Chapter IX

Cancellation Results for 2-Complexes and 4-Manifolds and Some Applications

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This is a survey chapter. The idea is to summarize some recent work which illustrates in one way or another the connection between topology in dimension 2 and the study of 4-dimensional manifolds. There are almost no new results and no result is proved completely in the paper. Instead, in each section we collect together some related statements and motivation, and give a sketch of some typical or important steps in the proofs.

1 A Cancellation Theorem for 2-Complexes

Any two finite 2-complexes K, K' with isomorphic fundamental groups become *simple* homotopy equivalent after wedging with a sufficiently large (finite) number of S^2's (see chapter I, (40)). Furthermore, if $\alpha : \pi_1(K, x_0) \to \pi_1(K', x'_0)$ is a given isomorphism and K, K' have the same Euler characteristic, then there is a simple-homotopy equivalence $f : K \vee rS^2 \to K' \vee rS^2$ inducing α on the fundamental groups. For a given group π, the minimal number r with the property above for all finite 2-complexes with this fundamental group is called the *stable range.*

It is known that for finite fundamental groups the stable range is always ≤ 2 ([Dy81], Theorem 3). The main result of this section is the following.

Theorem 1.1 ([HaKr92$_1$]) *Let K and K' be finite 2-complexes with the same Euler characteristic and finite fundamental group. Let $\alpha : \pi_1(K, x_0) \to \pi_1(K', x'_0)$ be a given isomorphism and suppose that $K \simeq K_0 \vee S^2$. Then there is a simple-homotopy equivalence $f : K \to K'$ inducing α on the fundamental groups.*

The analogous result for "homotopy type" instead of "simple-homotopy type" was proved by W. Browning ([Br78], 5.4; see also Chapter III, §2).

This is the best possible result in general ([Me76]; see also Chapter III, §2 and this chapter, §4); but for special fundamental groups like cyclic groups [Me76], [DySi73] or more generally finite subgroups of $SO(3)$ ([HaKr92$_1$]; see also [La91] for the groups $D(4n)$) it can sometimes be improved (see Theorem 1.3).

Proof: Let $h : K \vee rS^2 \to K' \vee rS^2$ be a simple-homotopy equivalence as above, inducing a given isomorphism α on the fundamental groups. We will prove the theorem inductively and thus we may assume that $r = 1$. Our strategy is to construct a simple self-equivalence of K such that, after composing with this, we obtain $h' : K \vee S^2 \to K' \vee S^2$ which fixes the element p_1 of π_2 represented by the S^2 factor. Then the composition of h' with the inclusion and projection gives a homotopy equivalence $f : K \to K'$, which by the additivity formula for the Whitehead torsion is simple.

To construct such a simple self-equivalence of K, one naturally first considers the corresponding algebraic problem of constructing an automorphism of π_2 preserving π_1 and then realizing it by a simple self-equivalence.

We fix some notation. Let $A = \mathbb{Z}[\pi_1(K)]$, $L = \pi_2(K_0)$ and let $P = P_0 \oplus P_1$ be the A-submodule of $\pi_2(K_0 \vee S^2 \vee S^2)$ generated by $\pi_2(S^2 \vee S^2)$. We note that the A-module L has $(A, \mathbb{Z})$-free rank ≥ 1 at all primes p not dividing the order of $\pi_1(K)$. This notion was introduced in [HaKr92$_1$] and means that there exists an integer r such that $(\mathbb{Z}^r \oplus L)_p$ has free rank ≥ 1 over A_p, where we consider $\mathbb{Z}$ as A-module via the augmentation map. In this case, the reason that $\pi_2(K_0)$ has $(A, \mathbb{Z})$-free rank ≥ 1 is that L fits into an exact sequence

(1) $0 \to L \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0$

with the modules $C_i = C_i(\tilde{K})$ finitely generated free A-modules.

More generally, any lattice L with a resolution (1) by finitely generated projective A-modules C_i is unique up to direct sum with projectives. The stable class is denoted $\Omega^3\mathbb{Z}$. Such lattices with minimal Z-rank need not contain any projective direct summands over $A = Z\pi$, but rationally contain all the representations of π except perhaps the trivial one. Then L has $(A, \mathbb{Z})$-free rank ≥ 1 at all primes not dividing the order of π.

We need the following notation. If $M = M_1 \oplus M_2$ is a direct sum splitting of an A-module, then $E(M_1, M_2)$ denotes the subgroup of $GL(M)$ generated by the elementary automorphisms ([Ba68], p.182). This is the group generated by automorphisms of the form $1 + f$ and $1 + g$, where $f: M_1 \to M_2$ and $g: M_2 \to M_1$ are arbitrary A-homomorphisms. An element of an A-module is called unimodular if there is a homomorphism to A mapping it to 1. The main algebraic ingredient of our proof is the following result whose proof we will sketch at the end of this section.

Theorem 1.2 [HaKr92$_1$], Corollary 1.12 and Lemma 1.16) *Let $M = P \oplus L$ be an A-lattice, where $P = p_0 A \oplus p_1 A = P_0 \oplus P_1$ and L has $(A, \mathbb{Z})$-free rank ≥ 1. Then the group $G = \langle E(P_0, L \oplus P_1), E(P_1, L \oplus P_0)\rangle$ acts transitively on unimodular elements in $L \oplus P$.*

To finish our proof, we have to realize elements in G by simple-homotopy self equivalences of $K_0 \vee 2S^2 = K \vee S^2$ inducing the identity on π_1. It is enough to do this for $E(P_1, L \oplus P_0)$. This group is generated by automorphisms of the form $1 + f$ and $1 + g$, where $f: L \oplus P_0 \to P_1$ and $g: P_1 \to L \oplus P_0$ are arbitrary A-homomorphisms. Recall that $P_1 = p_1 A$ and $L \oplus P_0 = \pi_2(K)$. Consider the map $Id \vee u: K \vee S^2 \to K \vee S^2$, where $u = (g(p_1), p_1) \in \pi_2(K \vee S^2) = \pi_2(K) \oplus p_1 A$. It realizes $1 + g$ and its restriction to K is the identity and it also induces the identity on $(K \vee S^2)/K = S^2$. Thus the additivity formula for the Whitehead torsion implies that the torsion of $Id \vee u$ vanishes.

To realize $1 + f$, we note that $f : L \oplus P_1 = \pi_2(K) = H_2(K; A) \to P_1 = A$ factors through $H_2(K, K^1; A)$, with K^1 the 1-skeleton. The reason for this is that we have an exact sequence

$$\mathrm{Hom}_A(H_2(K, K^1; A), A) \to \mathrm{Hom}_A(H_2(K; A), A) \to \mathrm{Ext}^1_A(H_1(K^1; A), A)$$

and the last group vanishes since $H_1(K^1; A)$ is $\mathbb{Z}$-torsion free. Choose a factorization map $\bar{f}: H_2(K, K^1; A) \to A$, where $H_2(K, K^1; A)$ is a free A-module generated by the 2-cells of K (appropriately connected to the base point). Denote this basis by $e_1, .., e_k$. Now write $K = K^1 \cup D^2 \cup ... \cup D^2$. Pinch off the 2-cells to obtain $K \vee rS^2$ and denote the projection map by $p: K \to K \vee kS^2$. Consider the composition map $\beta: K \to K \vee kS^2 \to K \vee S^2$, where the second map is $Id \vee \bar{f}(e_1) \vee ... \vee \bar{f}(e_k)$. By construction the induced map on π_2 is $1 \oplus f$ and the composition $K \to K \vee S^2 \to K$ is homotopic to Id. Finally, consider $\beta \vee Id: K \vee S^2 \to K \vee S^2$ realizing $1 + f$. Its restriction to

S^2 and the induced map on K are homotopic to the identity, implying from the additivity of the Whitehead torsion that $\beta \vee Id$ has trivial torsion. □

Without proof, we state the full classification result for 2-complexes with fundamental group a finite subgroup of $SO(3)$. Recall that the finite subgroups G of $SO(3)$ are cyclic, dihedral, A_4, S_4, and A_5.

Theorem 1.3 *Let π be a finite subgroup of $SO(3)$. If K and K' are finite 2-complexes with fundamental group π and the same Euler characteristic, and if $\alpha : \pi_1(K, x_0) \to \pi_1(K', x_0')$ is a given isomorphism, then there is a simple-homotopy equivalence $f : K \to K'$ inducing α on the fundamental groups.*

The proof runs along the same lines as above but needs several additional steps. For π cyclic or $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2$, this was proved in [Me76], [DySi73]. The result for $\pi = D(4n)$, the dihedral group of order $4n$, has recently been obtained by P. Latiolais [La91]. Our methods give a new proof in these cases.

Now, we give a sketch of the proof of Theorem 1.2.

Proof of Theorem 1.2: Recall that a *lattice* is a finitely generated right A-module that is torsion free over $\mathbb{Z}$. Our proof is based on an improvement of the Bass transitivity theorem ([Ba68], pp.178-184), which assumed that M has free rank $\geq d+2$ where d is the Krull dimension of the ring A. In our special case of lattices over group rings of finite groups (Krull dimension $= 1$), we are able to obtain a transitivity theorem assuming only free rank ≥ 2 when the lattice M contains a summand L which has $(A, \mathbb{Z})$-free rank ≥ 1. Thus, the improvement here is that a particular type of non-free modules which occurs geometrically as $\pi_2(K)$ can play the role of a free module in producing algebraic transitivity.

We denote the augmentation map by $\epsilon: A \to B = \mathbb{Z}$. This is a surjective ring homomorphism. If M is an A-lattice we get an induced homomorphism

$$\epsilon_*: M \to M \otimes_A \mathbb{Z}.$$

Recall that for an element $x \in M$, $O_M(x)$ is the left ideal in A generated by

$$\{f(x) \mid f \in \mathrm{Hom}_A(M, A)\}.$$

If $O_M(x) = A$, we say that x is unimodular. We need two easy facts whose proofs are omitted.

