Math. Z. 193, 191–214 (1986)

Mathematische Zeitschrift

Local Surgery Obstructions and Space Forms

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The space form problem, concerning the existence of free actions of finite groups on spheres, has now been substantially solved except in dimension three [3, 4, 8, 11, 13, 14]. This paper gives an effective method for the uniqueness question: which homotopy types are realized by such actions. Our results are fairly complete for 2-hyperelementary periodic groups and give new necessary conditions for actions even on S^3. In particular Corollary C below eliminates certain homotopically non-linear actions which presented an obstacle to knowing precise dimensional bounds (see [4; Conjecture D]). The methods also apply to the euclidean space form problem of semi-free actions on $\mathbb{R}^{n+k}$ fixing $\mathbb{R}^k$ [2] where the linear model is a free representation direct sum with a trivial representation. A free action on S^{n-1} yields a semi-free topological action on $\mathbb{R}^n$ by "coning".

The method involves the calculation of a certain local surgery obstructions (§§ 2, 3). The basic cases are the 2-hyperelementary type I groups: extensions of the form

$$1 \to \mathbb{Z}/m \to \pi \to \mathbb{Z}/2^k \to 1$$

where m is odd. We write σ for a Sylow 2-subgroup of π (and identify it with $\mathbb{Z}/2^k$). Let t be the twisting defining the extension, and let

$$t(\sigma) = \mathrm{Im}(t\colon \mathbb{Z}/2^k \to \mathrm{Aut}(\mathbb{Z}/m)) \cong \mathbb{Z}/2^l.$$

These are groups with periodic Tate cohomology of period 2^{l+1}. We assume throughout that $1 \leq l < k$ so that π is non-cyclic and satisfies Milnor's necessary condition (no dihedral subgroups) for acting freely on a sphere [8]. Under this assumption, π admits free linear representations in the period dimension so has linear actions on the sphere. In fact, let χ be a faithful complex character of

* Research partially supported by NSERC Grant A4000

$\mathbb{Z}/m\cdot 2^{k-l}$ and denote by $g\in(\mathbb{Z}/m)^\times$ a generator of $\operatorname{Im} t$. Then

$$V=\chi+\chi^g+\ldots+\chi^{g^{2^l-1}}$$

extends to a free π-representation of real dimension 2^{l+1} where a generator T of $\mathbb{Z}/2^k$ acts by:

$$T(z_1,z_2,\ldots,z_{2^l})=(\zeta_{2^{k-l}}z_{2^l},z_1,z_2,\ldots,z_{2^l-1}).$$

Here and everywhere below ζ_i denotes a primitive i'th root of one. The *homotopy type* of this action is that of the orbit space $N=S(V)/\pi$ and is determined completely by the generator (Chern class)

$$g=c_{2^l}(V)\in H^{2^{l+1}}(\pi;\mathbb{Z}).$$

According to [11], any other possible homotopy types in this dimension are given by the generators rg where r is an integer relatively prime to $|\pi|$. More generally, the homotopy types in dimension $2^{l+1}s-1$ are given by rg^s. Our problem is: as r varies, which of the generators rg^s describe the homotopy types of (i) free π-actions on S^{q-1} or (ii) semi-free π-actions on $(\mathbb{R}^{q+k},\mathbb{R}^k)$ for $q=2^{l+1}s$ and $k\geqq 0$?

The following results illustrate the method. They answer (i) in some cases and (ii) in general. From our earlier results [2] it is enough to set $k=0$ in (ii). Note also that reversing the orientation of S^{q-1} changes r to $-r$ so that we may assume $r\equiv 1 \pmod 4$ without loss of generality.

Theorem A. *Let $q=2^{l+1}\cdot s\geqq 5$ and $r\equiv 1 \pmod 4$. Let π be a 2-hyperelementary type* I *group with $-1\in t(\sigma)$ and* $\ker t$ *of order 2. Then π acts freely on S^{q-1} with homotopy type rg^s if and only if r is a* 2^l-th *power* (mod m), *where m is the odd part of $|\pi|$.*

Theorem B. *Let $q=2^{l+1}\cdot s\geqq 4$, and π a 2-hyperelementary type* I *group with $|\ker t|\geqq 2$. Then π acts semi-freely on* $(\mathbb{R}^q,0)$ *with homotopy type rg^s if and only if r is a square* (mod m), *for every odd divisor m of $|\pi|$ such that $\mathbb{Z}/m\rtimes\sigma$ has $-1\in t(\sigma)$.*

Remarks. (1) *When* $-1\in t(\sigma)$ the condition $r\in(\mathbb{Z}/m)^{\times 2^l}$ in Theorem A is necessary for the action to exist even if $|\ker t|>2$.

(2) A generator rg^s is linear if and only if r is a 2^l-th power (mod m) and minus a 2^l power (mod 2^k) [6], so as k increases many homotopically non-linear actions result.

An important special case occurs for $k=2$ and $l=1$ when $\pi=Q(4m)$ is a quaternion group.

Corollary C. *Let $\pi=Q(4m)$ and $q=4s\geqq 4$. Then if π acts semi-freely on* $(\mathbb{R}^q,0)$ *the action is homotopically linear.*

This result was conjectured in [4]. It is an essential step in analysing the actions of type II groups on spheres or euclidean spaces (cf. [2, 4]).

The first author would like to thank the Institut des Hautes Études Scientifiques at Bures-sur-Yvette (France) for its hospitality during Spring 1983 while part of this work was done.

1. Fibre Preserving Maps of Vector Bundles

Let ξ^q, η^q be vector bundles over a space X with structural group $O(q)$. The set of stable homotopy classes of fibre preserving maps

$$(1.1)\qquad \begin{array}{ccc} S(\xi^q) & \xrightarrow{t} & S(\eta^q) \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

is a representable homotopy functor [1, §4] with classifying space denoted QS^0/O. Since the degree of such a fibre-preserving map is well-defined up to sign (orientation-reserving bundle automorphism of ξ or η are allowed in the equivalence relation), there is a function

$$|\deg|: QS^0/O \to \mathbb{Z}^+.$$

The components $(QS^0/O)_r$ of QS^0/O are just the classifying spaces for maps of fixed degree $r \geqq 0$. Whitney sum then provides a pairing of classifying spaces:

$$(1.2)\qquad (QS^0/O)_r \times (QS^0/O)_{r'} \to (QS^0/O)_{rr'}$$

There is a map i: $QS^0/O \to BO$ induced by sending a triple (ξ^q, t, η^q) over X to the reduced KO element $[\xi^q - \eta^q]$, and a map j: $QS^0 \to QS^0/O$ obtained by regarding a map $X \to QS^0$ as a stable homotopy class of fibre preserving maps t: $X \times S^{q-1} \to X \times S^{q-1}$ for q large. Since the degree is only defined up to sign, the result is a fibration [1, 4.3]:

$$(1.3)\qquad Q_{\pm r}(S^0) \to (QS^0/O)_r \to BO$$

for each $r \geqq 0$. For $r=1$ we identify $Q_{\pm 1}(S^0) = G$, the H-space of stable homotopy classes of homotopy equivalences of S^{q-1} as $q \to \infty$ and $(QS^0/O)_1 = G/O$.

Let $[r]$ denote the element of $[X, Q_r S^0]$ represented by $id \times r$ on $X \times S^{q-1}$ where r: $S^{q-1} \to S^{q-1}$ is a degree r map. Then by composing with j: $Q_r S^0 \to (QS^0/O)_r$ and taking Whitney sum we define

$$(1.4)\qquad \delta_r: G/O \to (QS^0/O)_r.$$

(1.5) **Proposition.** *The map δ_r is a $\mathbb{Z}[1/r]$-homotopy equivalence.* □

This result can be applied to the situation where $\pi = \pi_1(X)$ is finite since $(QS^0/O)_r$ is an infinite loop space when r is inverted. Let $X(\sigma)$ denote the covering space of X with fundamental group $\sigma \subseteq \pi$.

(1.6) **Corollary.** *Let r be odd and suppose $\pi = \pi_1(X)$ is finite. Then the restriction map*

$$\mathrm{Res}_\pi^\sigma: [X, (QS^0/O)_r] \otimes \mathbb{Z}_{(2)} \to [X(\sigma), (QS^0/O)_r] \otimes \mathbb{Z}_{(2)}$$

is injective for σ a 2-Sylow subgroup of π. □

Finally, we recall the Adams map (cf. [1, §6])

(1.7) $$\alpha_r\colon BO(2)\to(QS^0/O)_r$$

defined on $[X,BO(2)]$ for r odd by:

$$\alpha_r(\xi)=[\xi,t_r,\psi^r\xi]$$

where ξ is a 2-plane bundle over X. Let $\mu_r\colon O(2)\to O(2)$ be defined by $\mu_r(z)=z^r$ for $z\in S^1=SO(2)$ and $\mu_r(c)=c$ for $c\in O(2)$ denoting "complex conjugation.. Then $\psi^r\xi$ is classified by

$$X\xrightarrow{\ \xi\ }BO(2)\xrightarrow{\ B\mu_r\ }BO(2)$$

and

$$t_r\colon S(\xi)\to S(\psi^r\xi)$$

is given by $t_r(x,z)=(x,z^r)$.

We remark that there is evidently a version of this theory for topological bundles and fibre-preserving maps. This leads to classifying spaces $(QS^0/TOP)_r$ and a map $(QS^0/O)_r\to(QS^0/TOP)_r$.

2. Degree r Normal Maps

In surgery theory the space $G/O=(QS^0/O)_1$ is used to describe the set of degree 1 normal maps which target a Poincaré duality space X of formal dimension n. More generally, for each integer $r\geqq 0$ let $\mathcal{N}_r(X)$ denote the set of normal cobordism classes of degree r normal maps

$$\begin{array}{ccc} \tau_N & \xrightarrow{\ \hat{p}_r\ } & \xi \\ \downarrow & & \downarrow \\ N^n & \xrightarrow[\ p_r\ ]{} & X \end{array}$$

where N^n is a smooth closed manifold of dimension n, τ_N is its stable tangent bundle and $\hat{p}_r$ is a vector bundle map covering the degree map p_r. If $r=1$, it follows that the vector bundle $-\xi$ is stable fibre homotopy equivalent to the Spivak normal fibre space ν_X of X. In general, this holds only when r is inverted.

