Proceedings of Symposia in Pure Mathematics
Volume 32, 1978

AN OBSTRUCTION TO POINCARÉ TRANSVERSALITY

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In [3] an invariant $A(X^{2n}, f)$ in $Z/2$ was defined for a double cover $\pi: \tilde{X} \to X$ of $2n$-dimensional Poincaré duality (PD) spaces classified by a map $f: X \to RP^{2n}$. If the homotopy class of the map f contains a representative which is Poincaré transverse to $RP^{2n-1} \subset RP^{2n}$ [10], we say that π is *Poincaré splittable*. The invariant $A(X, f)$ depends only on the bordism class of (X, f) in $\mathcal{N}_{2n}^{PD}(RP^\infty)$ and vanishes for Poincaré splittable covers. In particular, it vanishes for double covers of PL-manifolds. The authors pointed out that from the map $f: X \to RP^{2n}$, one can construct another obstruction to the existence of a Poincaré splittable double cover bordant to (X, f). Let $\gamma^q \to BG(q)$ be the universal $(q-1)$-spherical fibration and $S^{2n-1} \to RP^{2n-1}$ the double cover (an S^0-fibration η). Then $MG(q) \wedge RP^{2n}$ is the Thom Space of $\gamma \times \eta \to BG(q) \times RP^{2n-1}$ and a Pontrjagin-Thom construction gives a map

$$p(f): S^{q-2n} \to MG(q) \wedge RP^{2n}.$$

If $\pi: \tilde{X} \to X$ is bordant to a Poincaré splittable cover then $p(f)$ is homotopic to a Poincaré transversal map. According to Jones [6], Levitt [7] or Quinn [10], there is one obstruction $0p(f)$ (in $Z/2$) to homotoping $p(f)$ to a Poincaré transversal map. In [3] the authors conjectured that $0p(f) = A(X, f)$ in all dimension $2n$ ($n \geqq 2$), but when [3] was written there were no known examples for which the invariant $A(X, f)$ was nonzero.

In this note, we construct examples (X^{2n}, f) in all dimensions $2n \geqq 4$, for which $A(X^{2n}, f) = 1$, and outline the proof of the conjecture in dimension 4. This involves using the fact that $A(X, f) \neq 0$ to obtain the exotic characteristic classes of the

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AMS (MOS) subject classifications (1970). Primary 57B10; Secondary 55G99.

[1]This research was supported in part by MPS 74-07491A01.
[2]Partially supported by the National Research Council of Canada.

Spivak normal bundle to our basic example in dimension 4.

One can establish product formulas for $\theta p(f)$ as is done in [4] and for $A(X, f)$ on the basis of our results in §2, and both formulas have the same general shape. Moreover, both $\theta p(f)$ and $A(X, f)$ vanish on $\mathrm{im}(\mathcal{N}_*^{\mathrm{PL}}(RP^\infty)) \subset \mathcal{N}_*^{\mathrm{PD}}(RP^\infty)$; hence, writing $\mathcal{N}_*^{\mathrm{PD}}(RP^\infty)/\mathrm{im}(\mathcal{N}_*^{\mathrm{PL}}(RP^\infty))$ as a module over $\mathcal{N}_*^{\mathrm{PL}}(RP^\infty)$ we must evaluate $A(X, f)$ and $\theta(p(f))$ on generators and show they agree in order to prove their equality. According to Brumfiel and Morgan [4], the Pontrjagin-Thom map $\mathcal{N}_*^{\mathrm{PD}}(RP^\infty) \to \pi_*(RP^\infty \wedge MG)$ is an injection ($* \neq 2$). The problem then is to construct examples to realize enough exotic characteristic numbers. (One could begin by obtaining those which appear in the cohomological formula [4] for the transversality obstruction.) On the basis of the results of [2] and [7], this program seems feasible but we have not yet attempted it.

The invariant $A(X, f)$ is an Arf invariant based on a quadratic map $q\colon H^n(\tilde{X}; Z/2) \to Z/2$ refining the nonsingular bilinear form $l(a, b) = \langle a \cup T^*b, [\tilde{X}] \rangle$ where $a, b \in H^n(\tilde{X}; Z/2)$ and $T\colon \tilde{X} \to \tilde{X}$ is the free involution. We prove that this map q is the same as the Browder-Livesay map ψ used in [1] to define a desuspension obstruction for smooth involutions on homotopy spheres.