Proposition 1.4 *Let M be an A-lattice and $A' = A/A\mathfrak{t}$ for an ideal $\mathfrak{t} \in \mathbb{Z}$ such that the localized order $A_{\mathfrak{t}}$ is maximal. Then the induced map*

$$\mathrm{Hom}_A(M, A) \to \mathrm{Hom}_{A'}(M', A')$$

is surjective, where $M' = M/M\mathfrak{t}$.

Proposition 1.5 ([Ba73], (2.5.2), p.225) *If C is a semisimple algebra, then for each a, $b \in C$ there exists $r \in C$ such that $C(a+rb) = Ca + Cb$.*

Now let $x = p_0a + p_1b + v \in M$ be a unimodular element, with $p = p_0a + p_1b \in P$ and $v \in L$, so that $O(x) = Aa + Ab + O(v)$. Since the elementary matrices $E_n(\mathbb{Z})$ act transitively on unimodular elements in $\mathbb{Z}^n$ for $n \geq 2$, we may assume that $\epsilon_*(x) = \epsilon_*(p_0)$. In the proof, we use the stability assumption on L to move x so that its component in $p_0A \oplus L$ is unimodular. Then we move x to p_0 to prove the statement about unimodular elements in M. At each step, we use only elements σ of G fixing $\epsilon_*(p_0)$.

Lemma 1.6 *Let S be a finite set of (non-zero) primes in $\mathbb{Z}$, and $\bar{A} = A/\mathfrak{g}A$ where $\mathfrak{g}$ is the product of all the primes $\mathfrak{p} \in S$. Then after applying an element $\tau \in E(P_1 \oplus L, P_0)$ to x, $O(\bar{x}) = \bar{A}\bar{a} = \bar{A}$ and $\epsilon_*(x) = \epsilon_*(p_0)$.*

Proof: The semi-simple quotient ring $\bar{A}/\text{Rad}\,\bar{A} = \bar{C} \times \bar{C}'$, where $\bar{C} = \bar{B}/\text{Rad}\bar{B}$ and C' is a complementary direct factor. Here "Rad" denotes the Jacobson radical [CuRe62]. Since $\epsilon_*(x) = \epsilon_*(p_0)$, a projects to 1 in the $\bar{C}$ component of the semisimple quotient. Since $Aa + O(p_1b + v) = A$, there exists $c \in O(p_1b + v)$ such that $Aa + c$ contains 1, and c projects to zero in $\bar{B}$. By Proposition 1.5, there exists $z \in A$ with $A(a + zc) = A(\bmod\ \mathrm{g})$ and a map $g: P_1 \oplus L \to p_0A \subseteq M$ with $g(p_1b + v) = p_0zc$. Extend g to a map from M to M by zero on the complement. Then $\tau = 1 + g$ is an element of $E(P_1 \oplus L, P_0)$ and $\tau(x)$ has the desired properties. □

We apply Lemma 1.6 to the set S of primes $\mathfrak{p} \in \mathbb{Z}$ at which A is not maximal, or L does not have (A, B)-free rank ≥ 1.

Lemma 1.7 *If $x = p_0a + p_1b + v \in M$ is a unimodular element for which $Aa + \mathfrak{g}A = A$, then after applying an element $\tau \in E(P_1, L)$ we have $x = p_0a + p_1b + v$ with $p_0a + v$ unimodular and $\epsilon_*(x) = \epsilon_*(p_0)$.*

Proof: Let $\mathfrak{t} \subseteq \mathbb{Z}$ denote the ideal which is maximal among those such that $A\mathfrak{t} \subseteq Aa$. It is not hard to see that $\mathfrak{g}$ is relatively prime to $\mathfrak{t}$, and so $A_\mathfrak{t}$ is a maximal order.

Now we project to the semilocal ring $A' = A/A\mathfrak{t}$. This is a finite quotient ring of the maximal order $A_\mathfrak{t}$, and so the projection $\epsilon': A' \to B'$ splits and $A' = B' \times C'$. Since over the B' factor a projects to 1, we have $(Aa)' = A'$. Over the complementary factor C' we use a suitable $\tau \in E(p_1'C', L')$, so that after applying τ we achieve the condition

$$A'a' + O(v') = A'$$

over both factors of A'. This is an application of Proposition 1.5 to the component of x in $L' \oplus p_1' C'$ using the fact that $C' \subseteq L'$. The necessary homomorphism $g \in \mathrm{Hom}_{A'}(P_1', L')$, which is the identity over B', can be lifted to $\mathrm{Hom}_A(P_1, L)$ since P_1 is projective and extended to M by zero on $p_0 A \oplus L$.

We now lift the relation above to A using Proposition 1.4 and obtain

$$Aa + O(v) + A\mathfrak{t} = A.$$

But $A\mathfrak{t} \subseteq Aa$ so $v + p_0 a$ is unimodular. □

We now complete the proof of Theorem 1.2 by the following:

Lemma 1.8 *Let $x = p_0 a + p_1 b + v$ and $\epsilon_*(x) = p_0$. Suppose that $z = p_0 a + v$ is unimodular, and write $L \oplus P_0 = zA \oplus L_0$. Then there exist elementary automorphisms $\tau_1 \in E(zA, P_1)$, $\tau_2 \in E(P_1, P_0)$, $\tau_3 \in E(P_0, P_1)$ and $\tau_4 \in E(P_0, L)$ such that $\tau_4 \tau_2^{-1} \tau_3 \tau_2 \tau_1(x) = p_0$ and the product fixes $\epsilon_*(p_0)$.*

Proof: This is the argument of [Ba68, pp. 183-184]. Let $g_1(z) = p_1(1-a-b)$, with $g_1(L_0) = 0$. Define $g_2(p_1) = p_0$, $g_3(p_0) = p_1(a-1)$, and $g_4(p_0) = -v$, where the homomorphisms are extended to the obvious complements by zero.

If $\tau_i = 1 + g_i$, then

$$\tau_4 \tau_2^{-1} \tau_3 \tau_2 \tau_1(x) = p_0.$$

The product fixes $\epsilon_*(p_0)$ and lies in $E(P_1, P_0 \oplus L)$. □

This finishes the proof of Theorem 1.2.

2 Stable Classification of 4-Manifolds

There is a close analogy between the stable classification of homotopy types of 2-complexes and homeomorphism types of 4-manifolds. To indicate this analogy, consider the thickening functor from finite 2-complexes to closed 4-manifolds obtained by embedding a 2-complex X as polyhedron in $\mathbb{R}^5$ and taking the boundary of a smooth regular neighborhood (compare Chapter I, §3). If two 2-complexes are simple-homotopy equivalent, the corresponding 4-manifolds are s-cobordant (implying homeomorphic, if the fundamental groups are poly-(finite or cyclic) [Fr84]) and we denote the corresponding s-cobordism class by $M(X)$. If we replace the 2-complex by its 1-point unification with S^2, the corresponding 4-manifold changes by connected sum with $S^2 \times S^2$. This indicates the analogy of stable equivalence classes of 2-complexes with the following notation for 4-manifolds.

Definition 2.1 *Two smooth (topological) closed 4-manifolds M_0 and M_1 are stably diffeomorphic (homeomorphic) if the connected sums $M_0\#r(S^2 \times S^2)$ and $M_1\#r(S^2 \times S^2)$ are diffeomorphic (homeomorphic) for some integer r.*

Since the smooth stable s-cobordism theorem (implying that two s-cobordant 4-manifolds are stably diffeomorphic) holds [Qu83], the stable diffeomorphism class of $M(X)$ is determined by the stable simple-homotopy class of X and so (see §1) by $\pi_1(X)$.

Compared to the 2-complexes, it is not true that for 4-manifolds the stable classification needs only the fundamental group and the Euler characteristic as invariants. At least one has to control basic properties like orientability and existence of a spin-structure and in addition for oriented manifolds the signature.

The following definition turns out to be very useful for coding the fundamental group together with orientability and spin-structure information. Let M be a topological 4-manifold. Abbreviate $\pi_1(M) = \pi$. Let $u : M \to K(\pi, 1)$ be a classifying map of the universal covering $\tilde{M}$. Then we have an isomorphism $u^* : H^1(\pi; \mathbb{Z}/2) \to H^1(M; \mathbb{Z}/2)$ and an exact sequence $0 \to H^2(\pi; \mathbb{Z}/2) \to H^2(M; \mathbb{Z}/2) \to H^2(\tilde{M}; \mathbb{Z}/2)$ [Br82]. Thus we can pull back $w_1(M)$ by u from a class denoted $w_1 \in H^1(\pi; \mathbb{Z}/2)$ and, if $w_2(\tilde{M}) = 0$, $w_2(M)$ from a class denoted $w_2 \in H^2(\pi; \mathbb{Z}/2)$. If $w_2(\tilde{M}) \neq 0$, we define $w_2 = \infty$. There is an obvious notion of isomorphism classes of the triple (π, w_1, w_2) and we denote the isomorphism class by $[\pi, w_1, w_2]$.

