Smooth group actions on 4-manifolds

Ian Hambleton
Department of Mathematics & Statistics
McMaster University
Hamilton, ON L8S 4K1, Canada

Mathematische Arbeitstagung June 13-19, 2003
Max-Planck-Institut für Mathematik, Bonn, Germany

This talk is based on joint work with Ronnie Lee and Mihail Tanase. We are interested in a kind of "rigidity" for finite group actions on certain smooth 4-manifolds, namely those constructed by connected sums of geometric pieces such as algebraic surfaces. We can ask how closely a smooth, orientation-preserving, finite group action on such a connected sum resembles a equivariant connected sum of algebraic actions on the individual factors.

Following [2], we consider this question for the simplest case, where

$$X = \#_n \mathbb{C}P^2 = \mathbb{C}P^2 \# \dots \# \mathbb{C}P^2$$

is the connected sum of n copies of the complex projective plane, arranged so that X is simply-connected. In that paper we restricted ourselves to actions which induced the *identity* on the homology of X, but here we remove this assumption. Since the intersection form of X is the standard definite form $Q_X = \langle 1 \rangle \perp \cdots \perp \langle 1 \rangle$, we note that its automorphism group is given by an extension

$$1 \to \{\pm 1\}^n \to \mathrm{Aut}(H_2(X;\mathbf{Z}), Q_X) \to \Sigma_n \to 1$$

where Σ_n denotes the group of permutations of n elements. Therefore, if a finite group G of odd order acts smoothly on X, then $H_2(X;\mathbf{Z})$ is the direct sum of permutation modules of the form $\mathbf{Z}[G/H_\alpha]$, for various stabilizer subgroups $H_\alpha \subseteq G$.

We would like to understand how the following three invariants of such an action (X, G) are related:

(A) The permutation representation of G on $H_2(X;\mathbf{Z})$.

(B) The singular set of the action, meaning the collection of isotropy subgroups and fixed sets X^H for $H \subset G$.

(C) The tangential isotropy representations (T_xX, G_x) at all the singular points $x \in X$.

In the rest of the talk, we assume that $G = C_m$ is a finite cyclic group of *odd* order m, acting smoothly on $X = \#_n \mathbb{C}P^2$. We obtain many examples of such smooth actions by starting with linear actions of G on $\mathbb{C}P^2$, given by sending a generator

$$t \mapsto \begin{pmatrix} 1 & & \\ & \zeta^a & \\ & & \zeta^b \end{pmatrix}$$

where $\zeta = e^{\frac{2\pi i}{m}}$ is a primitive m^{th} root of unity, and a, b are integers such that the common divisor $(a, b, m) = 1$. In this case, G acts by the identity on homology, and the singular set always contains the three fixed points $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$. In addition, there can be up to three invariant 2-spheres with various isotropy subgroups, depending on the values of a and b. For example, if $(a, b) = (10, 3)$ and $m = 105$, then the action has 5 orbit types (the maximal number for $\mathbb{C}P^2$). The tangential representations at the three G-fixed points are given by the *rotation numbers* (a, b), $(-a, b-a)$ and $(a-b, -b)$ standing for the decomposition of $T_x\mathbb{C}P^2 = \mathbb{R}^4 = \mathbb{C}^2$ into eigenspaces under the action of t. These rotation numbers are well-defined modulo m up to identifying $(a, b) \equiv (b, a) \equiv (-a, -b)$.

Examples of smooth G-actions on a connected sum $X = \#_n \mathbb{C}P^2$ with more isotropy groups and various permuation actions on homology are constructed by a *tree* of equivariant connected sums, where we sum either at fixed points of two linear actions or along an orbit of singular points. In order to preserve orientation, the rotation numbers at the attaching points must be of the form (a, b) and $(a, -b)$. By this means, we can obtain a large supply of model actions, for which one should be able to work out the relation between (A), (B) and (C).

What can one say about a general smooth action (X, G) ? It is not hard to verify that the singular set consists of a configuration of isolated points and 2-spheres, as in the linear models. The main result of M. Tanase's thesis [3] generalizes [2, Thm. C]:

Theorem 1 *Let G be a cyclic group of odd order, acting smoothly on on $X = \#_n \mathbb{C}P^2$. Then there exists an equivariant connected sum of linear actions on $\mathbb{C}P^2$ with the same isotropy structure, singular set and rotation numbers, and the same permutation action on $H_2(X; \mathbf{Z})$ as for the given action (X, G).*

This result is proved by using the symmetries of the equivariant Yang-Mills moduli space [1] to produce a stratified equivariant cobordism between (X, G) and a connected sum of linear actions, relating the invariants (A), (B) and (C). The following result of [3] holds for smooth, but not topological actions, by an example of A. Edmonds.

Theorem 2 *Let (X, G) be a smooth action as above, with discrete singular set. Then the action (X, G) is semi-free.*

In principle, it should now be possible to say exactly which permutation modules can be realized by smooth actions on $\#_n \mathbb{C}P^2$ just by studying the equivariant connected sums. We observe that not all modules are realizable: for example if $G = C_{p^k}$ for $k \geq 3$, then the module $\mathbf{Z}[C_p] \oplus \mathbf{Z}[C_{p^2}]$ is not realizable by any smooth G-action on $\#_n \mathbb{C}P^2$. However, we have the following "stable" realization result:

Theorem 3 *Let $\mathcal{S}$ denote a set of subgroups of $G = C_m$ containing a unique maximal element H_0 (under inclusion of subgroups). There exists an integer $N = N(m)$ such that any permutation module*

$$\bigoplus \{\mathbf{Z}[G/H_\alpha]^{k_\alpha} : H_\alpha \in \mathcal{S}\}$$

is realizable by a smooth G-action on some $\#_n \mathbb{C}P^2$, provided that the multiplicity $k_0 \geq N$ for the maximal stabilizer subgroup H_0.

References

[1] Hambleton, I. and Lee, R.: Perturbation of equivariant moduli spaces, Math. Ann. **293** (1992), 17-37.

[2] Hambleton, I. and Lee, R.: Smooth group actions on definite 4-manifolds and moduli spaces, Duke Math. J. **78** (1995), 715-732.

[3] Tanase, M.: Smooth finite cyclic group actions on positive definite 4-manifolds, Ph.D. Thesis, McMaster University (2003).