Our basic example X^4 in dimension 4 is the orbit space of a free simplicial involution on $S^2 \times S^2$. The other examples are obtained from this one by forming the product with suitable smooth manifolds. In each case, the covering space is homotopy equivalent to a smooth manifold. We also indicate some generalizations of the construction using the results of [9] on projective homotopy. We sketch some proofs here; full details will appear elsewhere.

1. A quadratic map for double covers. In this section, we recall the definition of the quadratic map q and prove that it equals the Browder-Livesay map. All cohomology groups have $Z/2$ coefficients and $[X]$ denotes the fundamental class of a PD space X.

Let $\pi\colon \tilde{X} \to X$ be a double cover of $2n$-dimensional PD spaces classified by $f\colon X \to RP^\infty$. We denote the involution on $\tilde{X}$ by T and the map covering f by $\tilde{f}\colon \tilde{X} \to S^\infty$. Form $S^\infty \times_{Z/2} (\tilde{X} \times \tilde{X})$ where $Z/2$ acts on $\tilde{X} \times \tilde{X}$ by interchanging the factors and define $F\colon X \to S^\infty \times_{Z/2} (\tilde{X} \times \tilde{X})$ as the quotient of the equivariant map $\tilde{F}\colon \tilde{X} \to S^\infty \times (\tilde{X} \times \tilde{X})$ given by $\tilde{F}(x) = (\tilde{f}(x), (x, Tx))$. Now if $a_\sharp$ is a cocycle on $\tilde{X}$ representing $a \in H^n(\tilde{X})$ then $1 \otimes a_\sharp \otimes a_\sharp$ is an equivariant cocycle on $S^\infty \times (\tilde{X} \times \tilde{X})$ so represents a class $\alpha \in H^{2n}(S^\infty \times_{Z/2} (\tilde{X} \times \tilde{X}))$.

DEFINITION. $q(a) = \langle F^*(\alpha), [X] \rangle$.

Let $Y = S^\infty \times_{Z/2} \tilde{X}$ and define $\lambda\colon Y \to S^\infty \times_{Z/2} (\tilde{X} \times \tilde{X})$ by $\lambda[u, x] = [u, (x, Tx)]$. If $\rho\colon Y \to X$ is given by $\rho[u, x] = \pi(x)$, then ρ is a homotopy equivalence and $F \circ \rho \simeq \lambda$. We now describe a chain approximation for λ. Suppose that $T\colon \tilde{X} \to \tilde{X}$ is a simplicial map such that $T\sigma \cap \sigma = \varnothing$ for all simplices $\sigma \in X$ and partially order the simplices so that $T(a \cup_i b) = Ta \cup_i Tb$ where $\cup_i$ denotes the Steenrod cup-sub-i-product. We give S^∞ its usual equivariant cellular decomposition with cells e_i and Te_i in each dimension. In the statement below, $\Delta_j\colon C_k(\tilde{X}) \to C_{k+j}(\tilde{X} \times \tilde{X})$ is the jth Steenrod map [11] and $\tau\colon C_k(\tilde{X} \times \tilde{X}) \to C_k(\tilde{X} \times \tilde{X})$ is defined by $\tau(a \otimes b) = b \otimes a$. We recall the formulas

$$\partial \Delta_j = (1 + \tau)\Delta_{j-1} + \Delta_j \partial \quad \text{and} \quad \Delta_j \cdot T = (T \otimes T)\Delta_j.$$

THEOREM 1. *The map given by*

$$\lambda_\sharp(e_i \otimes c) = \sum_{0 \leq j \leq i} e_j \otimes (1 \otimes T)\tau^j \Delta_{i-j}(c)$$

and $\lambda_\sharp(Te_i \otimes c) = (T \otimes \tau)\lambda_\sharp(e_i \otimes Tc)$ *is a chain approximation to* λ *where* $c \in C_k(\tilde{X}; Z/2)$.