Definition 2.2 *For a topological 4-manifold M, we call the isomorphism class $[\pi, w_1, w_2]$ the algebraic normal 1-type.*

The algebraic normal 1-type determines the geometric normal 1-type , called the normal 1-type , as follows. We begin with the smooth case. Let M be a smooth manifold. If $w_2 = \infty$ (corresponding to $w_2(\tilde{M}) \neq 0$), then we define the normal 1-type as follows. Consider the real line bundle $E \to K(\pi, 1)$ with $w_1(E) = w_1$ and the composition

$$K(\pi,1) \times BSO \xrightarrow{E \times p} BO \times BO \longrightarrow \oplus BO,$$

where $E\colon K(\pi, 1) \to BO$ is the classifying map of the stable bundle given by E and $\oplus$ is the H-space structure on BO given by the Whitney sum. We denote the corresponding fibration by $B[\pi, w_1, \infty]$. If $w_2 \neq \infty$, we define the normal 1-type as the fibration $p : B(\pi, w_1, w_2) \longrightarrow BO$ given by the following pullback square

$$\begin{array}{ccc} B(\pi, w_1, w_2) & \longrightarrow & K(\pi, 1) \\ p \downarrow & & \downarrow w_1 \times w_2 \\ BO & \xrightarrow{w_1(EO) \times w_2(EO)} & K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2). \end{array}$$

where $w_i(EO)$ are the Stiefel-Whitney classes of the universal bundle and we interpret w_i as maps to $K(\mathbb{Z}/2, i)$. The fibre homotopy type of

$$p : B(\pi, w_1, w_2) \longrightarrow BO$$

is determined by the isomorphism class of (π, w_1, w_2) and is denoted by $B[\pi, w_1, w_2]$, the normal 1-type.

If $w_1 = 0$, $B[\pi, 0, w_2]$ factorizes over BSO and we choose one of the possible lifts. This way we consider $B[\pi, 0, w_2]$ as fibrations over BSO. To deal the oriented case ($w_1 = 0$) and the non-oriented case simultaneously we write $p : B(\pi, w_1, w_2) \longrightarrow B(S)O$.

For topological manifolds, one can make the obvious changes (replace the linear normal bundle by the topological normal bundle given by a map $\nu : M \to B(S)Top$) to obtain from the algebraic normal 1-type the normal 1-type $p : B(\pi, w_1, w_2) \longrightarrow B(S)Top$.

The following theorem plays a central role in the stable classification of 4-manifolds. Given a fibration $B \to B(S)O$, abbreviated for short as B, we consider the B-bordism group $\Omega_n(B)$ consisting of bordism classes of closed smooth n-manifolds, which are oriented, if the fibration is over BSO , together with a lift $\bar{\nu}$ over B of the normal Gauss-map $\nu : M \to B(S)O$ [St68]. Such a lift is called a *normal 1-smoothing* if $\bar{\nu}$ is a 2-equivalence. It is easy to check that, if the algebraic normal 1-type of M is $[\pi, w_1, w_2]$, by construction of $B[\pi, 0, w_2]$, M admits a normal 1-smoothing in $B[\pi, w_1, w_2]$. Similarly, for topological manifolds, one starts with a fibration $B \to B(S)Top$, abbreviated for short as B^{Top}, and introduces the analogous bordism group of topological manifolds denoted $\Omega_n(B^{Top})$.

Theorem 2.3 ([Kr85]) *Two smooth (topological) 4-manifolds M_0 and M_1 with the same algebraic normal 1-type $[\pi, w_1, w_2]$ are stably diffeomorphic (homeomorphic), if and only if they have the same Euler characteristic and if they admit normal 1-smoothings $\bar{\nu}_0$ and $\bar{\nu}_1$ respectively such that $(M_0, \bar{\nu}_0)$ and $(M_1, \bar{\nu}_1)$ represent the same bordism class in $\Omega_4(B[\pi, w_1, w_2])$ (in $\Omega_4(B^{Top}[\pi, w_1, w_2])$).*

If one wants to apply this theorem, one has to compute the bordism group $\Omega_4(B[\pi, w_1, w_2])$ or $\Omega_4(B^{Top}[\pi, w_1, w_2])$. In general, this is not easy; but, under some assumptions, it follows from the Atiyah-Hirzebruch bordism spectral sequence ([CoFl64]). For example, if $w_2 = \infty$ (i.e., $w_2(\tilde{M}) \neq 0$) and $w_1 = 0$, then $\Omega_4(B[\pi, w_1, w_2]) = \Omega_4(K(\pi, 1))$ and $\Omega_4(B^{Top}[\pi, w_1, w_2]) = \Omega_4^{(Top)}(K(\pi, 1))$, where the right side is the oriented smooth (topological) singular bordism group in $K(\pi, 1)$. In this situation the choice of a normal 1-smoothing is equivalent to the choice of a map $u : M \to K(\pi, 1)$ inducing an isomorphism on π or equivalently a representative of a classifying map of the universal covering. The different choices are obtained by composing with an automorphism of π acting as self equivalences of $K(\pi, 1)$. Now, $\Omega_i = 0$ for $1 \leq i \leq 3$ and $\Omega_0 \cong \Omega_4 \cong \mathbb{Z}$, where in the last case the isomorphism is given by the signature (cf [MiSt74]). In the topological case, one has an additional term $\mathbb{Z}/2$ detected by the Kirby-Siebenmann obstruction KS [KiSi77]. Thus, from the Atiyah-Hirzebruch spectral sequence, one has $\Omega_4(K(\pi, 1)) \cong \mathbb{Z} \oplus H_4(K(\pi, 1); \mathbb{Z})$ in the smooth case, and $\Omega_4^{(Top)}(K(\pi, 1)) \cong \mathbb{Z} \oplus H_4(K(\pi, 1); \mathbb{Z}) \oplus \mathbb{Z}/2$ in the topological case. The isomorphism is given by the signature of M, the image of the fundamental class $u_*([M])$ in $H_4(K(\pi, 1); \mathbb{Z})$, and, in the topological case, in addition the KS-invariant. This proves the first part of the following theorem.

Theorem 2.4 *Two oriented smooth (topological) 4-manifolds M_0 and M_1 with the same fundamental group and with $w_2(\tilde{M}_i) \neq 0$ are stably diffeomorphic (homeomorphic), if and only if they have the same Euler characteristic and signature, if $u_*(M_0) = u_*(M_1) \in H_4(K(\pi, 1); \mathbb{Z})/Out(\pi)$ and, in the topological case, $KS(M_0) = KS(M_1)$.*

Arbitrary values of the signature and the class in $H_4(K(\pi, 1); \mathbb{Z})/Out(\pi)$ and, in the topological case, of $KS \in \mathbb{Z}/2$ can be realized.

Proof: We are left with the realization statement. This follows since by surgery any element in the corresponding bordism group can be realized by a manifold, such that u induces an isomorphism of π and with $w_2(\tilde{M}) \neq 0$. □

One can use the same surgery method to say much more about the stable classification of 4-manifolds. For instance, if the manifolds M_i are equipped with spin-structures, they are stably diffeomorphic (homeomorphic) if and only if they have the same Euler characteristic and (M_0, u_0) and (M_1, u_1) represent the same element in the singular smooth (topological) bordism group of spin-manifolds together with maps to $K(\pi, 1)$. But this bordism group is much harder to compute and a general answer is not known. In the next theorems, we list some results for manifolds with special fundamental groups which can easily be obtained along these lines of arguments.

Theorem 2.5 *Let M_0 and M_1 be smooth (topological), oriented 4-manifolds with $w_2(\tilde{M}_i) = 0$ and $\pi_1(M_i) = \pi$. If $H_i(\pi, \mathbb{Z}/2) = 0$ for $1 \leq i \leq 3$, then M_0 and M_1 are stably diffeomorphic (homeomorphic) if and only if they have the same Euler characteristic, signature, $u_*(M_0) = u_*(M_1) \in H_4(K(\pi,1);\mathbb{Z})/Out(\pi)$ and, in the topological case, if $KS(M_0) = KS(M_1)$.*

If M is smooth, then the signature, abbreviated by σ, is by Rohlin's Theorem divisible by 16; and arbitrary values divisible by 16 of the signature and the class in $H_4(K(\pi,1);\mathbb{Z})/Out(\pi)$ and, in the topological case, of $KS \in \mathbb{Z}/2$ can be realized.

Theorem 2.5 in particular covers all finite fundamental groups of odd order.

Theorem 2.6 *Let M_0 and M_1 be smooth (topological), oriented 4-manifolds with $w_2(M_i) = 0$ and cyclic fundamental group $\pi_1(M_i) = \pi$. Then M_0 and M_1 are stably diffeomorphic (homeomorphic) if and only if both admit a spin structure or both do not admit a spin structure and they have same Euler characteristic, signature and, in the topological case, if $KS(M_0) = KS(M_1)$.*

The signature is always divisible by 8 and in the smooth case, if $w_2(M) \neq 0$, every integer divisible by 8 can be realized and, if $w_2(M) = 0$, all integers divisible by 16 can be realized. In the topological case, every integer divisible by 8 can be realized and, if $w_2(M) \neq 0$, one can prescribe $KS \in \mathbb{Z}/2$ arbitrarily, whereas, if $w_2(M) = 0$, $KS = \sigma(M)/8 \bmod 2$.

3 A Cancellation Theorem for Topological 4-Manifolds

In this section we prove a cancellation theorem for topological 4-manifolds which is analogous to Theorem 1.1.

Theorem 3.1 ([HaKr92$_2$], Theorem B) *Let X and Y be closed oriented topological 4-manifolds with finite fundamental group. Suppose that for some r the connected sum $X \# r(S^2 \times S^2)$ is homeomorphic to $Y \# r(S^2 \times S^2)$. If $X = X_0 \# (S^2 \times S^2)$, then X is homeomorphic to Y.*

Note that the assumption that X splits off one $S^2 \times S^2$ cannot be omitted, in general. There are, for example, even simply–connected closed topological 4-manifolds that are stably homeomorphic but not homeomorphic because they have non–isometric intersection forms. Examples of distinct but stably homeomorphic manifolds with finite fundamental group and the same equivariant intersection form were constructed in [KrSc84]. We will discuss these examples in the next section.

Before we prove this theorem, we formulate the following immediate corollary to it and Theorems 2.4, 2.5 and 2.6.

