyclic group. Let τ denote conjugation by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \theta \end{smallmatrix}\right)$ in $PGL_2(9)$, where θ generates $\mathbb{F}_9^{\times}$, and note that $PGL_2(9)$ is itself a semidirect product of $PSL_2(9)$ and the cyclic group of order 2 generated by

τ. It follows that the factor group $\mathrm{Aut}(PSL_2(9))/PSL_2(9)$ is a sum $C_2 \oplus C_2$ generated by σ and τ.

From our assumptions, G_1 is an extension of $PSL_2(9)$ by a cyclic group of order 2 and hence G_1 is also a subgroup of index 2 in $\mathrm{Aut}(PSL_2(9))$. There are three possibilities for G_1:

(i) G_1 is generated by $PSL_2(9)$ and σ;
(ii) G_1 is generated by $PSL_2(9)$ and τ;
(iii) G_1 is generated by $PSL_2(9)$ and $\sigma\tau$.

In the first two cases we consider the Borel subgroup B consisting of matrices

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \qquad ac = 1, \qquad a, b, c \in \mathbb{F}_9.$$

This subgroup together with either of the elements σ or τ generates a semidirect product:

$$1 \to C_3 \oplus C_3 \to \langle B, \sigma \rangle \to C_8 \to 1$$
$$1 \to C_3 \oplus C_3 \to \langle B, \tau \rangle \to C_4 \rtimes C_2 \to 1.$$

It is not difficult to show that $\langle B, \sigma \rangle$ and $\langle B, \tau \rangle$ are not subgroups of the Hessian group (G). Therefore they cannot operate effectively on $P^2(\mathbb{C})$.

In Case (iii) we consider the subgroup T of matrices:

$$\begin{pmatrix} a & b \\ \theta b & a \end{pmatrix} \quad a^2 - \theta b^2 = 1 \quad \text{and} \quad \begin{pmatrix} a & -b \\ \theta b & -a \end{pmatrix} \quad a^2 - \theta b^2 = -1$$

which is of order 10. Both the elements σ and τ stabilize this subgroup T and as a result $\langle T, \sigma\tau \rangle$ is a non-abelian group of order 20. This is the direct product of C_2 with a dihedral group of order 10. By considering the singular set of this action, we see that this group has a common fixed point with the 2-Sylow subgroup of $\langle PSL_2(9), \sigma\tau \rangle$ generated by

$$\begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma\tau.$$

However, T is a maximal subgroup of $PSL_2(9)$ so this would imply that $\langle PSL_2(9), \sigma\tau \rangle$ is a subgroup of $U(2)$. This contradiction completes the proof of (4.1).

Let us now return to the situation of a normal series

$$1 = G_{-1} \lhd G_0 \lhd G_1 \lhd \cdots \lhd G_n = G,$$

with G_0 a solvable group. Without loss of generality we can assume that G_1/G_0 is a non-abelian simple group.

PROPOSITION 4.2. *Suppose G_1 is a group extension of a solvable group G_0 by a nonabelian simple group*

$$1 \to G_0 \to G_1 \to G_1/G_0 \to 1,$$

where G_0 belongs to one of the types (A)–(G). *Then either G_1 is a subgroup of $U(2)$ or G_1 does not operate effectively on $P^2(\mathbb{C})$.*

Proof. We will separate our discussion into four different cases according to the structure of G_0:

Case (i): G_0 is a non-abelian solvable group in $U(2)$ with even order centre;

Case (ii): G_0 is an abelian group with $\mathrm{Fix}(G_0, P^2(\mathbb{C}))$ non-empty;

Case (iii): G_0 is an extension of an abelian group H_0 and a non-trivial subgroup G_0/H_0 of Σ_3, $\mathrm{Fix}(H_0)$ consists of three points and G_0/H_0 operates by permuting them;

Case (iv): G_0 is an extension of the abelian $C_3 \oplus C_3$ and a subgroup of $SL_2(3)$, the action of the abelian group has four invariant triangles.

The proof of Case (i) is easiest. For, in this case, G_0 contains in its centre a unique element t of order two. The cyclic group $C_2\langle t\rangle$ generated by this element is a normal subgroup in G_1. Since the fixed-point set $\mathrm{Fix}(C_2\langle t\rangle, P^2(\mathbb{C}))$ consists of a single point and a line, the group $G_1/C_2\langle t\rangle$ operates on this fixed set and therefore keeps the isolated point fixed. In other words, G_1 has a fixed point and belongs to a subgroup in $U(2)$.

In Case (ii), the group G_1/G_0 operates on the fixed-point set $\mathrm{Fix}(G_0)$ of G_0. Since there is no non-trivial homomorphism of a non-abelian simple group G_1/G_0 to the symmetric group Σ_3 on three letters, it follows that G_1 has an isolated fixed point. Thus G_1 is a subgroup of $U(2)$.

In Case (iii), we observe that if G_0 is chosen maximal with respect to H_0 then G_0 is "almost" maximal among the solvable groups: if $G_0 = H_0 \rtimes \Sigma_3$, then it is maximal and if G_0 is a proper subgroup of $H_0 \rtimes \Sigma_3$, then any further extension G_1 which is solvable coincides with $H_0 \rtimes \Sigma_3$. When G_0 is maximal we are done because any element g in $G_1 - G_0$ gives us a bigger solvable group $\langle G_0, g\rangle$ operating on $P^2(\mathbb{C})$, which is impossible. When G_0 is not maximal and not as in Case (i), then any element g in $G_1 - G_0$ gives us the maximal extension $\langle G_0, g\rangle \cong H_0 \rtimes \Sigma_3$. In particular this element g must keep $\mathrm{Fix}(H_0)$ invariant and must operate on this fixed-point set in a non-trivial manner. But this is impossible because there is no non-trivial homomorphism of the simple group G_1/G_0 to Σ_3.

For the proof of Case (iv), we use the same strategy as before. From (2.11), it is easy to see that any extension of G_0 by a solvable is a subgroup of the Hessian group $(C_3 \times C_3) \rtimes SL_2(3)$. Any element g in $G_1 - G_0$ operates on the four invariant triangles in a non-trivial manner.

Note that there is no nontrivial homomorphism of the simple group G_1/G_0 to the symmetric group Σ_4 of four elements.

Therefore, the action of G_1/G_0 keeps these four triangles invariant. For each of these triangles, we have the induced action on the three vertices. Once again, there is no non-trivial homomorphism of G_1/G_0 to Σ_3 and so this induced action is trivial, thus it is impossible to have any further extension of G_0.

Finally to complete the proof of our main theorem, it still remains to show that in the case when G_1 is a subgroup of $U(2)$ in (4.2) any further extension G_2 of such a group belongs to a subgroup of $U(2)$.

Recall that G_1 has an abelian normal subgroup G_0 and its factor group G_1/G_0 is a non-abelian simple group. From the description of subgroups in $U(2)$, such a group G_1 has a unique element t of order 2 in its centre. If G_1 is a normal subgroup of G_2, and this group G_2 operates effectively on $P^2(\mathbb{C})$, then the subgroup $C_2\langle t\rangle$ generated by t is also a normal subgroup in G_2. It follows as before that the isolated point in $\mathrm{Fix}(\langle t\rangle)$ is fixed by the entire group G_2. This implies that G_2 is a subgroup of $U(2)$, and the proof of our main theorem is complete.

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