COROLLARY 2. *For* $a \in H^n(\tilde{X})$,

$$q(a) = \left\langle \sum_{i=0}^{n} e^i \otimes (a_\sharp \cup_i Ta_\sharp), [Y] \right\rangle$$

where $a_\sharp$ *is a cocycle representing* a, e^i *is dual to* e_i *and* $\rho_*[Y] = [X]$.

With this explicit cochain formula, we can relate q to the Browder-Livesay map $\psi: H^n(\tilde{X}) \to Z/2$. First, we summarize their definition [**1**].

Let x be a cocycle in $C^n(\tilde{X}; Z/2)$. Then since $x \cup_{n+1} Tx = 0$, $(1 + T)(x \cup_n Tx) = 0$ so $x \cup_n Tx = (1 + T)\nu^n$. Assuming that ν^{n+j} are constructed for $0 \leqq j \leqq i < n$ so that

$$x \underset{n-j}{\cup} Tx + \delta\nu^{n+j-1} = (1 + T)\nu^{n+j}$$

they construct ν^{n+i+1} satisfying a similar formula. The cochain ν^{2n} turns out to be determined modulo $\delta C^{2n-1}(\tilde{X}; Z/2) + (1 + T)C^{2n}(\tilde{X}; Z/2)$, and so the class $(1 + T)\nu^{2n}$ represents a cohomology class in $H^{2n}_{Z/2}(C_*(\tilde{X}); Z/2) \cong H^{2n}(X)$. Then if $a = \{x\} \in H^n(X)$ they set $\psi(a) = \langle \{(1 + T)\nu^{2n}\}, [X] \rangle \in Z/2$.

THEOREM 3. *For all* $a \in H^n(\tilde{X})$, $\psi(a) = q(a)$.

PROOF. By construction, $(1 + T)\nu^{2n} = x \cup Tx + \delta\nu^{2n-1}$ where x is a cocycle representing a. Set

$$\nu = \sum_{i=0}^{n-1} e^{n-i-1} \otimes \nu^{n+i}$$

and compute

$$\delta\nu = \sum_{i=0}^{n} e^i \otimes \left(x \underset{i}{\cup} Tx\right) + e^0 \otimes (1 + T)\nu^{2n}.$$

Therefore,

$$\delta\nu = \lambda^\sharp(e^0 \otimes x \otimes x) + \rho^\sharp(1 + T)\nu^{2n}$$

so

$$\langle \lambda^\sharp(e^0 \otimes x \otimes x), [Y] \rangle = \langle e^0 \otimes (1 + T)\nu^{2n}, [Y] \rangle$$

and the result follows.

2. A product formula. For the construction of the next section, we need to compute q on $\tilde{X} \times N \to^{\pi \times 1} X \times N$ where N^{2m} is a PD space of dimension $2m$. Our main applications are the cases $N = CP^2$ and $N = RP^2$.

THEOREM 4. *Let* $\tilde{X} \times N \to^{\pi \times 1} X \times N$ *be the product covering and* $p + r = n + m$.

Let $a \in H^p(\tilde{X})$ *and* $b \in H^r(N)$, *then*

$$q(a \otimes b) = \left\langle \sum_{0 \le j \le r} F^*(1 \otimes a \otimes a) \cup f^*(u)^j \otimes \mathrm{Sq}_j(b), [X] \otimes [N] \right\rangle$$

where u *generates* $H^1(RP^\infty)$.

We now recall the definition of $A(X, f)$. ($f\colon X \to RP^\infty$ classifies $\pi\colon \tilde{X} \to X$.) According to [**1**] or [**3**],

$$q(a + b) - q(a) - q(b) = \langle a \cup Tb, [\tilde{X}] \rangle$$

for all $a, b \in H^n(\tilde{X})$. The bilinear form defined by the formula on the right-hand side is nonsingular and even, so there exists a symplectic base for $H^n(\tilde{X})$ with respect to this form. $A(X, f)$ is the Arf invariant associated to any such base. From the definition of q we easily verify that $A(X, f)$ depends only on the class of (X, f) in $\mathcal{A}_*^{PD}(RP^\infty)$ and vanishes for double covers of PL-manifolds. More generally,

PROPOSITION 5 [**3**]. *If* $\pi\colon \tilde{X} \to X$ *is a Poincaré splittable double cover of* $2n$*-dimensional* PD *spaces, then* $A(X, f) = 0$ *where* $f\colon X \to RP^\infty$ *classifies* π.