Corollary 3.2 *Let M_0 and M_1 be closed oriented topological manifolds with finite fundamental group π, such that one of the three conditions are fulfilled: i) $w_2(\tilde{M}_i) \neq 0$, ii) $w_2(\tilde{M}_i) = 0$ and π cyclic, iii) $H_i(\pi, Z/2) = 0$ for $1 \leq i \leq 3$. Suppose that $M_0 = X\#(S^2 \times S^2)$. Then M_0 is homeomorphic to M_1 if and only if both admit a spin structure or both do not admit a spin structure and they have same Euler characteristic, signature, Kirby-Siebenmann obstruction and $u_*(M_0) = u_*(M_1) \in H_4(K(\pi,1);\mathbb{Z})/Out(\pi)$.*

As in the proof of Theorem 1.1, there is an algebraic and a geometric part in the proof of Theorem 3.1. We begin by stating the algebraic input. As in the last section, we set $A = \mathbb{Z}[\pi]$ and we equip A with the anti-involution $a \mapsto \bar{a}$ mapping an element in π to its inverse. As common in algebra, we consider right A-modules but note that with the help of the anti-involution one can pass from right to left modules and vice versa. Thus, whenever the module comes naturally with a left action, we pass to the corresponding right action. In particular, we do this for the dual of a right module V , which we denote by $\bar{V}$. A quadratic A-module V is an A-module together with a hermitian form $\langle -, - \rangle$ and a quadratic refinement q in the sense of ([Wa70], Chapter 5) with values in $A/\{a - \bar{a}\}$. It has *$(A,\mathbb{Z})$-hyperbolic rank ≥ 1 at a prime $p \in \mathbb{Z}$* if there exists an integer r such that $(H(\mathbb{Z}^r) \oplus V)_p$ has free hyperbolic rank ≥ 1 over A_p. Here the hyperbolic form $H(W)$ of an A-module W is the form on $W \oplus \bar{W}$ which is trivial on W and $\bar{W}$ and evaluation between W and $\bar{W}$, and where the quadratic refinement vanishes on W and $\bar{W}$. The hyperbolic rank is $\geq s$ if the quadratic form splits off $H(A^s)$.

We need various subgroups of the isometries on a quadratic module. If $P = p_0A \oplus p_1A$ is A-free of rank 2, we denote by $E(P)$ the group generated by elementary triangular matrices having 1 on the diagonal and by $H(E(P)$ the induced isometries on the hyperbolic space $H(P)$. A *transvection* ([Ba73], p.91) of V is a unitary automorphism $\sigma = \sigma_{u,a,v} : V \to V$ given by

$$\sigma(x) = x + u\langle v, x\rangle - v\langle u, x\rangle - ua\langle u, x\rangle,$$

where u, $v \in V$ and $a \in A$ satisfy the conditions

$$q(u) = 0 \in A/\{a - \bar{a}\}, \langle u, v\rangle = 0, q(v) = a \in A/\{a - \bar{a}\}.$$

For any submodule $L \subseteq V$,

$$L^{\perp} = \{x \in V | \langle x, y\rangle = 0 \text{ for all } y \in L\}.$$

If $V = V' \perp V''$ is an orthogonal direct sum, with $L' \subseteq V'$ a *totally isotropic* submodule (*i.e.* $h(x, y) = 0 \pmod{\{a - \bar{a}\}}$ for all x, $y \in L'$), and $L'' \subseteq V''$, then we define

$$EU(V', L'; L'') = \langle \sigma_{u,a,v} | u \in L' \text{ and } v \in L''\rangle$$

and in the special case $V = P \perp \bar{P}$

$$EU(H(P)) = EU(P, P, \bar{P}).$$

A hyperbolic plane is a quadratic module isomorphic to $H(A)$. A hyperbolic pair consists of two vectors u and v with $q(u) = q(v) = 0$ and $< u, v >= 1$.

Theorem 3.3 ([HaKr92$_2$], Theorem 1.20 and Lemma 1.21) *Let V be a quadratic module which has $(A, \mathbb{Z})$-hyperbolic rank ≥ 1 at all but finitely many primes, and put $M = V \perp H(P)$, where $P = p_0A \oplus p_1A$ is A-free of rank 2. Then*

$$G = \langle EU(H(P), Q; V), H(E(P)) \cdot EU(H(P))\rangle$$

where $Q = P$ or $\bar{P}$, acts transitively on the set of q-unimodular elements of a fixed length, and the set of hyperbolic pairs and hyperbolic planes in M.

Here an element $x \in M$ is q-unimodular if there exists $y \in M$ such that $\langle x, y\rangle = 1$.

This theorem is the quadratic analogue of Theorem 1.2 and the proof uses the same philosophy. We will apply this algebraic cancellation theorem to prove Theorem 3.1. We need some preparations.

Proposition 3.4 *Let X be a closed oriented topological 4–manifold with finite fundamental group, and let $A = \mathbb{Z}[\pi_1(X)]$. There is an A–submodule V of $\pi_2(X)$ which supports a quadratic refinement of the intersection form on X. In addition, V has $(A, \mathbb{Z})$–hyperbolic rank ≥ 1 at all but finitely many primes.*

Proof: Since our algebraic result uses quadratic modules and the intersection form on $\pi_2(X)$ does in general not admit a quadratic refinement, we take the submodule $V = \ker\,(\langle w_2, -\rangle : \pi_2(X) \to \mathbb{Z}/2)$ on which the intersection form

S_X has a quadratic refinement $q : V \to A/\{\nu - \bar{\nu}\}$ defined as in [Wa70, Chapter 5].

Next we check that V has $(A, \mathbb{Z})$–hyperbolic rank ≥ 1 at all odd primes not dividing the order of $\pi_1(X)$. Since X is a closed manifold, the components of the multi–signature of S_X are all equal (compare [Le77]). On the other hand, from [HaKr88, 2.4] we know that $\pi_2(X)_{(p)}$ is isomorphic to the localization of $I \oplus I^* \oplus A^\ell$, where I denotes the augmentation ideal of A. It follows that the components of S_X are indefinite at all non–trivial characters of $\pi_1(X)$. Since S_X is unimodular when restricted to V_p, for p as above, we conclude that V has $(A, \mathbb{Z})$ hyperbolic rank ≥ 1 at all odd primes not dividing the order of $\pi_1(X)$. □

We need the following result of Cappell–Shaneson. In the statement a standard basis for the summand $H_2(S^2 \times S^2, \mathbb{Z})$ of $H_2(X\#(S^2 \times S^2), \mathbb{Z})$ is denoted by $\{p_0, q_0\}$.

Theorem 3.5 ([CaSh71],1.5) *Let X be a compact, connected smooth (topological) manifold of dimension four, and suppose $X = X_0\#(S^2 \times S^2)$ for some manifold X_0. Let $\omega \in H_2(X; A) \cong \pi_2(X)$ with $w_2(\omega) = 0$ and let $a \in A = \mathbb{Z}[\pi_1(X)]$ be any element such that $q(\omega) \equiv a(\text{mod } \{a - \bar{a}\})$. Then there is a base point preserving diffeomorphism (homeomorphism) ϕ of $X\#(S^2 \times S^2)$ with itself which preserves local orientations and induces the identity on $\pi_1(X\#(S^2 \times S^2))$, so that $\phi_*(p_0) = p_0$, $\phi_*(q_0) = q_0 + \omega - p_0 a$, and $\phi_*(\xi) = \xi - (\xi \cdot \omega)p_0$ for $\xi \in H_2(X; A)$.*

In order to prove Theorem 3.1, we need to realize transvections by homeomorphisms of $X\#r(S^2 \times S^2)$. For the rest of this section, we fix the notation

$$K\pi_2(X) = \ker\left(\langle w_2, -\rangle : \pi_2(X) \to \mathbb{Z}/2\right)$$

for the submodule of the intersection form on $H_2(X; A)$ on which a quadratic refinement is defined. We denote by $H(P_0)$, where $P_0 = p_0 A$, the summand of $H_2(X\#(S^2 \times S^2); A)$ given by $H_2(S^2 \times S^2; A)$. As further copies of $S^2 \times S^2$ are added to X by connected sum, we denote all these hyperbolic factors of the intersection form by $H(P)$. Note that Theorem 3.5 allows us to realize the transvections $\sigma_{p_0,a,v}$ by self-homeomorphisms of $X\#(S^2 \times S^2)$ for any $v \in K\pi_2(X_0)$, in the case when $X = X_0\#(S^2 \times S^2)$. Cappell and Shaneson use this to realize many isometries (see [CaSh71, Theorem A2]), but the conclusions given are not in the exact form we need.

Corollary 3.6 *Suppose that $K\pi_2(X) = V_0 \perp V_1$ with V_0 non-singular under the intersection form S_X. Then, for any transvection $\sigma_{p,a,v}$ on $K\pi_2(X) \perp H(P_0)$ with $p \in V_0 \perp P_0$ and $v \in K\pi_2(X)$, the stabilized isometry $\sigma_{p,a,v} \oplus Id_{2(S^2 \times S^2)}$ can be realized by a self-homeomorphism of $X\#3(S^2 \times S^2)$.*

Proof: First, we consider a unimodular isotropic element $p \in V_0 \perp P_0$. Since $V_0 \perp H(P_0)$ is non-singular, p is automatically a hyperbolic element and thus by Freedman [Fr84] we can re-split $X\#(S^2 \times S^2) = X'\#(S^2 \times S^2)$ such that p is represented by $S^2 \times *$. Thus $\sigma_{p,a,v} \oplus Id_{S^2 \times S^2}$ can be realized by a self-homeomorphism on $(X'\#(S^2 \times S^2))\#(S^2 \times S^2)$ for all $v \in K\pi_2(X)$ with $\langle v,p \rangle = 0$.