(2.2) **Theorem.** *Let X be a finitely-dominated Poincaré complex and ν be a vector bundle-reduction of ν_X. Then for any $r\geqq 0$ there is a bijection*

$$T_\nu\colon \mathcal{N}_r(X)\to[X,(QS^0/O)_r].$$

Proof. Let W be a regular neighborhood of X^n embedded in $\mathbb{R}^{n+k}$ (for k large). Then there is a diagram

$$\begin{array}{ccccc} \partial W & \longrightarrow & W & \xrightarrow{\ \gamma\ } & X \\ \uparrow & & \uparrow & & \| \\ S(\nu) & \longrightarrow & D(\nu) & \longrightarrow & X \end{array}$$

with $(D(\nu), S(\nu)) \to (W, \partial W)$ a homotopy equivalence. First we will define the map T_ν. We suppose that $(p\colon N \to X,\ \hat{p}\colon \tau_N \to \xi)$ is a degree r normal map and consider the composition:

$$f_0\colon N \xrightarrow{p} X \hookrightarrow W \to \gamma^*(\xi).$$

If $f_1 \simeq f_0$ is an embedding then

$$\nu(f_1) \oplus \tau_N = f_1^*(\tau_{\gamma^*(\xi)}) = f_1^*(\pi^* \gamma^*(\xi) \oplus \pi^* \tau_W)$$

where $\pi\colon \gamma^*(\xi) \to W$ is the bundle projection. It follows that

$$f_1^*(\tau_{\gamma^*(\xi)}) \cong p^*(\xi) \oplus \varepsilon^{n+k} \cong \tau_N \oplus \varepsilon_N^{n+k},$$

so $\nu(f_1)$ is stably trivial. Let

$$c\colon \gamma^*(\xi)^+ \to (\varepsilon_N^{n+k})^+$$

denote the collapse map on Thom spaces and note that there is a homotopy equivalence

$$(\xi \oplus \nu)^+ \to \gamma^*(\xi)^+.$$

The composite with the $(n+k)$-fold suspension of p,

$$t^+\colon (\xi \oplus \nu)^+ \to (\varepsilon_N^{n+k})^+ \to (\varepsilon_X^{n+k})^+ \tag{2.3}$$

is induced from a map

$$t\colon S(\xi \oplus \nu \oplus \varepsilon_X^1) \to S(\varepsilon_X^{n+k+1}).$$

If p has a degree r, then t has degree r on each fibre so is classified by a map $X \to (QS^0/O)_r$. Since this construction respects normal cobordism classes we get $T_\nu\colon \mathcal{N}_r(X) \to [X, (QS^0/O)_r]$.

The inverse map to T_ν is defined by transversality: each element of $[X, (QS^0/O)_r]$ can be represented by a triple $(\zeta, t, \varepsilon_N^{n+k})$ for k large. Then consider the induced map

$$t_0\colon S(\gamma^*(\zeta)) \to S(\varepsilon_W^{n+k})$$

and define $N = t_1^{-1}(X)$ where $t_1 \simeq t_0$ is transverse to $X \hookrightarrow W \hookrightarrow S(\varepsilon_W^{n+k})$. If $\pi^*\colon \gamma^*(\zeta) \to W$ is the projection, then

$$\pi^* \gamma^*(\zeta) \oplus \tau_W|_N \simeq \tau_N \oplus \nu$$

and so (after stabilizing) we get a degree r normal map

$$\begin{array}{ccc} \tau_N & \xrightarrow{\hat{p}_r} & \zeta - \nu \\ \downarrow & & \downarrow \\ N & \xrightarrow[p_r]{} & X \end{array}$$

as required. This construction gives an inverse map for T_ν. $\square$

Remark. When the bijection T_ν exists we will use it to define $\delta_r\colon \mathcal{N}(X)\to\mathcal{N}_r(X)$ relating the degree 1 normal invariants with the degree r normal invariants.

The remaining property of $\mathcal{N}_r(X)$ we need is the existence of a surgery obstruction map (where $\pi=\pi_1(X)$, and $A=\hat{\mathbb{Z}}_2^\times\oplus\pi^{ab}\oplus SK_1(\hat{\mathbb{Z}}_2\pi)$):

$$\lambda_2^A\colon \mathcal{N}_r(X)\to L_n^A(\hat{\mathbb{Z}}_2\pi)$$

whenever X^n is (weakly) simple and r is odd. This means that the finitely-dominated Poincaré space X is equipped with a simple base measured in $Wh'(\hat{\mathbb{Z}}_2\pi)$ for $C_*(\tilde{X})\otimes\hat{\mathbb{Z}}_2$ (cf. [2, §8]). This will be the case for example when X is a finite simple Poincaré space. If $\dim X$ is odd, $L^A=L'$. For any r, the construction of Ranicki [9: 3.8, 4.1] produces a based quadratic structure over $\mathbb{Z}[1/r]\pi$ on the cone complex $C_*(p_r^!)\otimes\mathbb{Z}[1/r]$ and therefore an element $\lambda'(p_r,\hat{p}_r)\in L_n'(\mathbb{Z}[1/r]\pi)$. If r is odd, completion at 2 gives the desired element.

To adapt Ranicki's construction to our setting note that the "geometric Umkehr" map [9, 4.2] needed exists only after inverting r. A degree r normal map determines the map t^+ of (2.3) which is a $\mathbb{Z}[1/r]$-homotopy equivalence. Let

(2.4) $$F\colon \Sigma^{n+k}\tilde{X}_+\to\Sigma^{n+k}\tilde{N}_+$$

be the composition of $(\tilde{t}^+)^{-1}$ *with the* $S\pi$-dual of $T\pi(\hat{p})\colon T\pi(\nu_{\tilde{N}})\to T\pi(-\xi)$.

Now assume that X^n is an odd dimensional finite Poincaré complex with ν_X reducible. In §4 we will need to investigate the diagram:

(2.5) $$\begin{array}{ccc} [X,(QS^0/O)_r] & \xrightarrow{\lambda_2'} & L_n'(\hat{\mathbb{Z}}_2\pi) \\ \uparrow{\scriptstyle\delta_r} & & \uparrow{\scriptstyle i_2'} \\ [X,G/O] & \xrightarrow{\lambda'} & L_n'(\mathbb{Z}\pi) \end{array}$$

where i_2' is induced by completing at 2. We will use the results of [2, §6]: if $\pi=\mathbb{Z}/m\rtimes\sigma$ is a 2-hyperelementary group (m odd, σ a 2-group) the L-groups have a functorial splitting indexed by the divisors of m. The top component of $\lambda_2'(p_r,\hat{p}_r)$ is denoted $\lambda_2'(p_r,\hat{p}_r)(m)$.

(2.6) **Proposition.** *Let X^n be a finite (weakly) simple Poincaré complex with ν_X reducible and $\pi=\pi_1(X)$ a 2-hyperelementary group. If $(f\colon M\to X,\ \hat{f}\colon \tau_M\to\xi)$ is a degree 1 normal map, then*

$$\lambda_2'(\delta_r\cdot(f,\hat{f}))(m)=i_2'\lambda'(f,\hat{f})(m)\in L_3'(\hat{\mathbb{Z}}_2\pi)(m)$$

in the top component

$$L_n'(\hat{\mathbb{Z}}_2\pi)(m)\subseteq L_n'(\hat{\mathbb{Z}}_2\pi).$$

Proof. The element $\delta_r \cdot (f,\hat{f})$ is represented by the product degree r normal map [1, §5]:

$$(M \xrightarrow{f} X) \times (r \cdot pt \to pt).$$

This is just the disjoint union $r \cdot M$ of r copies of M each mapped by f (and covered by $\hat{f}$ on τ_M). This normal map can be written as a composite:

$$r \cdot M \xrightarrow{r \cdot \mathrm{id}} M \xrightarrow{f} X \tag{2.7}$$

satisfying the local version of [9, 4.3]. Therefore

$$\lambda_2^A(f,\hat{f}) + \lambda_2^A(r \cdot M \to M) = \lambda_2^A(\delta_r \cdot (f,\hat{f})).$$

However $\lambda_2^A(r \cdot M \to M)$ arises from the product formula [9, 8.1]:

$$L_0^A(\hat{\mathbb{Z}}_2) \otimes L^n(\mathbb{Z}\pi) \to L_n^A(\hat{\mathbb{Z}}_2 \pi) \tag{2.8}$$

applied to $\lambda^A(r \cdot pt \to pt) \in L_0^A(\hat{\mathbb{Z}}_2)$ and the symmetric signature $\sigma^*(M) \in L^n(\mathbb{Z}\pi)$. By definition of the top component [2, 6.9] and the fact that $\sigma^*(M)$, when localized at 2, is in the image of $L^n(\mathbb{Z}\sigma)$, we obtain that $\lambda^A(r \cdot M \to M)(m) = 0$. In top component $L' = L^A$. □

We conclude this section with another property of the top component of a surgery obstruction.

(2.9) **Proposition.** *Let X^n be a finite (weakly) simple Poincaré space with $\pi = \pi_1(X)$ 2-hyperelementary and ν_X reducible. If $(f_0,\hat{f}_0)$ and $(f_1,\hat{f}_1)$ are degree 1 normal maps to X, then*

$$\lambda'(f_0,\hat{f}_0)(m) = \lambda'(f_1,\hat{f}_1)(m) \in L_2'(\mathbb{Z}\pi)(m)$$

where $m = |\pi|_{\mathrm{odd}}$.

Proof. This is a formal consequence of the Ranicki sequences:

$$\begin{array}{ccccccc}
\dots \to H_n(X;\mathbb{L}_0) & \longrightarrow & L_n'(\mathbb{Z}\pi) & \to & \mathscr{S}_n(X) & \to & H_{n-1}(X;\mathbb{L}_0) \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\dots \to H_n(B\pi;\mathbb{L}_0) & \xrightarrow{\lambda} & L_n'(\mathbb{Z}\pi) & \to & \mathscr{S}_n(B\pi) & \to & H_{n-1}(B\pi;\mathbb{L}_0).
\end{array}$$

The elements $(f_0,\hat{f}_0)$ and $(f_1,\hat{f}_1)$ arise from different reductions of ν_X; according to [10] their surgery obstructions $\lambda'(f_i,\hat{f}_i)$ have the same image, namely the total surgery obstruction, $s(X) \in \mathscr{S}(X)$. From the diagram

$$\lambda'(f_0,\hat{f}_0) - \lambda'(f_1,\hat{f}_1) \in \mathrm{Im}(\lambda\colon H_n(B\pi;\mathbb{L}_0) \to L_n'(\mathbb{Z}\pi))$$

But $\mathrm{Im}\,\lambda$ is detected by restriction to a 2-Sylow subgroup $\sigma \subseteq \pi$ and so its top component is zero. □

Remark. If X is only finitely-dominated, the same conclusion holds in the top component of $L_n^p(\mathbb{Z}\pi)$.

3. Degree r Maps of Swan Complexes

Let π be a group with periodic cohomology. A Swan complex for π in dimension n is a finitely-dominated oriented Poincaré complex X of formal dimension n with $\pi_1 X=\pi$ and $\tilde{X}\simeq S^n$. From [11] the homotopy type of X is determined by its first k-invariant $g(X)\in H^{n+1}(\pi;\mathbb{Z})$. Furthermore, if X_1, X_2 are two Swan complexes in dimension n then for any integer r (prime to $|\pi|$) such that $g(X_2)=rg(X_1)$ there exists a degree r map $p_r\colon X_1\to X_2$ which is unique up to homotopy.