Using the product formula, we establish

COROLLARY 6. *If* $\tilde{X} \times CP^2 \to^{\pi \times 1} X \times CP^2$ *is the product covering,* $A(X \times CP^2, fp_1) = A(X, f)$ *where* $p_1\colon X \times CP^2 \to X$ *is the projection.*

3. The examples. We will now describe the basic example in dimension 4. It is a PD space X^4 with fundamental group $Z/2$ and nonzero A-invariant.

The complex X^4 is among those constructed in [**12**, p. 240]. Let K^3 be the 3-skeleton of $RP^2 \times S^2$ in a normal cell decomposition and note that $\tilde{K}^3 \simeq S_1^2 \vee S_2^2 \vee S^3$. We obtain X^4 by attaching the 4-cell e^4 by a different map than that used to get $RP^2 \times S^2$. To describe the map, we need to denote generators of $\pi_2 S_i^2$, $\pi_3 S^3$ and $\pi_3 S_i^2$ by I_i, J and η_i, respectively, for $i = 1, 2$. Then, according to the Hilton-Milnor theorem, $\pi_3 K^3$ is generated by J, η_1, η_2 and $[I_1, I_2]$. The $Z/2$ action on these is given by

$$T(J) = J - [I_1, I_2], \quad T\eta_i = \eta_i, \quad T[I_1, I_2] = -[I_1, I_2]$$

and the attaching map used to obtain $RP^2 \times S^2$ has class J. To construct X^4 we use a map in the class $J + \eta_1$ where the notation is chosen so that S_1^2 is the sphere covering RP^2 in $(RP^2 \times S^2)^{(3)} = \tilde{K}^3$. Since $(1 - T)e^4$ is then attached with class $[I_1, I_2]$, $\tilde{X}^4 \simeq S^2 \times S^2$. Observe that X^4 is nonorientable. In fact, there is no orientable example in dimension four.

This PD space has $A(X^4, f) = 1$ where the map $f\colon X \to RP^\infty$ induces the universal covering $\pi\colon \tilde{X} \to X$. To see this we need to describe the generators of $H^2(\tilde{X}^4)$. By construction, $X^4 \simeq (RP^2 \vee S^2) \cup e^3 \cup e^4$. Let a denote the cohomology dual of the class represented by the cover of $RP^2 \subset RP^2 \vee S^2 \subset X^4$, and b denote the dual of the class represented by one cover of $S^2 \subset RP^2 \vee S^2 \subset X^4$. Then $b = \pi^*\bar{b}$ for some $\bar{b} \in H^2(X^4)$. Since $\{a, b\}$ forms a symplectic base, it is enough to show $q(a) = q(b) = 1$.

LEMMA 7. *Let* $\pi\colon \tilde{X} \to X$ *be a double cover of* $2n$*-dimensional* PD *spaces and* $b \in H^n(X)$. *Then*

$$q(\pi^*\bar{b}) = \left\langle \sum_{i=0}^{n} (f^*u)^i \cup \mathrm{Sq}_i(\bar{b}), [X] \right\rangle$$

where u generates $H^1(RP^\infty)$.

From this lemma,

$$q(b) = \left\langle \sum_{i=0}^{2} (f^*u)^i \cup \mathrm{Sq}_i(\bar{b}), [X] \right\rangle = \left\langle \bar{a} \cup \bar{b}, [X] \right\rangle = 1$$

where $\bar{a}$ is dual to the class represented by $RP^2 \subset X^4$. To prove $q(a) = 1$ it is necessary to compute $a_\sharp \cup_i Ta_\sharp$ where $a_\sharp$ is the obvious cochain representing a. We omit the details.

One can generalize the construction of X^4 to higher dimensions in several ways. Here is one direction. Let K^{n+1} be the $(n+1)$-skeleton of $RP^n \times S^n$ in a normal cell decomposition. Since

$$\pi_{n+1} K^{n+1} = \pi_{n+1} RP^n \oplus \pi_{n+1} S^n \oplus \pi_{n+1} S^{n+1},$$

we can construct K^{n+2} by attaching an $(n+2)$-cell to K^{n+1} using a map representing $\eta + \alpha$ where $\eta \in \pi_{n+1} RP^n$ is the nontrivial element and $\alpha \in \pi_{n+1} K^{n+1}$ is the class of the attaching map for the normal $(n+2)$-skeleton of $RP^n \times S^n$.