Next, we consider the transvection $\sigma_{p,0,v}$ for an arbitrary $p \in V_0 \perp P_0$, but assume that $v \in K\pi_2(X)$ is isotropic. Then we write $p = \sum p_i$ with $p_i \in V_0 \perp P_0$ unimodular and $\langle v, p_i \rangle = 0$. This uses the fact that $A = \mathbb{Z}[\pi_1(X)]$ and $P_0 \cong A$. We obtain: $\sigma_{p,0,v} = \sigma_{v,0,-p} = \sigma_{v,0,-\sum p_i} = \prod \sigma_{p_i,0,v}$. Thus $\sigma_{p,0,v} \oplus Id_{S^2 \times S^2}$ is realizable by a self-homeomorphism on $(X\#(S^2 \times S^2))\#(S^2 \times S^2)$, since $\sigma_{p_i,0,v}\#Id_{S^2\times S^2}$ is realizable.

Finally, we realize an arbitrary transvection $\sigma_{p,a,v}\,\#Id_{2(S^2\times S^2)}$, of the form required, by a homeomorphism on $(X\#(S^2 \times S^2))\#(S^2 \times S^2)$. We use the fact that v can be expressed as $v = \sum v_i$ with $v_i \in K\pi_2(X) \perp H_2(S^2 \times S^2; A)$ isotropic and $\langle v_i, p \rangle = 0$. Thus $\sigma_{p,a,v} \oplus Id_{2(S^2\times S^2)} = \prod \sigma_{p,0,v_i} \oplus Id_{S^2 \times S^2}$ which by the considerations above is realizable. □

Corollary 3.7 *Let X_0 be a topological 4-manifold, $V = K\pi_2(X_0)$ and consider an element $\varphi \in EU(H(P),Q;V)$, for $Q = P, \bar{P}$, as an isometry of the intersection form of $X_0\#2(S^2 \times S^2)$. Then the stabilized isometry $\varphi \oplus Id_{2(S^2\times S^2)}$ can be realized by a self-homeomorphism of $X_0\#4(S^2 \times S^2)$.*

Proof: By definition, the group $EU(H(P),Q;V)$ is generated by transvections $\sigma p, a, v$ with $p \in P$ or $\bar{P}$ and $v \in V$ fulfilling the conditions of a transvection. It is enough to consider the case $p \in P$. Now Corollary 3.6 applies with the splitting $K\pi_2(X) = V \perp H(A)$ with $H(A)$ the first summand of $H(P)$. This shows that for each $\varphi \in EU(H(P),Q;V)$, the isometry $\varphi \oplus Id_{2(S^2\times S^2)}$ can be realized by a self-homeomorphism on $(X_0\#2(S^2 \times S^2))\#2(S^2 \times S^2)$. □

Proof of Theorem 3.1: By induction, it is enough to consider the case $r = 1$. Let $f : X\#(S^2 \times S^2) \to Y\#(S^2 \times S^2)$ be a homeomorphism. We will apply Theorem 3.3 and Corollary 3.6 to show that there is a self-homeomorphism g of $X\#3(S^2 \times S^2)$ such that $(f\#Id) \cdot g$ induces the identity on the hyperbolic form corresponding to $\# \, 3(S^2 \times S^2)$ in $H_2(X\#3(S^2 \times S^2); A)$. Then it follows that X and Y are s-cobordant ([Kr85], Theorem 3.1). By Freedman [Fr84], X and Y are homeomorphic.

To begin, we apply Theorem 3.3 to

$$V \oplus H(P) \subseteq H_2(X_0\#2(S^2 \times S^2); A),$$

where $P = A \oplus A$ and $V = K\pi_2(X_0)$. This gives an isometry

$$\varphi \in G = \langle EU(H(P), Q; V), H(E(P)) \cdot EU(H(P))\rangle,$$

where $Q = P$ or $\bar{P}$, such that $f_* \cdot \varphi$ induces the identity on $H_2(2(S^2 \times S^2); A) \subseteq H_2(X_0 \# 2(S^2 \times S^2); A)$. We finish the proof by showing that for each $\varphi \in G$, $\varphi \oplus Id$ can be realized by a self-homeomorphism on $X_0 \# 4(S^2 \times S^2)$. Note that by definition $G \subseteq \mathrm{Aut}(H_2(X_0 \# 2(S^2 \times S^2); A))$.

The elements of $EU(H(P), Q; V)$ are handled by Corollary 3.7. In addition, we have to realize an arbitrary element in $H(E(P)) \cdot EU(H(P))$, stabilized by the identity, by a self-homeomorphism of $(X_0 \# 4(S^2 \times S^2))$. This follows again from Corollary 3.6 and the considerations above since this group is generated by transvections $\sigma_{p,a,x}$ with $p \in P_0$ or P_1 ([Ba73], p.142-143). □

As in the case of 2-complexes we want to finish this section by stating without proofs two classification results for oriented 4-manifolds with special fundamental groups which follow from more refined cancellation results and Theorems 2.5 and 2.6.

We begin with the complete classification for finite cyclic fundamental groups. The following notation is useful for encoding the different possibilities of the vanishing of the second Stiefel-Whitney class. The *w_2-type* is I, if $w_2(\tilde{M}) \neq 0$, II, if $w_2(M) = 0$, or III, if $w_2(\tilde{M}) = 0$ and $w_2(M) \neq 0$.

Theorem 3.8 ([Fr84], 1-connected case; [HaKr92$_3$], general case) *Let M be a closed, oriented 4-manifold with finite cyclic fundamental group. Then M is classified up to homeomorphism by the fundamental group, the intersection form on $H_2(M, \mathbb{Z})/Tors$, the w_2-type, and the Kirby-Siebenmann invariant. Moreover, any isometry of the intersection form can be realized by a homeomorphism. All invariants can be realized except in the case of w_2-type II, where KS is determined by the intersection form.*

Next we give an explicit bound for the difference between the Euler characteristic e and the absolute value of the signature σ for odd order fundamental groups guaranteeing cancellation. Combined with Theorem 2.5, this gives a homeomorphism classification under these stability assumptions. For any finite group π, let $d(\pi)$ denote the minimal $\mathbb{Z}$-rank for the abelian group $\Omega^3\mathbb{Z} \otimes_{\mathbb{Z}\pi} \mathbb{Z}$. Here we minimize over all representatives of $\Omega^3\mathbb{Z}$, obtained from a free resolution of length three (see section 1) of $\mathbb{Z}$ over the ring $\mathbb{Z}\pi$. Let $b_2(M)$ denote the rank of $H_2(M; \mathbb{Z})$.

Theorem 3.9 [HaKr92$_3$]) *Let M be a closed oriented manifold of dimension four, and let $\pi_1(M) = \pi$ be a finite group of odd order. When $w_2(\tilde{M}) =$*

0 (resp. $w_2(\tilde{M}) \neq 0$), assume that $b_2(M) - |\sigma(M)| > 2d(\pi)$, (resp. $> 2d(\pi) + 2$). Then M is classified up to homeomorphism by the signature, Euler characteristic, type, Kirby-Siebenmann invariant, and fundamental class in $H_4(\pi, \mathbb{Z})/Out(\pi)$.

The *type* is the parity (even or odd) of the intersection form on M.

4 A Homotopy Non-Cancellation Theorem for Smooth 4-Manifolds

In the case of 2-complexes, it was not easy to give non-cancellation examples, e.g., of 2-complexes X and Y such that $X \vee S^2$ is (simple-) homotopy equivalent to $Y \vee S^2$ but X not (simple-) homotopy equivalent to Y. The first examples were only published in 1976 (see references in Chapter I, following (40)).

In the case of topological 4-manifolds, the existence of closed topological 4-manifolds X and Y such that $X\#(S^2 \times S^2)$ is homeomorphic to $Y\#(S^2 \times S^2)$ but X not homeomorphic or equivalently not homotopy-equivalent to Y follows easily from Freedman's classification of 1-connected 4-manifolds (see Theorem 3.8). There are for instance 1-connected topological 4-manifolds X with intersection form $E_8 \oplus E_8$ and Y with intersection form E_{16}, where E_8 and E_{16} are the indecomposable even negative definite unimodular forms over $\mathbb{Z}$ with signature 8 and 16, respectively. These forms become isometric after adding a hyperbolic plane [Se73] and thus by Theorem 3.8, $X\#(S^2 \times S^2)$ is homeomorphic to $Y\#(S^2 \times S^2)$ but X is not homeomorphic to Y.

In the case of smooth 4-manifolds with finite fundamental group, it is not so easy to find non-cancellation examples, which here means manifolds X and Y such that $X\#(S^2 \times S^2)$ is diffeomorphic to $Y\#(S^2 \times S^2)$ but X is not diffeomorphic to Y. The method used above in the topological category finding manifolds with non-isometric definite intersection form which are stably homeomorphic cannot work in the smooth category since by Donaldson's Theorem [Do83] the only definite forms realized as intersection forms of smooth 4-manifolds are up to sign the standard Euclidean forms.

In this situation it is natural to try to make use of the non-cancellation examples of 2-complexes by applying the thickening construction (see the beginning of §2). This was carried out in [KrSc84] and we summarize these examples.

Here is the main result. Recall that we denote the boundary of a thickening of a 2-complex X in $\mathbb{R}^5$ by $M(X)$.

Theorem 4.1 ([KrSc84], Theorem III.3) *Suppose $G = (\mathbb{Z}/p)^s$ is elementary abelian where p is a prime congruent to 1 mod 4 and $s > 1$ is odd. Then there exist finite 2-dimensional CW complexes X and Y such that $M(X)$ and $M(Y)$ are not homotopy equivalent but $M(X)\#r(S^2 \times S^2)$ is diffeomorphic to $M(Y)\#r(S^2 \times S^2)$ for $r > 0$.*

Remark: The homotopy type of 4-manifolds with odd order fundamental group is determined by the quadratic 2-type consisting of the quadruple (π_1, π_2, k, s), where π_2 has to be considered as module over π_1, $k \in H^3(\pi_1; \pi_2)$ is the first k-invariant and s is the equivariant intersection form on π_2 [HaKr88], [Ba88]. In the examples that we will describe in the following, the triple (π_1, π_2, s) is isomorphic for $M(X)$ and $M(Y)$ ([KrSc84], p.21) and thus the manifolds are distinguished by the k-invariant, but this is not the way we prove our result.