(3.1) **Proposition.** *Suppose that N^n is a smooth oriented manifold of dimension n with $\pi_1 N=\pi$ and $\tilde{N}\simeq S^n$. If X^n is any Swan complex in dimension n and $g(X)=rg(N)$, then there exists a bundle map $\hat{p}_r\colon \tau_N\to\xi$ covering the degree r map $p_r\colon N\to X$.*

Proof. Choose s so that $rs\equiv 1 (\mathrm{mod}\,|\pi|)$ and notice that $g(N)=sg(X)$. Therefore there exists a degree s map $p_s\colon X\to N$ and we set $\xi=p_s^*(\tau_N)$. To obtain $\hat{p}_r$ we note that $p_r^*p_s^*(\tau_N)=(p_s\circ p_r)^*(\tau_N)$. Localized at $|\pi|$, $(p_s\circ p_r)^*(\tau_N)\cong\tau_N$ and localized away from $|\pi|$ both bundles are trivial. □

Since we know that any Swan complex X^n has a vector bundle reduction for ν_X, the degree 1 normal invariant set is non-empty. We wish now to compare the surgery obstruction of a degree 1 normal map $(f_1\colon M\to X, \hat{f}_1\colon \tau_M\to\nu)$ to that of $(p_r\colon N\to X, \hat{p}_r\colon \tau_N\to\xi)$ in the case when π is a type I group and N is the orbit space of a linear action. More precisely, let π be a metacyclic group

(3.2)
$$1\to\mathbb{Z}/n\to\pi\to\mathbb{Z}/2^k\to 1$$

where m is odd and the action map $t\colon \mathbb{Z}/2^k\to(\mathbb{Z}/m)^\times$ has image $\mathbb{Z}/2^l$ for some l, $1\leqq l<k$. Let χ be a faithful character of $\mathbb{Z}/m\cdot 2^{k-l}$ and set

(3.3)
$$V=\chi=\chi^g+\chi^g+\ldots+\chi^{g^{2^l-1}}$$

where g generates $\operatorname{Im} t$. This gives a free representation of π in the period dimension 2^{l+1}. To obtain the situation of the last section, let $N=S(V)/\pi$ and X be another Swan complex in this dimension $n=2^{l+1}-1$ with

(3.4)
$$g(X)=r\cdot g(N)=r\cdot c_{2^l}(V).$$

We now fix a reduction ν of the Spivak normal fibre space ν_X and use it to identify

(3.5)
$$\mathcal{N}_r(X)\cong[X,(QS^0/O)_r] \quad\text{and}\quad \mathcal{N}(X)\cong[X,G/O]$$

by T_ν as in (2.2).

It will be convenient to vary r by a multiple of $|\pi|$ so that r is prime to all the torsion in $\pi_i(Q_1S^0)$, the i^{th} stable stem, for $i\leqq\dim X+1$. Since the homotopy type of the Swan complex X depends only on the residue class of r $(\mathrm{mod}\,|\pi|)$ this choice of r involves no loss of generality for our applications.

Moreover, under this assumption the map

$$\delta_r\colon [X, G/O] \to [X, (QS^0/O)_r] \tag{3.6}$$

is a bijection by (1.5). In fact, since $\dim X$ is odd the rational localizations of both sides are trivial. From the fibration (1.3) and the "loop-sum" identification [1] of $Q_r S^0 \simeq Q_1 S^0$, it follows that the r-localizations of both sides are also trivial.

We now take $r \equiv 1 \pmod 4$ to fix the orientation on X.

(3.7) **Proposition.** *Let $r \equiv 1 \pmod 4$ be prime to $\pi_i(Q_1 S^0)$ for $i \leqq \dim X + 1$. Then there exists a degree 1 normal map $(f, \hat{f})\colon M \to X$ such that*

$$\mathcal{N}(p_r, \hat{p}_r) = \delta_r \cdot \mathcal{N}(f, \hat{f}) \in \mathcal{N}_r(X).$$

Proof. From our identification (3.5), the normal invariant $\mathcal{N}(p_r, \hat{p}_r)$ gives an element in $[X, (QS^0/O)_r]$. The result follows from (3.6). □

This Proposition is all we need for our calculations of surgery obstructions in this paper. However, in some situations it may be necessary to fix the integer r (not only its residue class $\bmod |\pi|$). This can occur for example in studying the possible linking numbers of fixed-point sets in non-free actions (this was pointed out to us by P. Löffler and T. tom Dieck). To control the normal invariant in these situations, one needs more precise information about the restriction of our degree one normal map $(p_r, \hat{p}_r)$ to a 2-Sylow subgroup.

We indicate how such information can be obtained, using Adams' map α_r from (1.7).

Let $\sigma \equiv \mathbb{Z}/2^k$ be a 2-Sylow subgroup of π and θ a suitable faithful character of σ. Then

$$\operatorname{Res}^\sigma_\pi(V) = \sum_{i=0}^{2^l-1} \theta^{i\cdot 2^{k-l}+1}.$$

If W denotes the right-hand side of this expression, let

$$W^{(r)} = \theta^r + \sum_{i=1}^{2^l-1} \theta^{i\cdot 2^{k-l}+1}$$

and define $Y = S(W^{(r)})/\sigma$. Although $W^{(r)}$ does not in general extend to a free π-representation,

$$\operatorname{Res}^\sigma_\pi(r c_{2^l}(V)) = r c_{2^l}(W) = c_{2^l}(W^{(r)})$$

so there is an orientation-preserving homotopy equivalence

$$g\colon Y \to X(\sigma)$$

where $X(\sigma)$ denotes the σ-covering space of X. Let $\hat{g}\colon \tau_Y \to (g^{-1})^* \tau_Y$ be a bundle map covering g.

(3.8) **Theorem.** *Let $h\colon X(\sigma) \to Y$ be a homotopy inverse for g and let $[\psi_r] \in \mathcal{N}_r(Y)$ be represented by $\alpha_r(S(W^{(r)}) \times_\sigma \theta) \in [Y, (QS^0/O)_r]$. Then*

$$\mathcal{N}((h, \hat{h}) \circ \operatorname{Res}^\sigma_\pi(p_r, \hat{p}_r)) = [\psi_r].$$

Proof. Consider the triple

$$(3.9)\qquad \begin{array}{ccc} S(\theta^r)\times_\sigma \theta & \xrightarrow{t_r} & S(\theta^r)\times_\sigma \theta^r \\ & \searrow \quad \swarrow & \\ & S(\theta^r)/\sigma & \end{array}$$

as in (1.7) where $t_r(w,z)=(w,z^r)$. Deform t_r to a map transversal to the zero section by

$$H_s(w,z)=(w,z^r-sw)\qquad 0\leqq s\leqq 1.$$

Then H_1 is transverse and

$$H_1^{-1}(w,0)=\{(w,z)\,|\,z^r=w\}=\{(z^r,z)\,|\,z\in S(\theta)\}$$

$$(3.10)\qquad \begin{array}{ccc} S(\theta)\times_\sigma \theta & \xrightarrow{\hat{q}_r^0} & S(\theta^r)\times_\sigma \theta \\ \downarrow & & \downarrow \\ S(\theta)/\sigma & \xrightarrow{q_r^0} & S(\theta^r)/\sigma. \end{array}$$

Now let $W=\theta\oplus W_0$ and take the join of (3.9) with $S(W_0)/\sigma$. The result is

$$\begin{array}{ccc} S(W^{(r)})\times_\sigma W & \xrightarrow{t_r\oplus \mathrm{id}} & S(W^{(r)})\times_\sigma W^{(r)} \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

The map $H\oplus\mathrm{id}$ is a homotopy of $t_r\oplus\mathrm{id}$ to a transverse map and the resulting normal map is the join of (3.10) with $S(W_0)/\sigma$, namely:

$$(3.11)\qquad \begin{array}{ccc} \mathrm{Res}(\tau_N) & \xrightarrow{\hat{q}_r} & \zeta \\ \downarrow & & \downarrow \\ N(\sigma) & \xrightarrow{q_r} & Y \end{array}$$

where $\zeta=S(W^{(r)})\times_\sigma W$. By construction (check first for q_s^0) $q_s^*\mathrm{Res}(\tau_N)\cong\zeta$ where $q_s\colon Y\to N$ is the degree s map for $rs\equiv 1(\mathrm{mod}\,|\pi|)$. On the other hand, the normal map $(h,\hat{h})\circ\mathrm{Res}(p_r,\hat{p}_r)$ is represented by the diagram

$$(3.12)\qquad \begin{array}{ccccc} \mathrm{Res}(\tau_N) & \xrightarrow{\mathrm{Res}\,\hat{p}_r} & \mathrm{Res}(\xi) & \xrightarrow{\hat{h}} & g^*\mathrm{Res}(\xi) \\ \downarrow & & \downarrow & & \downarrow \\ N(\sigma) & \xrightarrow[\mathrm{Res}\,p_r]{} & X(\sigma) & \xrightarrow[h]{} & Y \end{array}$$

where $\xi=p_s^*\tau_N$. Therefore

$$g^*\mathrm{Res}(\xi)=g^*(\mathrm{Res}\,p_s)^*\mathrm{Res}(\tau_N)=(\mathrm{Res}\,p_s\circ g)^*\mathrm{Res}(\tau_N)\cong\zeta$$

since $\mathrm{Res}\, p_s \circ g \simeq q_s$ by uniqueness of these degree s maps. Similarly $h \circ \mathrm{Res}\, p_r \simeq q_r$ so (3.12) has the same normal invariant as (3.11). □

We can now combine (3.7) with (2.6), (2.9) to obtain the desired relation between the surgery obstruction of $(f, \hat{f})$ and $(\hat{p}_r, \hat{p}_r)$ evaluated in $L'_3(\hat{\mathbb{Z}}_2\pi)$.

(3.13) **Proposition.** *Let* $(f, \hat{f})$: $M \to X$ *be a degree* 1 *normal map and* $(p_r, \hat{p}_r)$: $N \to X$ *be the degree r normal map in* (3.1). *Assume that* X *is a finite weakly simple complex and* $r \equiv 1 \pmod 4$. *Then the image of* $\lambda'(f, \hat{f}) \in L'_3(\mathbb{Z}\pi)$ *in the top component is*

$$\lambda'_2(f, \hat{f})(m) = \lambda'_2(p_r, \hat{p}_r)(m) \in L'_3(\hat{\mathbb{Z}}_2\pi)(m).$$

Proof. Since $n = 2^{l+1} - 1 \equiv 3 \pmod 4$ we have surgery obstructions in $L'_3(\hat{\mathbb{Z}}_2\pi)$. Also $L'_3(\hat{\mathbb{Z}}_2\pi)$ is 2-local and so

$$\lambda'_2(p_r, \hat{p}_r) = \lambda'_2(\delta_r \cdot (f, \hat{f}))$$

by (3.7). The rest follows immediately from (2.6) and (2.9). □

(3.14) **Corollary.** *If* X *is finite and* $r \equiv 1(4)$ *then*

$$\lambda'_2(f, \hat{f})(m) = \Delta(N)(m) - \Delta(X)(m) \in \mathrm{Im}(H^0(Wh'(\hat{\mathbb{Z}}_2\pi)) \to L'_3(\hat{\mathbb{Z}}_2\pi)).$$

Proof. Since p_r is a 2-local equivalence this is just [2, 8.6]. Recall that $\Delta(N)$, $\Delta(X)$ are the Reidemeister torsions of these complexes. □

(3.15) *Remark.* When $L^h_3(\mathbb{Z}\pi) \to L_3(\hat{\mathbb{Z}}_2\pi)$ is injective (e.g. for m prime) where

$$U = \mathrm{Im}(Wh'(\mathbb{Z}\pi) \to Wh'(\hat{\mathbb{Z}}_2\pi))$$

we have determined also the image $\lambda^h(f, \hat{f})(m)$ of $\lambda'(f, \hat{f})(m)$ in $L^h_3(\mathbb{Z}\pi)(m)$.