PROPOSITION 8. *If $n \equiv 2 \pmod 4$, then there exists a PD space X^{2n} with $\tilde{X} \simeq S^n \times S^n$, $\pi_1 X = Z/2$, $X^{(n+2)} \simeq K^{n+2}$ in a normal cell decomposition; and $A(X,f) = 1$.*

The point here is that $\eta \in \pi_{n+1} RP^n$ is a projective element if and only if $n \equiv 2\ (4)$ (see [9]). Similarly, by using other projective elements in $\pi_{n+k} RP^n$ for $k < n$, one can construct more examples. For $n = 3$, even though η is not projective, we can obtain a PD space X^6 with $A(X^6, f) = 1$ by this construction. This is described in [5].

4. Realization of the transversality obstruction. Our main result is

THEOREM 9. *In each dimension $2n \geqq 4$ there exists a PD space X^{2n} and a map $f\colon X^{2n} \to RP^\infty$ such that $A(X^{2n}, f) = 1$, and $\tilde{X}$ has the homotopy type of a smooth manifold.*

PROOF. The method of proof is clear. The example X^4 of §2 is crossed with copies of CP^2 to obtain examples in dimensions $\equiv 0\ (4)$. From Corollary 6, all these PD spaces have nonzero A-invariant. In addition, we note that the above examples X^{4k} provide examples X^{4k+2} $(k \geqq 1)$. Consider $\tilde{X}^{4k} \times RP^2 \to^{\pi \times 1} X^{4k} \times RP^2$. By an argument similar to that of Corollary 6 we see that $A(X^{4k} \times RP^2, fp_1) = 1$ and these give the examples in dimensions $4k + 2$.

5. The Spivak normal bundle to X^4. Define an injection $\rho\colon \mathcal{N}_*^{PD}(\mathrm{pt}) \to \mathcal{N}_*^{PD}(RP^\infty)$ by $\rho\{X^n\} = \{X^n, w_1\}$ where $w_1\colon X^n \to RP^\infty$ classifies the first Stiefel-Whitney class of X^n. We need the following

LEMMA 10 [3]. *The Pontrjagin-Thom map $\mathcal{N}_*^{PD}(RP^\infty) \to \pi_*(RP^\infty \wedge MG)$ is an injection, so every class in $\mathcal{N}_*^{PD}(RP^\infty)$ is detected by characteristic numbers $(* \neq 2)$.*

Consider the class of $\{X^4\}$ in $\mathcal{N}_4^{PD}(\mathrm{pt})$. We calculate that the Stiefel-Whitney class of X^4 is $1 + e^1$. On the other hand, the fact given in §3 that $A(X^4, f) = 1$ together with the fact that $\rho\{X^4\} = (X^4, f)$ shows that $\{X^4\} \neq 0$ in $\mathcal{N}_4^{PD}(\mathrm{pt})$. This gives

COROLLARY 11. *$K_3(X^4) \neq 0$ and X^4 generates the cokernel of* ($\mathcal{N}_4^{\mathrm{Diff}}(\mathrm{pt}) \to \mathcal{N}_4^{PD}(\mathrm{pt})$). (*Since the only further characteristic classes in dimensions* $\leqq 4$ *are* K_3 *and* $\mathrm{Sq}^1 K_3$ [**8**].)

Let $\kappa\colon X^4 \to^{\tau} S^3 \cup_2 e^4 \to^{\lambda} B_G$ be the composition where τ is the pinching map and λ satisfies $\lambda^*(\kappa_3) \neq 0$, $\lambda^*(w_4) = 0$. If (κ) is the induced bundle we have that the Spivak normal bundle of X^4 is the Whitney sum $\xi_1 \oplus (\kappa)$ where ξ_1 is the nontrivial line bundle.

COROLLARY 12. *In dimension* 4, *$A(X, f)$ coincides with the stable transversality obstruction of* [**4**].

PROOF. The calculations of [**4**] show the stable transversality obstruction is given by $e_1 K_3$, and the result follows from Corollary 11.

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