The simplest examples for our theorem are derived from Metzler's theorem, a special case of which is stated below (see also Chapter III, §§1 and 2). Note, that a presentation of a group defines a 2-complex with this group as fundamental group by attaching to a wedge of r circles, r the number of generators, 2-cells according to the relations.

Theorem 4.2 ([Me76]) *For $s \geq 2$ and $(q, p) = 1$, the presentations*

$$< a_1, ..., a_s; a_i^p = 1, [a_1^q, a_2] = 1, [a_i, a_j] = 1, 1 \leq i < j \leq s, (i,j) \neq (1,2) >$$

of $(\mathbb{Z}/p)^s$ determine 2-complexes $X(q)$. $X(q)$ and $X(q')$ are not homotopy-equivalent, if $q \neq \pm k^{s-1} q'$ mod p for all k.

If one considers the boundary $M(X(q))$ of a thickening of $X(q)$ one gets examples of non-cancellation examples of smooth 4-manifolds, if Metzler's invariant or some weakening of it survives as invariant of the thickening. We don't know, if the full invariant survives but some partial information does. Theorem 4.1 is a consequence of the following Proposition.

Proposition 4.3 *Let $X(q)$ be as in Theorem 4.2. Then, if $s > 1$ is odd and p is a prime congruent to 1 mod 4, $M(X(q))$ and $M(X(q'))$ are not homotopy equivalent if qq'^{-1} is not a square mod p.*

Since $M(X(q))$ and $M(X(q'))$ are stably diffeomorphic, Theorem 4.1 follows.

Proof: In the following, we give a sketch of the proof of Proposition 4.3. For the details see [KrSc84].

Since $\pi_1(X(q)) = \pi_1(M(X(q))$ and $\pi_1(X(q')) = \pi_1(M(X(q'))$ are isomorphic we choose an isomorphism, a polarization, between them and denote the group by π.

Denote the cellular chain complex over $A = \mathbb{Z}[\pi]$ of the universal covering of $M(X(q))$ and $M(X(q'))$ by C and C'. Then it is easy to show by standard homological algebra that there is a chain map $h : C \to C'$ inducing the identity on $H_0(..;\mathbb{Z})$ and $H_4(..;\mathbb{Z})$. Denote the 0th Tate cohomology of an A-module M by $\hat{H}^0(M) = M^\pi/N(M)$, where M^π is the fixed point set and $N(M)$ consists of the norm elements. If f is an A-module homomorphism we denote the induced map between the Tate cohomologies by $\hat{H}^0(f)$. If h is a chain map as above, then $\hat{H}^0(h_*)$ is an isomorphism.

Consider the equivariant intersection form on the middle homology of the universal covering. This induces an equivariant symmetric bilinear form on $H_2(\tilde{X}(q))^\pi = H_2(C)^\pi$. Any orientation preserving homotopy equivalence which induces the given isomorphism on π_1 induces a map respecting this bilinear form.

Thus, if $M(X(q))$ and $M(X(q'))$ are orientation preserving homotopy equivalent inducing the given isomorphism on π_1, then $\hat{H}^0(h_*)$ is induced by an **isometry** from $H_2(C)^\pi$ to $H_2(C')^\pi$.

Thus we get an invariant of polarized oriented homotopy types by the set of all isomorphisms $\hat{H}^0(h_*)$ modulo those induced by isometries from $H_2(C)^\pi$ to $H_2(C')^\pi$. Dividing out the different choices of polarizations equivalently of automorphisms of π and using the fact that $M(X)$ always admits an orientation reversing diffeomorphism ($M(X)$ can be described as a double of a 4-dimensional thickening and interchanging the two halves gives the orientation reversing diffeomorphism), one gets a homotopy invariant.

The main work of [KrSc84] is to show that this invariant is non-trivial if qq'^{-1} is not a square mod p. For this one can rather easily compute a representative of this invariant but it is not so easy to decide when it is non-trivial. We get our result by weakening the invariant, namely we pass to an L-theoretic invariant. More precisely, it is not difficult to show that the restriction of the intersection form to $H_2(C)^\pi$ is up to scaling by a constant a hyperbolic form over $\mathbb{Z}$. It induces the hyperbolic form over $\mathbb{Z}/p$ on $\hat{H}^0(H_2(C))$. After appropriately identifying $H_2(C)$ with $H_2(C')$ our invariant given by $\hat{H}^0(h_*)$ gives an automorphism of determinant 1 of $\hat{H}^0(H_2(C))$, which turns out to be an isometry. Stable equivalence classes of isometries of determinant 1 represent elements in the Wall group $L_1^0(\mathbb{Z}/p)$ [Wa70]. If $M(X(q))$ and $M(X(q'))$ are homotopy equivalent, this element in $L_1^0(\mathbb{Z}/p)$ is induced from an isometry of $H_2(C)$. Since an isometry of a scaled hyperbolic form over $\mathbb{Z}$ is an isometry of the hyperbolic form itself it is in the image of the reduction map from $L_1^0(\mathbb{Z})$ to $L_1^0(\mathbb{Z}/p)$.

For the manifolds $M(X(q))$ and $M(X(q'))$, $\hat{H}^0(H_2(C))$ is isometric to the hyperbolic form on $\mathbb{Z}/p$ and the invariant in $L_1^0(\mathbb{Z}/p)$ is represented by a diagonal matrix of rank 2 over $\mathbb{Z}/p$ with entries (qq'^{-1}) and $(qq'^{-1})^{-1}$. The different choices of a polarization of the fundamental groups correspond to an action of $Aut(\pi)$. It turns out that $Aut(\pi)$ acts on $\hat{H}^0(H_2(C))$ by diagonal matrices of rank 2 over $\mathbb{Z}/p$ with entries r^{s-1}, r^{1-s} for some r prime to p. Thus, if s is odd, the action is trivial.

To finish the proof, we need the following information from [Wa76]. The Wall group $L_1^0(\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$ generated by the diagonal matrix of rank 2 with entries $(-1,-1)$. The Wall group $L_1^0(\mathbb{Z}/p)$ is isomorphic to $\mathbb{Z}/2$ generated by a diagonal matrix of rank 2 with entries (r, r^{-1}), where r is a non-square mod p. Thus, if p is congruent to 1 *mod* 4, the reduction map is trivial finishing the argument. □

Remark: Comparing Theorem 4.2 and Proposition 4.3 one sees that the invariant used there is considerably weaker than Metzler's. It would be interesting to know if one actually is losing information by passing from 2-complexes to boundaries of 5-dimensional thickenings.

5 A Non-Cancellation Example for Simple-Homotopy Equivalent Topological 4-Manifolds

The non-cancellation examples in Section 4 were non-homotopy equivalent but stably diffeomorphic smooth 4-manifolds. As mentioned before one can get other examples from exotic structures on closed smooth oriented 4-manifolds. They are homeomorphic, not diffeomorphic but stably diffeomorphic. Such examples are much more complicated than the ones described in Section 4 since the only known way to distinguishing them is by Donaldson invariants. We will describe many exotic structures in Section 6.

The most delicate question one can ask in the topological category in connection with non-cancellation examples is whether there are simple- homotopy equivalent non-homeomorphic but stably homeomorphic topological closed 4-manifolds. Recently, in joint work with Peter Teichner, we found the first examples of this type. We will describe them here. The examples constitute another link between 2-dimensional topology and 4-manifolds since they are distinguished by a codimension 2 invariant.

We begin with a notation. According to Freedman there exists a unique non-smoothable 4-manifold which is homotopy-equivalent to $\mathbb{CP}^2$, the Chern manifold denoted $\mathbb{CH}$. We will see that there is a similar manifold corresponding to $\mathbb{RP}^4$, a unique non-smoothable 4-manifold homotopy equivalent to $\mathbb{RP}^4$, denoted $\mathbb{RH}$.

Theorem 5.1 ([HaKrTe92]) *The simple-homotopy equivalent closed 4-manifolds $\mathbb{RP}^4 \# \mathbb{CP}^2$ and $\mathbb{RH} \# \mathbb{CH}$ are not homeomorphic but homeomorphic after connected sum with r copies of $S^2 \times S^2$.*

Remark: In [HaKrTe92] it is actually shown that $r = 1$ works, but we don't need this to get our non-cancellation examples. We don't know whether $\mathbb{RH} \# \mathbb{CH}$ admits a smooth structure. The only known obstruction, the Kirby-Siebenmann obstruction vanishes, since it is non-trivial on both summands and is additive under connected sum. If a smooth structure exists, then one gets examples of stably diffeomorphic simple-homotopy equivalent smooth 4-manifolds that are not homeomorphic.

Proof: We begin with the construction of $\mathbb{RH}$. According to Freedman [Fr84], there exists a unique simply connected topological 4-manifold with intersection form isomorphic to E_8, the unique negative definite form with signature -8. We denote this manifold by $M(E_8)$. The Kirby-Siebenmann obstruction of $M(E_8)$ is $KS(M(E_8)) = 1$. This follows since the Kirby Siebenmann obstruction of a *TopSpin*-manifold (i.e. w_1 and w_2 vanish) is equal to $1/8\sigma(M)$ *mod* 2. Consider $\mathbb{RP}^4 \# M(E_8)$. The quadratic intersection form of this manifold is $E_8 \otimes_{\mathbb{Z}} A$, where $A = \mathbb{Z}[\mathbb{Z}/2]$ equipped with the anti-involution which here in the non-oriented case maps the nontrivial element τ in $\mathbb{Z}/2$ to $-\tau$. This form is stably (i.e. after adding a hyperbolic form) isomorphic to a hyperbolic form. This follows for instance from the fact that the map of Wall groups $L_0(\mathbb{Z}) \to L_0(\mathbb{Z}[\mathbb{Z}/2])$ is trivial [Wa70]. By Freedman [Fr84], one can decompose the manifold as the connected sum of some topological manifold M' and $\# \ r(S^2 \times S^2)$, if the quadratic intersection form of a manifold M splits off a hyperbolic form of rank $2r$. Applying this to $\mathbb{RP}^4 \# M(E_8) \# r(S^2 \times S^2)$ one can decompose this as the connected sum of $(r+8)(S^2 \times S^2)$ and some manifold which we will denote by $\mathbb{RH}$. By construction this manifold has fundamental group $\mathbb{Z}/2$ and Euler characteristic 1. Thus the manifold is homotopy equivalent to $\mathbb{RP}^4$. One can prove that this manifold is unique up to homeomorphism but for our context we don't need this and call any manifold constructed this way by the same name. Since

$KS(M(E_8)) = 1$ and the Kirby-Siebenmann obstruction is additive under connected sum, $KS(\mathbb{RH}) = 1$.