When X is not finite but the cohomology class $\{\sigma(X)\}$ of its finiteness obstruction is trivial in $H^0(\tilde{K}_0(\mathbb{Z}\pi))$, we have defined an invariant $\Delta_0(X)$ in [2, 8.12] by considering the cohomology long exact sequence induced by

$$0 \to Wh(\mathbb{Q}\pi)/Wh'(\mathbb{Z}\pi) \to Wh(\hat{\mathbb{Q}}\pi)/Wh'(\hat{\mathbb{Z}}\pi) \to \tilde{K}_0(\mathbb{Z}\pi) \to 0.$$

Using this invariant and the δ-invariant of [2, 4.3, 4.22] we can give a formula for $\lambda^p(f, \hat{f})(m)$. Recall that $L^p_3(\mathbb{Z}\pi)$ is determined by the semicharacteristic, the cohomology finiteness obstruction and this δ-invariant which takes values in a quotient of $H^0(Wh(\hat{\mathbb{Q}}_2\pi)_+)$. Here + denotes the part of $Wh(\hat{\mathbb{Q}}_2\pi)$ given by the type O representations. Since we have assumed Milnor's condition ($\ker t \neq 1$) the semicharacteristic vanishes.

(3.16) **Proposition.** *Let* X *be finitely-dominated and* $\{\sigma(X)\} = 0 \in H^0(\tilde{K}_0(\mathbb{Z}\pi))$. *Let* $(f, \hat{f})$ *and* $(p_r, \hat{p}_r)$ *be the normal maps used in* (3.13) *and* $r \equiv 1 \pmod 4$. *Then*

$$\delta(\lambda^p(f, \hat{f})(m)) = \Delta(N) - \Delta_0(X) \in H^0(Wh(\hat{\mathbb{Q}}_2\pi)_+)/I$$

where

$$I = L^K_0(\hat{\mathbb{Z}}_2\pi) + H^0(Wh'(\mathbb{Z}\pi)) + d^* H^1(\tilde{K}_0(\mathbb{Z}\pi))$$

and

$$d^*\colon H^1(\tilde{K}_0(\mathbb{Z}\pi)) \to H^0(Wh(\mathbb{Q}\pi)/Wh'(\mathbb{Z}\pi)) \to H^0(Wh(\hat{\mathbb{Q}}_2\pi)_+)/H^0(Wh'(\mathbb{Z}\pi)).$$

Proof. The argument is contained in [2, 8.17–8.23] so we include only an outline. Replace $(f,\hat{f})$, $(p_r,\hat{p}_r)$ by suitable normal maps $(\psi,\hat{\psi})$: $\bar{M}\to\bar{X}$ and $(q_r,\hat{q}_r)$: $\bar{N}\to\bar{X}$ where $\bar{X}$ is finite. The construction of these normal maps (cf. [2, 8.18]) ensures that (3.13) still holds and so we get

(3.17) $$\lambda_2'(\psi,\hat{\psi})(m)=\Delta(\bar{N})-\Delta(\bar{X})\in\operatorname{Im}(H^0(Wh'(\hat{\mathbb{Z}}_2\pi))\to L_3'(\hat{\mathbb{Z}}_2\pi)).$$

Now the argument for [2, 8.21–8.23] proves that the image of (3.17) in the group $H^0(Wh(\hat{\mathbb{Q}}_2\pi)_+)/I$ is the formula required. □

4. Reidemeister Torsions

The results of the previous section expressed the surgery obstructions of a degree one normal map $(f,\hat{f})$: $M\to X$ over a Swan complex X in terms of Reidemeister torsion invariants. We now attempt to calculate these invariants.

To each Swan complex X with k-invariant $g(X)\in H^{n+1}(\pi;\mathbb{Z})$ there corresponds an extension with finitely generated projective $\mathbb{Z}\pi$-modules

$$0\to\mathbb{Z}\xrightarrow{\eta}P_n\to\ldots\to P_0\xrightarrow{\varepsilon}\mathbb{Z}\to 0$$

and an invariant $\hat{\Delta}(g(X))=\prod\{\hat{\Delta}_p(g(X))\colon p\,\big|\,|\pi|\}$, where

$$\hat{\Delta}_p(g(X))\in K_1(\hat{\mathbb{Q}}_p\pi)/K_1(\hat{\mathbb{Z}}_p\pi),$$

defined in [4] or [13]. Often we write $\hat{\Delta}$(X) instead of $\hat{\Delta}$(g(X)) and consider it as an element of $K_1(\hat{\mathbb{Q}}\pi)/K_1(\hat{\mathbb{Z}}\pi)$ by giving it the value zero at primes which do not divide $|\pi|$. In the exact sequence

(4.1) $$K_1(\mathbb{Q}\pi)\to K_1(\hat{\mathbb{Q}}\pi)/K_1(\hat{\mathbb{Z}}\pi)\xrightarrow{\partial}\tilde{K}_0(\mathbb{Z}\pi)\to 0$$

$\partial(\hat{\Delta}(X))$ is the obstruction to choosing X finite (given $g(X)$) and if $\partial(\hat{\Delta}(X))=0$ then any preimage $\Delta(X)\in K_1(\mathbb{Q}\pi)$ is the Reidemeister torsion for a choice of (simple) homotopy type for X.

Suppose N is a manifold representing a generator $g(N)\in H^{n+1}(\pi;\mathbb{Z})$, and suppose further that

$$g(X)=r\cdot g(N)\in H^{n+1}(\pi;\mathbb{Z})$$

where r is prime to the group order, and $r\equiv 1(mod\,4)$. Then $\partial(\hat{\Delta}(X)-\Delta(N))$ is the Swan image of r in $\tilde{K}_0(\mathbb{Z}\pi)$, represented by the projective ideal (r,Σ), $\Sigma=\Sigma\{g\,|\,g\in\pi\}$.

Let $e=1/|\pi|\Sigma$ be the usual idempotent in $\mathbb{Q}\pi$. Then $r\cdot e+(1-e)\in\mathbb{Q}\pi$ is a unit and defines an element in $K_1(\mathbb{Q}\pi)$, whence in $K_1(\hat{\mathbb{Q}}_p\pi)/K_1(\hat{\mathbb{Z}}_p\pi)$ for each p.

(4.2) **Lemma.** *With X, N as above*

$$\begin{aligned}\hat{\Delta}_p(X)-\hat{\Delta}_p(N)&=[r\cdot e+(1-e)] &&\text{for } r\,\big|\,|\pi|\\ &=0 &&\text{for } r\nmid|\pi|.\end{aligned}$$

Proof. We must compare the multiple extensions associated with k-invariants $g(X)$ and $g(N)$. Let C_* be the based complex representing $g(N)$. The construction of [11, §2] shows that one may choose a complex P_* realizing $g(X)$ with $C_i = P_i$ for $i<n$. Moreover there is a chain map

$$\begin{array}{ccccccccccccc} 0 \to & \mathbb{Z} & \to & C_n & \to & C_{n-1} & \to \dots \to & C_0 & \to & \mathbb{Z} & \to 0 \\ & \downarrow \gamma & & \downarrow \phi & & \| & & \| & & \downarrow & \\ 0 \to & \mathbb{Z} & \to & P_n & \to & P_{n-1} & \to \dots \to & P_0 & \to & \mathbb{Z} & \to 0. \end{array}$$

For each prime p which divides the group order $\phi\colon C_n \otimes \hat{\mathbb{Z}}_p \to P_n \otimes \hat{\mathbb{Z}}_p$ is an isomorphism, so the base for C_n defines a base for $P_n \otimes \hat{\mathbb{Z}}_p$. The required formula now follows directly from the definitions. □

It is convenient for explicit calculations to use Fröhlich's description of (4.1) in terms of character homomorphisms. We recall the setting. Let $\bar{\mathbb{Q}} \supset \mathbb{Q}$ be any sufficiently large number field (containing the $|\pi|$'th roots of 1) and let Ω be the Galois group for $\bar{\mathbb{Q}}/\mathbb{Q}$. Let $R\pi$ denote the complex representation ring of π. Then (4.1) is isomorphic to the sequence

$$\text{(4.3)} \qquad \operatorname{Hom}_\Omega^+(R\pi, \bar{\mathbb{Q}}^\times) \xrightarrow{\Phi} \frac{\operatorname{Hom}_\Omega(R\pi, J(\bar{\mathbb{Q}}))}{\operatorname{Image} K_1(\hat{\mathbb{Z}}\pi)} \xrightarrow{\partial} \tilde{K}_0(\mathbb{Z}\pi) \to 0.$$

Here $J(\bar{\mathbb{Q}})$ denotes the group of finite ideles and $\operatorname{Hom}_\Omega^+(,)$ denotes the subset of all character homomorphism which map symplectic characters into totally positive elements.

Below we shall need the explicit isomorphism between (4.1) and (4.3), so we recall it (see [2, §6]). Given a representation $\rho\colon \pi \to Gl_n(\bar{\mathbb{Q}})$ there is an induced representation of $\mathbb{Q}\pi$ in $M_n(\bar{\mathbb{Q}})$, whence a homomorphism

$$K_1(\mathbb{Q}\pi) \to K_1(M_n(\bar{\mathbb{Q}})) \cong \bar{\mathbb{Q}}^\times.$$

This defines a monomorphism

$$\text{(4.4)} \qquad \theta\colon K_1(\mathbb{Q}\pi) \to \operatorname{Hom}_\Omega(R\pi, \bar{\mathbb{Q}}^\times)$$

which maps isomorphically onto $\operatorname{Hom}_\Omega^+(R\pi, \bar{\mathbb{Q}}^\times)$. If $\mathbb{Q}$ is replaced with its finite completion $\hat{\mathbb{Q}}_p$, θ becomes an isomorphism. With these facts it is clear that (4.1) and (4.3) are isomorphic sequences.

We are only interested in computing the 2-local part of the finiteness obstruction and of the Reidemeister torsion. The 2-local K-theories have induction and restriction homomorphisms. In technical terms, they are Mackey functors over the 2-local Burnside ring $\Omega(\pi)_{(2)} = \Omega(\pi) \otimes \mathbb{Z}_{(2)}$. Now, $\Omega(\pi)_{(2)}$ decomposes into a product of rings corresponding to the minimal 2-local primes of $\Omega(\pi)$. Hence any functor over $\Omega(\pi)_{(2)}$ (e.g. the K-theories) decomposes accordingly.

For 2-hyperelementary groups,

$$1 \to \mathbb{Z}/m \to \pi \to \sigma \to 1,$$

with m odd and σ a 2-group, the above 2-local decomposition of, say $K_1(\mathbb{Q}\pi)_{(2)}$ becomes a decomposition indexed by the divisors of m,

$$K_1(\mathbb{Q}\pi)_{(2)} \cong \sum_{d|m}^{\oplus} K_1(\mathbb{Q}\pi)(d).$$

Moreover, $K_1(\mathbb{Q}\pi)(d) \cong K_1(\mathbb{Q}\pi_d)(d)$ where π_d is the subgroup of π generated by $\mathbb{Z}/d$ and σ. Hence it suffices to consider the top component corresponding to the divisor m.

Let $\{p_1, \dots, p_t\}$ be the prime divisors of m and let $m_i = m/p_i$. The idempotent $\tilde{E}$ which corresponds to the top component $K_1(\mathbb{Q}\pi)(m)$ is given by

$$\tilde{E} = \prod_{i=1}^{t} (1 - \tilde{E}_i), \quad (\tilde{E}_i)^{p_i} = \mathrm{Ind}_i \circ \mathrm{Res}_i \tag{4.5}$$

where $\pi_i = Z/m_i \rtimes \sigma$, $(\tilde{E})^{p_i} = \tilde{E}_i \circ \dots \circ \tilde{E}_i$ (p_i times) and

$$K_1(\mathbb{Q}\pi)_{(2)} \underset{\mathrm{Ind}_i}{\overset{\mathrm{Res}_i}{\rightleftarrows}} K_1(\mathbb{Q}\pi_i)_{(2)}.$$

The monomorphism θ in (4.4) is natural with respect to both induction and restriction; in view of (4.2) we must evaluate $\tilde{E}\theta(re + (1-e))$. First we specify necessary notations:

$$\begin{aligned} & m = p_1^{l_1} \dots p_t^{l_t}; \quad m_i = m/p_i \\ & \pi = \mathbb{Z}/m \rtimes \sigma; \quad t\colon \sigma \to \mathrm{Aut}(\mathbb{Z}/m),\ \kappa = \mathrm{Ker}(t) \\ & \bar{\chi} \text{ is a faithful character of } \mathbb{Z}/m \\ & \{\bar{\chi}_0, \dots, \bar{\chi}_s\} = \mathrm{irr}_C(\kappa)/\Omega, \quad \bar{\chi}_0 = \text{trivial characters of } \kappa. \\ & \chi_j = \mathrm{Ind}_\tau^\pi(\bar{\chi} \otimes \bar{\chi}_j), \quad \tau = \mathbb{Z}/m \times \kappa. \end{aligned} \tag{4.6}$$

Let $R_m\pi \subseteq R\pi$ be the Ω-submodule of the complex representation ring generated by $\{\chi_j\}$. This inclusion defines an isomorphism of the top component:

$$\tilde{E}\,\mathrm{Hom}_\Omega(R\pi, \bar{\mathbb{Q}}^\times) \xrightarrow{\cong} \mathrm{Hom}_\Omega(R_m\pi, \bar{\mathbb{Q}}^\times)$$

and we have

(4.7) **Lemma.** *In* $\mathrm{Hom}_\Omega(R_m\pi, \bar{\mathbb{Q}}^\times)$,

$$\begin{aligned} & \tilde{E}(\theta(\mathrm{re} + (1-e)))(\chi_0) = r^{(-1)^t m}, && m \text{ square-free} \\ & \tilde{E}(\theta(\mathrm{re} + (1-e)))(\chi_j) = 1, && \text{otherwise.} \end{aligned}$$

Proof. It follows from the definition (4.4) that for the central idempotent $e \in \mathbb{Q}\pi$, $\theta(e)(\rho) = 1$, unless $\rho\colon \pi \to \mathrm{Gl}_n(\bar{\mathbb{Q}})$ is the trivial representation with $\rho(g) = 1$ for all g. Hence

$$\tilde{E}\theta(\mathrm{re} + (1-\mathrm{e}))(\chi_j) = \theta(\mathrm{re} + (1-e))(\tilde{E}^*(\chi_j)) = 1$$

for $j>0$. For $j=0$ we get by (4.5) that

$$\tilde{E}\theta(\mathrm{re}+(1-e))(\chi_0)^{p_1\dots p_t}=\prod_{i=1}^{t}(\tilde{E}_i^{p_i})\theta(\mathrm{re}+(1-e))^{(-1)^t}$$
$$=\theta(\mathrm{re}+(1-e))\left(\prod_{i=1}^{t}\mathrm{Ind}_i\mathrm{Res}_i(\chi_0)\right)^{(-1)^t}.$$

It is easy to verify that

$$\prod_{i=1}^{t}\mathrm{Ind}_i\mathrm{Res}_i(\chi_0)=\mathrm{Ind}_{\bar{\pi}}\circ\mathrm{Res}_{\bar{\pi}}(\chi_0)$$

where $\bar{\pi}=\mathbb{Z}/\bar{m}\rtimes\sigma$, $\bar{m}=m/p_1\dots p_t$. Since the multiplicity of the trivial character in the right-hand side is 1 if $\bar{m}=1$ and 0 otherwise the lemma follows. □

It is inconvenient (and not necessary for us) to work with the top component of (4.3) directly. Instead we consider the subsequence

$$(4.8)\qquad \mathrm{Hom}_\Omega^+(R_m\pi,\mathcal{O}^\times)_{(2)}\to\prod_{p\,|\,|\pi|}\frac{\mathrm{Hom}_\Omega(R_m\pi,\hat{\mathcal{O}}_p^\times)_{(2)}}{\mathrm{Image}(K_1(\hat{\mathbb{Z}}_p\pi)_{(2)})}\xrightarrow{\partial}D(\mathbb{Z}\pi)(m)_{(2)}\to 0$$

where $\mathcal{O}$ denotes the ring of integers of $\bar{\mathbb{Q}}$ and $D(\mathbb{Z}\pi)\subseteq\tilde{K}_0(\mathbb{Z}\pi)$ is the subgroup of elements which vanish in $\tilde{K}_0$ of the maximal order. We have

$$\mathrm{Hom}_\Omega(R_m\pi,\hat{\mathcal{O}}_p^\times)=\prod_{j=0}^{s}(\hat{\mathbb{Z}}_p\otimes A_j)^\times$$

$$\mathrm{Hom}_\Omega(R_m\pi,\mathcal{O}^\times)=A_0^\times\times A_1^*\times\prod_{j\geq 2}^{s}A_j^\times$$

where A_j denotes the ring of integers in the character field $\mathbb{Q}(\chi_j)$. Thus the only problem is to calculate Image $(K_1(\hat{\mathbb{Z}}_p\pi))$.

Each of the representations χ_j of $\mathbb{Q}\pi$ factors over $\mathbb{Q}(\zeta_m)^t\sigma=\mathbb{Q}(\bar{\chi})^t\sigma$:

$$\chi_j\colon\ \mathbb{Q}\pi=\mathbb{Q}[\mathbb{Z}/m]^t\sigma\longrightarrow\mathbb{Q}(\zeta_m)^t\sigma\xrightarrow{\chi_j^0}M_n(\mathbb{Q}(\chi_j))$$

and we must determine the image of the induced homomorphism

$$(4.9)\qquad \begin{array}{ccc} K_1(\hat{\mathbb{Z}}_p\pi)(m) & \xrightarrow{(\chi_j)_*} & (\hat{\mathbb{Q}}_p\otimes\mathbb{Q}(\chi_j))_{(2)}^\times \\ & \searrow\quad\nearrow & \\ & K_1(\hat{\mathbb{Z}}_p\otimes\mathbb{Z}[\zeta_m]^t\sigma) & \end{array}$$

For a p-adic field L, U_L denotes the units (of the ring of integers) and $U_L^1\subseteq U_L$ the units congruent to 1 modulo the uniformising parameter.

(4.10) **Proposition.** *For $p|m$,*

$$(\chi_j)_*(K_1(\hat{\mathbb{Z}}_p\pi)(m))=\Pi(U_L^1)_{(2)}$$

where L runs over the p-adic completions of $\mathbb{Q}(\chi_j)$.

Proof. Let $m=p^a m_1$ with m_1 prime to p. If T is a generator of $\mathbb{Z}/m$, let $J\subseteq\hat{\mathbb{Z}}_p\pi$ be the two-sided ideal generated by $(1-T^{m_1})$. This is contained in the radical of $\hat{\mathbb{Z}}_p\pi$, so there is an exact sequence

$$(1+J)^\times \longrightarrow K_1(\hat{\mathbb{Z}}_p\pi)\xrightarrow{q_*} K_1(\hat{\mathbb{Z}}_p\pi_1)\longrightarrow 0$$

where $\pi_1=Z/m_1\rtimes\sigma$. The composition

$$K_1(\hat{\mathbb{Z}}_p\pi_1)\xrightarrow{\mathrm{Ind}} K_1(\hat{\mathbb{Z}}_p\pi)\xrightarrow{q_*} K_1(\hat{\mathbb{Z}}_p\pi_1)$$

is an isomorphism, so the top component $K_1(\hat{\mathbb{Z}}_p\pi)(m)\cong\tilde{E}K_1(\hat{\mathbb{Z}}_p\pi)_{(2)}$ is contained in the kernel of q_*. In fact, the image of $(\chi_j)_*$ in (4.9) is equal to the image of

$$1+(1-\zeta_{p^a})(\hat{\mathbb{Z}}_p\otimes\mathbb{Z}[\zeta_m]^t\sigma)\to K_1(\hat{\mathbb{Q}}_p\otimes\mathbb{Q}(\zeta_m)^t\sigma)\xrightarrow{(\chi_j^0)_*}(\hat{\mathbb{Q}}_p\otimes\mathbb{Q}(\chi_j))^\times$$

Since σ is a 2-group $\hat{\mathbb{Z}}_p\otimes\mathbb{Z}[\zeta_m]^t\sigma$ is a maximal order in $\hat{\mathbb{Q}}_p\otimes\mathbb{Q}(\zeta_m)^t\sigma$, so if $(\chi_j^0)_*$ maps $K_1(\hat{\mathbb{Q}}_p\otimes\mathbb{Q}(\zeta_m)^t\sigma)$ into $\prod L^\times$ then it maps $K_1(\hat{\mathbb{Z}}_p\otimes\mathbb{Z}[\zeta_m]^t\sigma)$ into $\prod U_L$. On the other hand, $(\mathbb{F}_p\otimes\mathbb{Z}[\zeta_m])^t\sigma$ is semi-simple and is mapped by χ_j into the product of the units of the residue fields. The exact sequence

$$1+(1-\zeta_{p^a})(\hat{\mathbb{Z}}_p\otimes\mathbb{Z}[\zeta_m]^t\sigma)\to K_1(\hat{\mathbb{Z}}_p\otimes\mathbb{Z}[\zeta_m]^t\sigma)\to K_1(\mathbb{F}_p\otimes\mathbb{Z}[\zeta_m]^t\sigma)$$

completes the argument since U_L^1 is the kernel of $U_L\to\bar{L}^\times$. □

We have left to consider the difficult case:

$$K_1(\hat{\mathbb{Z}}_2\pi)(m)\cong K_1(\hat{\mathbb{Z}}_2\otimes\mathbb{Z}[\zeta_m]^t\sigma)_{(2)}.$$

A general calculation of this group (and of its image in $\mathrm{Hom}_\Omega(R_m\pi,\hat{\mathcal{O}}_2^\times)$) seems to be out of reach at present. Even when the Sylow 2-subgroup σ is cyclic we are not able to get complete information, so we content ourselves with some general remarks which give an easy calculation in some special cases.