Next we show that $\mathbb{RP}^4\#\mathbb{CP}^2$ and $\mathbb{RH}\#\mathbb{CH}$ are stably homeomorphic. For this we apply Theorem 2.3. Obviously, both manifolds have the same normal 1-type: $[\mathbb{Z}/2, x, \infty]$, where x generates $H^1(\mathbb{Z}/2;\mathbb{Z}/2)$. The geometric normal 1-type is the trivial fibration $Id : BO \to BO$. Thus the relevant bordism group is the non-oriented topological bordism group $\mathfrak{N}_4^{Top}$, which is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, detected by w_1^4, w_4 and KS. This follows, since, if $KS = 0$, the manifold is bordant to a smooth manifold ([Fr84], [FrQu90]) and the smooth non-oriented bordism group is detected by w_1^4 and w_4 [Th54]. By construction, all these invariants agree for $\mathbb{RP}^4\#\mathbb{CP}^2$ and $\mathbb{RH}\#\mathbb{CH}$. Thus, by Theorem 2.3, they are stably homeomorphic.

To finish, we have to show that they are not homeomorphic. This will follow from the construction and computation of an invariant which roughly speaking is defined as follows. Let M be one of the manifolds we want to distinguish. $H^2(M;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. Let $c \in H^2(M;\mathbb{Z})$ be a class which reduces to $w_2\nu(M)$ and which generates $H^2(M;\mathbb{Z})/Tors$. Such a class is unique up to sign. Now, represent c by a map to $\mathbb{CP}^N$ for some large N. After making this map transversal to $\mathbb{CP}^{N-1}$, the inverse image of $\mathbb{CP}^{N-1}$ is a surface Σ in M (transversality holds in the topological category, see e.g. [FrQu90]) and it inherits from M a so called normal Pin^+-structure, which is unique up to sign in the corresponding bordism group (for details see [HaKrTe92], §2). Here Pin^+ is the central extension

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow Pin^{\pm}(n) \longrightarrow O \longrightarrow 0$$

classified by $w_2 + w_1^2$. We obtain a fibration

$$p : BPin^+ \longrightarrow BO.$$

A normal Pin^+-structure is a lift of the normal Gauss map to Pin^+. According to Brown [Br72], a Pin^+-structure on a surface Σ determines a quadratic refinement with values in $\mathbb{Z}/4$ of the intersection form on $H^2(\Sigma;\mathbb{Z}/2)$. The Witt group of such forms is isomorphic to $\mathbb{Z}/8$ and the corresponding element represented by the quadratic refinement on Σ is denoted by $\pm\text{arf}\ (M) \in \mathbb{Z}/8$.

This is our invariant and it is obviously a homeomorphism invariant. Note that one can define the same sort of invariant on $M\#r(S^2 \times S^2)$ after choosing a cohomology class c reducing to w_2. But, if $r > 0$, this invariant depends on the choice of c (not only up to sign) and loses all its information (to indicate the dependence on c we denote the invariant by arf (M, c)). But it turns out that it takes different values for $\mathbb{RP}^4\#\mathbb{CP}^2$ and $\mathbb{RH}\#\mathbb{CH}$.

The reason for this is the following. The class c is the sum c_1+c_2 corresponding to the connected sum decomposition of our manifolds. The arf-invariant is additive under connected sum. For oriented manifolds, one has the following formula ([KiTa89], Cor.9.3):

$$2 \cdot \text{arf } (M,c) = c \circ c - \sigma(M) + 8 \cdot KS(M) \text{mod } 16,$$

where $\sigma(M)$ is the signature of M. Thus, arf $(\mathbb{CP}^2, c_2) = 0$ and arf $(\mathbb{CH}) = 4$ (mod)8. By construction of $\mathbb{RH}$ we see that $(\mathbb{RH}, c_1)\#(4(S^2 \times S^2), 0) = (\mathbb{RP}^4, c_1)\#(E_8, 0)$. Thus, from the formula above, $\pm$arf $(\mathbb{RP}^4, c_1) = \pm$arf $(\mathbb{RH}, c_1)$. □

6 Application of Cancellation to Exotic Structures on 4-Manifolds

In this section, we study the existence of exotic structures on many algebraic surfaces with finite fundamental group. From the point of view of cancellation problems for 4-manifolds the construction of exotic structures on oriented closed 4-manifolds is equivalent to the construction of homeomorphic smooth manifolds which are stably diffeomorphic but not diffeomorphic. The reason for this is that homeomorphic oriented smooth closed 4-manifolds are automatically stably diffeomorphic, a result that can rather easily be derived from Theorem 2.3 by comparing the topological and the smooth bordism group of the corresponding normal 1-type ([Kr84$_1$], for another proof see [Go84]). To distinguish stably diffeomorphic smooth oriented closed 4-manifolds, one has to find rather delicate invariants. These are provided by the Donaldson polynomials [Do90], which are defined for closed oriented smooth 4-manifolds with some additional restrictions. For instance, these restrictions are fulfilled for all 1-connected algebraic surfaces. We will base our examples of exotic structures on the following result of Donaldson.

Theorem 6.1 ([Do90]) *Let X be a 1-connected compact algebraic surface without singularities. Then X is not diffeomorphic to a connected sum $M_1\#M_2$ unless M_1 or M_2 have negative definite intersection form.*

To apply this theorem to the construction of exotic structures on closed 4-manifolds, it is sufficient to find an algebraic surface with finite fundamental group X and a smooth 4-manifold M, such that X and M are homeomorphic but the universal covering $\tilde{M}$ is diffeomorphic to a connected sum $M_1\#M_2$ where M_1 and M_2 do not have negative definite intersection form.

The following result is an application of this method showing the existence of an exotic structure on surfaces where the sum of the signature σ and the Euler characteristic e is sufficiently large.

Theorem 6.2 ([HaKr90]) *Let π be a finite group. Then there is a constant $c(\pi)$ such that a compact non-singular algebraic surface X with $\pi_1(X) \cong \pi$ and $\sigma(X) + e(X) \geq c(\pi)$ has at least two smooth structures.*

Note that by a construction of Shafarevic ([Sh74, p. 402 ff]) for each finite group π there are algebraic surfaces with fundamental group π and arbitrarily large $\sigma(X) + e(X)$ (compare, [HaKr90, p. 109] and the following remark).

Remark: In [HaKr90], we used instead of $\sigma(X) + e(X) \geq c(\pi)$ the condition $c_1^2(X) \geq 0$ and $e(X)$ sufficiently large. We thank Stefan Bauer for pointing out that our proof works under this slightly better condition.

Proof: The first ingredient in the proof is the following Proposition. We say that two closed topological 4-manifolds M_0 and M_1 are *weakly stably homeomorphic* if there exists a natural number r and and integers s_0 and s_1 such that $M_0 \# r(S^2 \times S^2) \# s_0 K$ is homeomorphic to $M_1 \# r(S^2 \times S^2) \# s_1 K$. Here K is the Kummer surface (K_3-surface), the quartic in $\mathbb{CP}^3$, and for s negative we mean by sK the connected sum of $-s$ copies of K with its negative orientation. Recall that K is a 1-connected 4-manifold with signature -16 and Euler characteristic 24.

Proposition 6.3 *Let π be a finite group. Then the set of weakly stable homeomorphism classes of closed smooth oriented 4-manifolds with fundamental group π is finite.*

With this proposition we proceed as follows. For each weakly stable homeomorphism class α, choose a representative M_α with $e(M_\alpha)$ minimal and $-8 \leq \sigma(M_\alpha) < 8$ and suppose $M_\alpha \cong M'_\alpha \# S^2 \times S^2$, if π is trivial. Then, for each closed oriented smooth 4-manifold X with fundamental group isomorphic to π, there exist α and s such that X is stably homeomorphic to $M_\alpha \# sK$. If $e(X) > e(M_\alpha \# sK)$, then Theorem 3.1 implies that X is homeomorphic to $Y = M_\alpha \# r(S^2 \times S^2) \# sK$ for some $r > 0$. Now, Donaldson's Theorem 6.1 implies that, if X is an algebraic surface, then X and Y are not diffeomorphic, since $\tilde{X}$ is again a compact algebraic surface and for π non-trivial $\tilde{Y} = Y' \# \widetilde{S^2} \times S^2$ decomposes as the connected sum of two smooth manifolds with indefinite intersection forms, and for π trivial we assumed that the same holds for Y. Now the proof of Theorem 6.2 is finished if we can find a number $c(\pi)$ such that $e(X) \geq e(M_\alpha \# sK)$ for any algebraic surface X with fundamental group π and $\sigma(X) + e(X) \geq c(\pi)$. It is actually enough to do this for minimal surfaces X since the condition $\sigma(X) + e(X) \geq c(\pi)$ is invariant under blow ups and also $X \# k \cdot \overline{\mathbb{CP}}^2$ and $Y \# k \cdot \overline{\mathbb{CP}}^2$ remain non-diffeomorphic by Donaldson's Theorem.