If the kernel $\kappa=\mathrm{Ker}\{t\colon\sigma\to\mathbb{Z}/m^\times\}$ is cyclic of order 2^s then we have a Morita equivalence

$$\hat{\mathbb{Z}}_2\otimes\mathbb{Z}[\zeta_m]^t\sigma\cong M_n(\hat{\mathbb{Z}}_2\otimes A[\kappa])$$

with $A=\mathbb{Z}[\zeta_m]^\sigma$. Thus

$$K_1(\hat{\mathbb{Z}}_2\pi)(m)\cong K_1(\hat{\mathbb{Z}}_2\otimes A[\kappa])_{(2)}\cong(\hat{\mathbb{Z}}_2\otimes A[\kappa])^\times_{(2)}.$$

and we must calculate the cokernel of

(4.11) $$(\hat{\mathbb{Z}}_2\otimes A[\kappa])^\times_{(2)}\to\prod_{j=0}^{s}(\hat{\mathbb{Z}}_2\otimes A_j)^\times_{(2)}$$

where as above A_j denotes the ring of integers in $\mathbb{Q}(\chi_j)=\mathbb{Q}(\zeta_m)^\sigma[\zeta_{2^j}]$.

(4.12) **Lemma.** *The cokernel in* (4.11) *is trivial if* $\kappa\subseteq\mathbb{Z}/2$ *and in general has exponent* 2^{s-1}.

Proof. For $\kappa=\mathbb{Z}/2$, look at the diagram

$$\begin{array}{ccccccc} 1\to 1+(1-T)(\hat{\mathbb{Z}}_2\otimes A[\mathbb{Z}/2])_{(2)} & \longrightarrow & (\hat{\mathbb{Z}}_2\times A[\mathbb{Z}/2])^{\times}_{(2)} & \longrightarrow & (\hat{\mathbb{Z}}_2\otimes A)^{\times}_{(2)}\to 1 \\ \downarrow & & \downarrow & & \| \\ 1\to \quad (\hat{\mathbb{Z}}_2\otimes A)^{\times}_{(2)} & \longrightarrow & 2\cdot(\hat{\mathbb{Z}}_2\otimes A)^{\times}_{(2)} & \longrightarrow & (\hat{\mathbb{Z}}_2\otimes A)^{\times}_{(2)}\to 1 \end{array}$$

In general by induction it is enough to look at

$$\operatorname{cok}\{1+(1-T^{2^{s-1}})(\hat{\mathbb{Z}}_2\otimes A[\mathbb{Z}/_{2^s}])^{\times}\to(\hat{\mathbb{Z}}_2\otimes A[\zeta_{2^s}])^{\times}\}_{(2)}.$$

For each prime $\mathfrak{p}|2$ in A, we obtain a local ring $R_{\mathfrak{p}}$ with maximal ideal $\mathfrak{p}$ such that $\mathfrak{p}^{2^{s-1}}=(2)$ which contributes a factor $(R_{\mathfrak{p}}/(2))^{\times}$ to the cokernel. Since $(R_{\mathfrak{p}}/\mathfrak{p})^{\times}$ is of odd order and

$$(1+\mathfrak{p}^{2^i})^{\times}/(1+\mathfrak{p}^{2^{i+1}})^{\times}$$

has exponent 2 the result follows. □

We return to the geometric situation of Lemma 4.2: X and N are Swan complexes for the group π in (4.6), N is a manifold and the k-invariants are related by the equation $g(X)=rg(N)$. Following the notation in (4.6) we let $A=\mathbb{Z}[\zeta_m]^{\sigma}$ be the ring of integers in $\mathbb{Q}(\chi_0)$ and consider the reduction homomorphism

$$\Phi\colon (A^{\times})_{(2)}\to(A/m)^{\times}_{(2)}\supseteq(\mathbb{Z}/m)^{\times}_{(2)}.$$

We can collect our calculations from (4.2), (4.7), (4.10) and (4.12) in

(4.13) **Theorem.** *Suppose m is square-free.*

(i) *A necessary condition for* $\partial\hat{\Delta}(X)(m)=0$ *in* $\tilde{K}_0(\mathbb{Z}\pi)(m)$ *is that* $r\in\operatorname{Image}(\Phi)$.

(ii) *If the kernel* $\kappa=\mathbb{Z}/2$ *then the condition in* (i) *is also sufficient, and when it is satisfied,*

$$\hat{\Delta}(X)(m)-\Delta(N)(m)\in\operatorname{Hom}_{\Omega}(R_m\pi,\mathcal{O}^{\times})_{(2)}$$

maps χ_0 *into* $\Phi^{-1}(r)$ *and* χ_j *into* 1 *for* $j>0$. *If m is not square-free then* $\hat{\Delta}(X)(m)=\Delta(N)(m)$.

The reader should notice in connection with the above theorem that the finiteness obstruction $\partial\hat{\Delta}(X)\in\tilde{K}_0(\mathbb{Z}\pi)$ is in fact 2-primary, so there is no harm in only considering the 2-local situation. This follows from induction since $\tilde{K}_0(\mathbb{Z}\pi)_{(p)}$ is detected by p-hyperelementary subgroups (i.e. cyclic subgroups for $p\neq 2$) and each k-invariant on a cyclic subgroup is realized by a suitable lens space.

(4.14) **Lemma.** *If m is a prime and* $t(\sigma)\subseteq(\mathbb{Z}/m)^{\times}$ *has order* 2^l, *then the image of* Φ *consists of the* 2^{l-1}*-st powers in* $(\mathbb{Z}/m)^{\times}$.

Proof. Since $u_r(m)=\zeta_m^{(r-1)/2}(\zeta_m^r-1/\zeta_m-1)\in\mathbb{Z}[\eta_m]^{\times}$, where $\eta_m=\zeta_m+\zeta_m^{-1}$, reduces to r in the residue field $\mathbb{F}_m$ it is clear that $r^{2^{l-1}}\in\operatorname{Im}\Phi$. Indeed, if

$$N\colon \mathbb{Z}[\eta_m]^{\times}\to A^{\times}$$

denotes the norm then $\Phi(N(u_r(m))=r^{2^{l-1}}$.

Conversely, if $r \in \operatorname{Im} \Phi_A$ consider the diagram

$$\begin{array}{ccc} A^{\times} & \longrightarrow & (\mathbb{Z}/m)^{\times} \\ {\scriptstyle N_0}\downarrow & & \downarrow{\scriptstyle N_1} \\ \mathbb{Z}^{\times} & \longrightarrow & (\mathbb{Z}/m)^{\times} \end{array}$$

where $N_1(r) = r^{(m-1)/2^l}$. Since $(\mathbb{Z}/m)^{\times}$ is cyclic of order $(m-1)$ we can only have $N_1(\Phi_A(r)) = \pm 1$ when $r \in (\mathbb{Z}/m)^{\times 2^{l-1}}$. $\square$

(4.15) *Remark.* If $-1 \in t(\sigma)$, $|t(\sigma)| = 2^l$ then 2^{l-1}-st powers in $(\mathbb{Z}/m)^{\times}$ are always in the image of Φ even when m is not a prime. Indeed, it suffices to check the square-free case. If $m = p_1 \ldots p_t$ and S is a non-empty subset of $\{p_1, \ldots, p_t\}$ define $u_r(S) = (\zeta_S - 1)/(\zeta_S^r - 1)$, $\zeta_S = \Pi\{\zeta_{p_i} | p_i \in S\}$. Then

$$u_r(m) = \Pi(u_r(S))^{(-1)^{|S|}}$$

has $\Phi(u_r(m)) = r$. Note that since σ is cyclic and $-1 \in t(\sigma)$ the order of $t(\sigma)$ is 2^l in each factor $(\mathbb{Z}/p_i)^{\times}$.

(4.16) **Theorem.** *Suppose $-1 \in t(\sigma)$ and that $t\colon \sigma \to (\mathbb{Z}/m)^{\times}$ has image of order 2^l.*

(i) *When* $\ker t$ *has order 2, X is homotopy equivalent to a finite complex if and only if r is a 2^{l-1}-st power* $(\operatorname{mod} m)$.

(ii) *For $l \geqq 2$ the cohomology class of $\partial\hat{\Delta}(X)$ in $H^0(\tilde{K}_0(\mathbb{Z}\pi)(m))$ is zero if and only if r is a square* $(\operatorname{mod} m)$.

(iii) *For $l = 1$ and* $|\ker t| > 2$, *the cohomology class of $\partial\hat{\Delta}(X)$ in $H^0(\tilde{K}_0(\mathbb{Z}\pi)(m))$ is zero if and only if r is a square* $(\operatorname{mod} m)$.

Proof. Assume first that $\ker t$ has order 2. Part (i) is proved in (4.13)–(4.15). For (ii) we notice from (4.8) that the action of $\mathbb{Z}/2$ on $D(\mathbb{Z}\pi)(m)$ is trivial, so that the Tate cohomology group $H^0(D(\mathbb{Z}\pi)(m)) = D(\mathbb{Z}\pi)(m)/2$. Hence when $r \equiv s^2 (\operatorname{mod} m)$ we see that $\partial\hat{\Delta}(X) = 0$.

Conversely, if $r \not\equiv s^2 (\operatorname{mod} m)$ then $\partial\hat{\Delta}(X) = [(r, 1)]$ is non-trivial in $H^0(D(\mathbb{Z}\pi)(m))$ when $l \geqq 2$. It suffices to consider the case of a prime m, since we can restrict to such a subgroup. We have the sequence (see (4.3) and (4.8)):

$$0 \to D(\mathbb{Z}\pi)(m) \to \tilde{K}_0(\mathbb{Z}\pi)(m) \to \Gamma(A)_{(2)} \oplus \Gamma^*(A)_{(2)} \to 0$$

where $\Gamma(A)$, $\Gamma^*(A)$ denote the class group, strict class group of $\mathbb{Q}(\chi_0)$. The sequence decomposes into two sequences corresponding to χ_0 and χ_1 and $\partial\hat{\Delta}(X)$ lies entirely in the χ_0 part (denoted the $+$ part in [2]). It follows from [2, (9.28)] that $\partial\hat{\Delta}(X) = [(r, 1)]$ maps to zero in $H^0(\tilde{K}_0(\mathbb{Z}\pi)(m))$ if and only if

$$\{r\} \in \operatorname{Image}(\Phi_A\colon L^{(2)} \to (A/m)^{\times}_{(2)})$$

where $L = \mathbb{Q}(\chi_0)$ and $L^{(2)}$ denotes the elements in L with even valuation. By taking norms and noting that $\mathbb{Q}^{(2)} = \mathbb{Q}^{\times 2}$ it is clear that the condition cannot be satisfied.

Now suppose $|\ker t|>2$. We prove first that the condition $r\equiv s^2 (\mathrm{mod}\, m)$ is sufficient for the vanishing of $\{\partial\hat{\Delta}(X)\}$ in $H^0(\tilde{K}_0(\mathbb{Z}\pi)(m))$. Actually we show the element vanishes already in $H^0(D(\mathbb{Z}\pi)(m))$.

Set $\kappa=\ker t$. From (4.8) we obtain

$$(4.17)\quad A^\times \times A^* \times \prod_{j\geqq 2}(A_j)^\times \xrightarrow{\Phi} \prod_{j\geqq 0}(\hat{\mathbb{Z}}_2\otimes A_j)^\times/(\mathbb{Z}_2\otimes A[\kappa])^\times_{(2)} \times \prod_{j\geqq 0}(A_j/m)^\times_{(2)},$$

and, modulo $\mathrm{Im}\,\Phi$, $\hat{\Delta}(X)(m)$ is represented by $(r,1,\dots,1)$ in each product on the right-hand side (see (4.2) and note that $\Delta(N)\in\mathrm{Im}\,\Phi$). The finiteness obstruction is trivial in $D(\mathbb{Z}\pi)(m)=\mathrm{cok}\,\Phi$ if and only if $\hat{\Delta}(X)(m)\in\mathrm{Im}\,\Phi$. For the cohomology class note first that when $r\equiv 1 (\mathrm{mod}\, 8)$ and $r\in(\mathbb{Z}/m)^{\times 2}$ the class $\{(r,1,\dots,1)\}=0$ in H^0 of the right-hand side of (4.17), hence $\{\partial\hat{\Delta}(X)(m)\}=0$ from (4.2) and (4.12). If $r\equiv 5(\mathrm{mod}\, 8)$, let T denote a generator of $\kappa\cong\mathbb{Z}/2^s$ and consider the unit in $(\hat{\mathbb{Z}}_2\otimes A[\kappa])^\times$:

$$-(1+T+T^{-1})\mapsto(-3,1,-1,\dots,-(1+\eta_{2^s}),\dots)\in\prod_{j\geqq 0}(\hat{\mathbb{Z}}_2\otimes A_j)^\times_{(2)},$$

where $\eta_i=\zeta_i+\zeta_i^{-1}$. Let

$$u=(1,1,-1,\dots,-(1+\eta_{2^s}),\dots)\in\prod_{j\geqq 0}(A_j)^\times$$

and now alter our representative for $\hat{\Delta}(X)(m)$ by $\Phi(u)$. Since $H^0((A_j/m)^\times)=0$ for $j\geqq 2$ it follows that

$$(4.18)\qquad \{\hat{\Delta}(X)(m)-\Delta(N)(m)\}=\Phi_*\{u\}$$

and so $\{\partial\hat{\Delta}(X)(m)\}=0$.

Conversely, if $r\not\equiv s^2(\mathrm{mod}\, m)$ we must show that $\{\partial\hat{\Delta}(X)(m)\}\neq 0$. For (ii) this follows from the case $|\ker t|=2$, established above. Indeed, we simply project π to the quotient group $\bar{\pi}=\mathbb{Z}/m\times\bar{\sigma}$ with $\ker\bar{t}=\mathbb{Z}/2$.

For (iii) the argument is more complicated. We project to $\bar{\pi}=\mathbb{Z}/m\times\bar{\sigma}$ with $\ker\bar{t}=\mathbb{Z}/4$, and assume that m is a prime. Explicating (4.12) in this case gives

$$\prod_{j=0}^{2}(\hat{\mathbb{Z}}_2\otimes A_j)^\times_{(2)}/(\mathbb{Z}_2\otimes A[\kappa])^\times_{(2)}\cong A/2.$$

We have $A_0=A_1=A=\mathbb{Z}[\eta_m]$, $A_2=A[i]$, and

$$\Phi\colon A_0^\times\times A_1^*\times A_2^\times\to A/2\times\prod_{j=0}^{2}(A_j/m)^\times_{(2)}$$

is given as the pair $\Phi=(\Phi',\Phi'')$, where Φ' maps into $A/2$ and Φ'' maps into $\prod(A_j/m)^\times_{(2)}$. Using the isomorphisms

$$(A_j/4)^\times_{(2)}\cong A_j/2=A/2 \quad \text{for } j=0,1$$
$$(A_j/2)^\times_{(2)}\cong A/2 \quad \text{for } j=2,$$

$\Phi'(a_0,a_1,a_2)=\rho_4(a_0)+\rho_4(a_1)+\rho_2(a_2)$, the sum of the reduction modulo the square of the uniformiser. The second component is

$$\Phi''(a_0,a_1,a_2)=(\rho_m(a_0),\rho_m(a_1),\rho_m(a_2)).$$

Let $E \subset \mathbb{Q}(\eta_m)$ be the maximal 2-power degree extension of $\mathbb{Q}$ and let B be its ring of integers. Set $B_0 = B_1 = B$ and $B_2 = B[i]$. Consider the B-analogue of Φ above,

$$\Phi_B\colon B_0^\times \times B_1^* \times B_2^\times \to B/2B \times \prod_{j=0}^{2} (B_j/m)_{(2)}^\times .$$

The norm homomorphism $N\colon A \to B$ induces norms of $A_j^\times$, $(A_j/m)_{(2)}^\times$ and $(A_j/4)_{(2)}^\times$. Under the identification of $(A/4)_{(2)}^\times$ with $A/2$, the norm corresponds to the trace, and

$$\Phi_B \circ N^{(3)} = (\mathrm{Tr} \times N^{(3)}) \circ \Phi.$$

Since $|A{:}B|$ is odd, $N(r) = r^{|A:B|}$ is a square (mod m) if and only if r is. So it suffices to prove that the class of $(0, r, 1, 1)$ in $H^0(\operatorname{coker} \Phi_B)$ is non-zero when $r \not\equiv s^2 (\operatorname{mod} m)$.

There are two cases to consider according to the congruence class of $m (\operatorname{mod} 4)$.

For $m \equiv 3 (\operatorname{mod} 4)$, $B = \mathbb{Z}$ and

$$\Phi_B\colon \mathbb{Z}^\times \times \mathbb{Z}[i]^\times \to \mathbb{Z}/2 \times (\mathbb{F}_m^\times \times \mathbb{F}_m^\times \times \mathbb{F}_{m^2}^\times)_{(2)}.$$

The map is $\Phi_B(x, y) = (x \cdot y, x, 1, y)$, so

$$\operatorname{cok} \Phi_B = (\mathbb{F}_{m^2}^{\times 2})_{(2)} \times (\mathbb{F}_m^\times)_{(2)}.$$

The natural projection into the cokernel,

$$\mathbb{Z}/2 \times (\mathbb{F}_m^\times \times \mathbb{F}_m^\times \times \mathbb{F}_{m^2}^\times)_{(2)} \to \mathbb{F}_{m^2}^{\times 2} \times \mathbb{F}_m^\times,$$

sends (x_1, x_2, x_3, x_4) into $(i(x_1) x_2 x_4^2, x_3)$ where $i\colon \mathbb{Z}/2 \to \mathbb{F}_{m^2}^{\times 2}$ is the injection. In particular $(0, r, 1, 1)$ maps into $(r, 1)$.

The involution on $\operatorname{cok} \Phi_B$ sends (u, v) to (u^{-1}, v) so $H^0(\operatorname{cok} \Phi_B) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Consequently, the class of $(0, r, 1, 1)$ is non-trivial.

For $m \equiv 1 (\operatorname{mod} 4)$, we take the norm, (trace) from B to $\mathbb{Z}$,

$$\begin{array}{ccc} B_0^\times \times B_1^* \times B[i]^\times & \xrightarrow{\Phi_B} & B/2B + \prod_{j=0}^{2} (B_j/m)_{(2)}^\times \\ \big\downarrow N & & \big\downarrow \mathrm{Tr} \times N \\ \mathbb{Z}^\times \times (1) \times \mathbb{Z}^\times & \xrightarrow{\Phi} & \mathbb{Z}/2 \times \langle \pm 1 \rangle \times \langle \pm 1 \rangle \times (\langle \pm 1 \rangle \times \langle \pm 1 \rangle). \end{array}$$

We have used that $(\mathbb{Z}[i]/m)^\times = \mathbb{F}_m^\times \times \mathbb{F}_m^\times$ (interchanged by the involution) and that the norms

$$N\colon (B_j/m)_{(2)}^\times \to \mathbb{F}_m^\times, \qquad N\colon B[i]^\times \to \mathbb{Z}[i]^\times$$

both have images $\langle \pm 1 \rangle$. There is an induced

$$\bar{N}\colon \operatorname{cok} \Phi_B \to \operatorname{cok} \bar{\Phi} = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

The involution on $\operatorname{cok} \bar{\Phi}$ is trivial, and $\bar{N}$ clearly maps the class of $(0, r, 1, 1)$ non-zero.

Thus for all primes m, $\{\partial\Delta(X)\}\neq 0$ in $H^0(D(\mathbb{Z}\pi)(m))$ for case (iii). We have left to check that the element maps nontrivially even into $H^0(\tilde{K}(\mathbb{Z}\pi)(m))$, but this is clear from the proof, since it consisted of taking norms down to $\mathbb{Q}$ and $\mathbb{Q}(i)$, where the class group is trivial. □

(4.19) **Corollary.** *When* $r\in(\mathbb{Z}/m)^{\times 2}$,

$$\Delta_0(X)(m)=\Delta(N)(m)\in H^0(Wh(\hat{\mathbb{Q}}_2\pi)_+)/I.$$

Proof. See (3.16) for notation. The $(+)$-part corresponds to the $A^\times$ factor of the domain of Φ where u is trivial. □

(4.20) *Remark.* From (4.17) and (4.12) we can see that $\partial\hat{\Delta}(X)(m)=0$ if $r\in(\mathbb{Z}/m)^{\times 2^l}$ and $r\in(\mathbb{Z}/2^k)^{\times 2^d}$ where $d=\min(l,k-l-1)$ without assuming $|\ker t|=2$ or $-1\in t(\sigma)$. If in fact $d=l$ then $g(X)$ is a linear k-invariant so there is nothing to check. If $d=k-l-1<l$ then $\hat{\Delta}(X)(m)-\Delta(N)(m)$ is trivial 2-adically (by (4.12)) and a 2^l-th power $(\mathrm{mod}\, m)$. Choose $r_1\equiv 1(\mathrm{mod}\, 2^k)$ and $r_1\equiv r(\mathrm{mod}\, m)$ and let N_1 be the linear space form with k-invariant $g(N_1)=r_1 g(N)$. Then $\hat{\Delta}(X)(m)=\Delta(N_1)(m)$. □

5. Surgery Obstructions

Let $\pi=\mathbb{Z}/m\rtimes\sigma$ be a 2-hyperelementary type I group such that $t\colon \sigma\to\mathrm{Aut}(\mathbb{Z}/m)$ has image $\mathbb{Z}/2^l$. It was pointed out in the introduction that there exists an orthogonal space form in the period dimension 2^{l+1}.

Let X denote the Swan complex with k-invariant $g(X)=rg(N)$ where $(r,|\pi|)=1$. The effect of changing orientation on X is to replace r by $-r$, so there is no loss of generality in assuming $r\equiv 1(\mathrm{mod}\,4)$.

Let $(f,\hat{f})\colon M\to X$ be the degree 1 normal map from (3.7). We shall use (3.14) and (3.16) to evaluate its surgery obstructions in $L_3^h(\mathbb{Z}\pi)$ and $L_3^p(\mathbb{Z}\pi)$.

Suppose $\kappa=\ker t$ has order 2. The arithmetic sequence

$$\text{(5.1)}\qquad \dots\xrightarrow{\psi_0} L_0^S(\hat{\mathbb{Q}}\pi)\longrightarrow L_3'(\mathbb{Z}\pi)\longrightarrow L_3'(\hat{\mathbb{Z}}\pi)\oplus L_3^S(\mathbb{Q}\pi)\xrightarrow{\psi_3}\dots$$

was evaluated in [4, §4]. Its top component decomposes into a direct sum of 2 sequences (denoted the $(+)$ part and the $(-)$ part) compatible with the decomposition of (4.8) corresponding to χ_0,χ_1.

Let A be the ring of integers in $\mathbb{Q}(\chi_0)=\mathbb{Q}(\chi_1)$ and consider the reduction maps

$$\varphi_A\colon A_{(2)}^\times\to(A/4A)_{(2)}^\times=A/2A$$
$$\Phi_A\colon A_{(2)}^\times\to(A/mA)_{(2)}^\times.$$

The top component of the finiteness obstruction for X is given as the class of r in the cokernel of Φ_A, when m is square-free and $|\ker t|=2$ (Theorem 4.13).

If X is homotopy equivalent to a finite complex then each choice of finite cell-structure, or rather of simple homotopy type, gives a surgery obstruction $\lambda_2'(f,\hat{f})\in L_3'(\hat{\mathbb{Z}}_2\pi)$, which only depends on X and $(f,\hat{f})$.

(5.2) **Proposition.** *Suppose X is homotopy equivalent to a finite complex and $|\ker t|=2$. If m is square free and $U=\mathrm{Im}(K_1'(\mathbb{Z}\pi)\to K_1'(\hat{\mathbb{Z}}_2\pi))$ then*

$$\lambda_2^U(f,\hat f)=0 \quad \text{in} \quad L_3^U(\hat{\mathbb{Z}}_2\pi)(m)$$

if and only if $0\in\varphi_A(\Phi_A^{-1}(r))$ modulo $H^0(K_1'(\mathbb{Z}\pi)(m)$. If m is not square-free then $\lambda_2^U(f,\hat f)(m)=0$.

Proof. It is clear from (4.13(ii)) and (3.14) that it suffices to consider the $(+)$ part. By [4, Lemma 4.11] one has

$$L_3^U(\hat{\mathbb{Z}}_2\pi)_+(m)\cong A/2A \text{ modulo } H^0(K_1'(\mathbb{Z}\pi)_+(m))$$

and from (4.11), $K_1(\hat{\mathbb{Z}}_2\pi)_+(m)\cong(\hat{\mathbb{Z}}_2\otimes A)_{(2)}^\times$. The natural map from $H^0(K_1(\hat{\mathbb{Z}}_2\pi)_+(m))$ into $L_3^U(\hat{\mathbb{Z}}_2\pi)_+(m)$ is induced by φ_A and the result follows by (3.14). □

(5.3) **Corollary.** *If $-1\in t(\sigma)$ and $|\ker t|=2$ then $\lambda_2^U(f,\hat f)=0$ if and only if $r\in(\mathbb{Z}/m)^{\times 2^l}$.*

Proof. Let $N\colon \mathbb{Z}[\eta_m]^\times\to A^\times$ be the norm homomorphism. In the course of proving Lemma 4.14 we found that when m is a prime,

$$\Phi_A(N(u))=s^{2^{l-1}}, \qquad u=\zeta^{s-1/2}(\zeta_m^s-1/\zeta_m-1).$$

Thus $\Phi_A(N(u)^2)=s^{2^l}$ so that $0\in\mathcal{S}_A(\Phi_A^{-1}(s^{2^l}))$. When m is composite use (4.15) instead for the required units. To see that the condition is necessary, suppose m is a prime and consider the diagram

$$\begin{array}{ccccc}
\mathbb{Z}[\eta_m]/2 & \longleftarrow & \mathbb{Z}[\eta_m]^\times & \xrightarrow{\ \rho\ } & \mathbb{F}_m^\times \\
\downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle \bar N} \\
A/2A & \xleftarrow{\ \varphi_A\ } & A^\times & \longrightarrow & \mathbb{F}_m^\times \\
\downarrow{\scriptstyle \mathrm{Tr}_0} & & \downarrow{\scriptstyle N_0} & & \downarrow{\scriptstyle \bar N_0} \\
\mathbb{Z}/2 & \longleftarrow & \mathbb{Z}^\times & \longrightarrow & \mathbb{F}_m^\times
\end{array}$$

The composite $\bar N_0\circ\bar N$ is the $(m-1)/2$ power map, so if s is a generator of $\mathbb{F}_m^\times$ then $s^{(m-1)/2}\equiv -1 \pmod m$ and hence $\mathrm{Tr}_0\,\varphi_A(N(u))\neq 0$.

Finally, $K_1'(\mathbb{Z}\pi)_+(m)=\mathrm{Ker}\{\Phi_A\colon A_{(2)}^\times\to(A/m^\times)_{(2)}\}$ since the kernel of Φ in (4.3) is precisely $K_1'(\mathbb{Z}\pi)(m)$. Thus $H^0(K_1'(\mathbb{Z}\pi)_+(m))\to L_3'(\hat{\mathbb{Z}}_2\pi)(m)$ has image the subgroup $\varphi_A(\mathrm{Ker}\,\Phi_A)$. Since $\mathrm{Tr}_0(\varphi_A(\mathrm{Ker}\,\Phi_A))=0$, $\varphi_A(N(u))\neq 0$ modulo $H^0(K_1(\mathbb{Z}\pi)_+(m))$. □

Proof of Theorem A. Since the λ_2^U-obstruction calculated in (5.3) is the image of $\lambda^h(f,\hat f)$, the condition $r\in(\mathbb{Z}/m)^{\times 2^l}$ is necessary for the action to exist, regardless of which finite cell-structure is chosen on X.

Conversely if the condition is satisfied we can assume that $\hat\Delta(X)(m)=\Delta(N)(m)$ by varying the linear space form N as in (4.20). The surgery obstruction group is given by an extension

$$0 \to \operatorname{cok} \psi_0 \to L'_3(\mathbf{Z}\pi) \to \ker \psi_3 \to 0$$

from (5.1). But by [4, Theorem A] the image of $\lambda'(f,\hat{f})$ in $\ker \psi_3$ is trivial. Now if $r \equiv 1 \pmod 4$, the k-invariant becomes linear over the subgroup $\rho \subseteq \pi$ with $\operatorname{Im} t \equiv \mathbf{Z}/2$ since $r \in (\mathbf{Z}/4m)^{\times 2}$ and so $\operatorname{Res}^{\rho}_{\pi} \lambda'(f,\hat{f}) = 0$. However, the calculation of [4, 4.15] give

$$\operatorname{cok} \psi_0(\pi) = H^1((A/m)^{\times})/\langle \pm 1 \rangle$$

and

$$\operatorname{cok} \psi_0(\rho) = H^1((\mathbf{Z}[\eta_m]/m)^{\times})/\langle \pm 1 \rangle$$

with the restriction map induced by the inclusion $A \subseteq \mathbf{Z}[\eta_m]$. □

Proof of Theorem B. A necessary condition for the action to exist is that $\{\partial \hat{\Delta}(X)(m)\} = 0$ in $H^0(\tilde{K}_0(\mathbf{Z}\pi)(m))$ whenever $\pi_0 = \mathbf{Z}/m \rtimes \sigma$ is a subgroup of π with $-1 \in t(\sigma)$. For $l \geqq 2$ or $l = 1$ and $|\ker t| > 2$ this happens precisely when $r \in (\mathbf{Z}/m)^{\times 2}$ by (4.16). If $l = 1$ and $|\ker t| = 2$ (5.3) shows that $r \in (\mathbf{Z}/m)^{\times 2}$ is necessary for the λ^U_2-obstruction to vanish. But from [2, 5.18, 9.31] the λ^U_2-obstruction determines the λ^p-obstruction with indeterminacy

$$\ker(\Phi_A \colon F^{(2)}/F^{\times 2} \to H^0((A/m)^{\times})).$$

Now the same "norm argument" as in (5.3) proves that $r \in (\mathbf{Z}/m)^{\times 2}$ is necessary for $\lambda^p(f,\hat{f}) = 0$ also.

For the converse, recal that the surgery obstruction to an action of $\pi = \mathbf{Z}/m \rtimes \sigma$ lies in

$$L^p_3(\mathbf{Z}\pi) = \sum_{d|m}^{\oplus} L^p_3(\mathbf{Z}\pi)(d)$$

by [2, §6] where the d-component is determined by restriction to the subgroup $\mathbf{Z}/d \rtimes \sigma \subseteq \pi$. We may assume that $d > 1$ since the lens spaces cover all possible homotopy types. In addition from the arithmetic sequence [2, 7.2]:

$$\to L^K_0(\hat{S}(d)) \to L^p_3(\mathbf{Z}\pi)(d) \to \prod_{l \nmid d} L^K_3(\hat{R}_l(d)) \oplus L^K_3(S(d)) \to$$

it is easy to see that $L^p_3(\mathbf{Z}\pi)(d) = 0$ if $-1 \notin t(\sigma)$ for $\mathbf{Z}/d \rtimes \sigma$. Indeed under this assumption $S(d)$ is type U and $\hat{R}_l(d)$ is type U or GL. For the remaining components we use (4.19) and (3.16) to conclude that the surgery obstruction vanishes. □

(5.4) *Remark.* In (4.20) we showed that $\hat{\Delta}(X)(m) = \Delta(N_1)(m)$ if $r \in (\mathbf{Z}/m)^{\times 2^l}$ and $r \in (\mathbf{Z}/2^k)^{\times 2^d}$ where $d = \min(l, k-l-1)$. Therefore X is finite and the surgery obstruction is given by (3.14). It follows that π acts freely on S^{q-1} with this k-invariant.

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Received January 3, 1985