To compare for a minimal surface X, $e(X)$ with $e(M_\alpha \# s \cdot K)$, we express $e(M_\alpha \# s \cdot K)$ in terms of $e(M_\alpha)$, $\sigma(M_\alpha)$ and $\sigma(X)$:

$$e(M_\alpha \# s \cdot K) = e(M_\alpha) + 22 \cdot \mid s \mid$$

and

$$\sigma(X) = \sigma(M_\alpha \# s \cdot K) = \sigma(M_\alpha) - 16s.$$

This implies

$$e(M_\alpha \# s \cdot K) = e(M_\alpha) + (11/8) \mid \sigma(X) - \sigma(M_\alpha) \mid .$$

We have the following inequality for algebraic surfaces:

$$e(X) - (11/8) \mid \sigma(X) \mid \geq (1/12)e(X).$$

If $\sigma(X) < 0$, this is an immediate consequence of the signature theorem ($\sigma(X) = \frac{c_1^2(X) - 2e(X)}{3}$) and the fact that a minimal surface has $c_1^2 \geq 0$. If $\sigma(X) \geq 0$, the signature theorem implies

$$e(X) - (11/8) \mid \sigma(X) \mid \geq (1/12)e(X) + (11/24)(4e(X) - c_1^2(X)).$$

Thus we are finished for surfaces fulfilling $(4e - c_1^2) \geq 0$. For surfaces of general type, this is a consequence of the inequality of Miyaoka-Yau ([BaPeVa84], p. 212). The only minimal surfaces with finite fundamental group and $\sigma(X) \geq 0$ are diffeomorphic to $\mathbb{C}P^2$, $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ (this follows from the Enriques-Kodaira classification ([BaPeVa84], p. 187ff)), for which the inequality holds.

Using this inequality together with the formula for $e(M_\alpha \# s \cdot K)$ above (note that $\sigma(X) = \sigma(M_\alpha)$ mod 16 and $-8 \leq \sigma(M_\alpha) < 8$) we get:

$$\begin{aligned} e(X) - e(M_\alpha \# s \cdot K) &= e(X) - (11/8) \mid \sigma(X) \mid - e(M_\alpha) \pm (11/8)\sigma(M_\alpha) \\ &\geq (1/12)e(X) - e(M_\alpha) - 11. \end{aligned}$$

Since $2e(X) > \sigma(X) + e(X)$, we see that if $\sigma(X) + e(X) \geq 24(e(M_\alpha) + 11)$ we have $e(X) > e(M_\alpha \# s \cdot K)$.

As there are only finitely many M_α's, we can define

$$c(\pi) := 24 \cdot \max\{e(M_\alpha) + 11\},$$

finishing the proof of our theorem. □

Proof: (of Proposition 6.3) The proof is an application of Theorem 2.3. First, we note that for a fixed algebraic normal 1-type $[\pi, w_1, w_2]$, the bordism group $\Omega_4(B^{Top}[\pi, w_1, w_2]) \otimes \mathbb{Q}$ is isomorphic to $\mathbb{Q}$, the isomorphism is given by the signature. This is an easy consequence of the Atiyah-Hirzebruch spectral sequence. K is a 1-connected spin-manifold and thus the connected sum with K does not change the algebraic normal 1-type. Since $\sigma(K) = -16$, the set of weakly stable homeomorphism classes of manifolds with fixed algebraic normal 1-types is finite. But, if we fix π, the set of algebraic normal 1-types is finite since $H^1(\pi;\mathbb{Z}/2)$ and $H^2(\pi;\mathbb{Z}/2)$ are finite. □

This result and stronger results for special fundamental groups led us to the following conjecture.

Conjecture: *A compact non-singular algebraic surface with finite fundamental group has at least two smooth structures.*

We note that a minimal surface with finite fundamental group has $c_1^2 \geq 0$ (this follows from the classification, e.g. [BaPeVa84], p. 188). But if $c_1^2 \geq 0$ and $\sigma(X) + e(X) < c(\pi)$, the Euler characteristic can only take finitely many values. On the other hand, there are only finitely many homeomorphism types of closed oriented 4-manifolds with prescribed finite fundamental group π and fixed Euler characteristic ([HaKr88], Corollary 1.5). Thus we obtain:

Corollary 6.4 *Let π be a finite group. Then the conjecture holds for all but perhaps a finite number of homeomorphism types of minimal algebraic surfaces X with fundamental group π.*

Based on similar arguments as above and some more delicate computations of Donaldson invariants one gets the following result, which we state without proof.

Theorem 6.5 ([HaKr92$_3$]) *(i) The conjecture holds for all algebraic surfaces with finite non-trivial cyclic fundamental group.*

(ii) The conjecture holds for all elliptic surfaces X with finite fundamental group except perhaps if X has geometric genus 0, where the statement holds after blowing up once replacing X by $X\#\overline{\mathbb{C}P}^2$.

7 Topological Embeddings of 2-Spheres into 1-Connected 4-Manifolds and Pseudo-free Group Actions

We finish this paper with two further applications of cancellation to 4-dimensional topology. The first is again a link between 2- and 4-dimensional topology and concerns the existence and uniqueness of locally flat simple embeddings of 2-spheres in a 1-connected 4-manifold N. These problems were substantially settled in [LeWi90] for homology classes of odd divisibility. Let $x \in H_2(N;\mathbb{Z})$. Then $x = dy$ with y primitive and d is called the *divisibility* of x. Such embeddings are called *simple* if the fundamental group of the complement is abelian (and hence isomorphic to $G = \mathbb{Z}/d$). Denote $y \cdot y$ by m, and let $b_2(N)$ and $\sigma(N)$ denote the rank and signature of the intersection form on $H_2(N;\mathbb{Z})$. A homology class x is called *characteristic*, if its reduction mod 2 is dual to w_2.

Theorem 7.1 ([HaKr92$_2$]) *Let N be a closed 1-connected topological 4-manifold.*

i) Let $x \in H_2(N;\mathbb{Z})$ be a homology class of divisibility $d \neq 0$. Then x can be represented by a simple locally flat embedded 2-sphere in N if and only if

$$KS(N) = (1/8)(\sigma(N) - x \cdot x) \bmod 2$$

when x is a characteristic class, and if

$$b_2(N) \geq \max_{0 \leq j < d} |\sigma(N) - 2j(d-j)(1/d^2)x \cdot x|.$$

ii) Any two locally flat simple embeddings of S^2 in N representing the homology class x are ambiently isotopic if $b_2(N) > |\sigma(N)| + 2$ and

$$b_2(N) > \max_{0 \leq j < d} |\sigma(N) - 2j(d-j)(1/d^2)x \cdot x|.$$

The proof will be based on the original idea of V. Rochlin (as in [LeWi90]). The embedding problem will be studied via an associated semi-free cyclic group action which is the same as a branched covering: if $f : S^2 \to N$ is an embedding representing a homology class of divisibility d, then there is a d-fold branched cyclic covering (M, G) over N, branched along $f(S^2)$.

This correspondence connects the embedding problem with the second topic of this section. It is the classification of actions of finite cyclic groups on 1–connected 4–manifolds, where we assume that the group action has a singular set consisting of isolated points. We also assume that the singular set of the action is *non-empty*: free actions, or equivalently 4-manifolds with finite cyclic fundamental group, were classified in Theorem 3.8. For earlier work in this direction compare [EdEw90], [Wi90]. The following result is a slight generalization of ([HaKr92$_2$], Corollary 4.1).

Theorem 7.2 (compare [HaKr92$_2$], Corollary 4.1) *Let M be a closed, oriented, simply-connected topological 4–manifold. Let G be a finite cyclic group* *acting locally linearly on M, preserving the orientation, with non-empty finite singular set. Let M_0 denote the complement of a set of disjoint open G-invariant 4-disks around the singular set, and assume that $X = M_0/G = W\#(S^2 \times S^2)$, where $\partial W = \partial(M_0/G)$. Then the action (M, G) is classified up to equivariant homeomorphism by the w_2-type, the local singular data, the signature and Euler characteristic of M and the Kirby–Siebenmann invariant of M_0/G.*

The "w_2-type" is I, II or III, if $w_2(M) \neq 0$, if $w_2(X) = 0$ or if $w_2(M) = 0$ and $w_2(X) \neq 0$ resp. The "local singular data" is the equivalence class of pairs consisting of the tangential G–representations at the singular set together with, when M is spin and $|G|$ is even, a preferred set of spin structures on the lens spaces bounding $X = M_0/G$. To describe this preferred set note that the w_2-type determines the normal 1-type of X. If M is spin and $|G|$ is even, then a normal 1-smoothing on X determines a spin-structure on $\nu(X) - L$, where L is a complex line bundle with $w_2(L) = w_2(X)$ and both possible spin-structures occur. Now, consider the boundary components $\partial_i X$. If the map from $H^2(X;\mathbb{Z}/2)$ to $H^2(\partial_i X;\mathbb{Z}/2)$ is non-trivial for some i then it is an isomorphism and, since $\partial_i X$ is spin, X is spin. In this case we choose L the trivial bundle and the preferred set of spin structures is the restriction of any spin structure on X to ∂X. If the map from $H^2(X;\mathbb{Z}/2)$ to $H^2(\partial_i X;\mathbb{Z}/2)$ is trivial for all i, then the restriction of L to $\partial_i X$ is stably trivial for all i and a normal 1-structure on X determines a spin structure on ∂X. Any of these gives the preferred set in this second case.

We also remark that $KS(M_0/G) = KS(M_0) = KS(M)$ when G has odd order, since connected sum with the Chern manifold changes the $\mathbb{Z}/2$-valued Kirby-Siebenmann invariant.

The proof of both theorems is similar in spirit but the proof of Theorem 7.1 is rather lengthy. We will prove Theorem 7.2 in detail and only give a sketch for Theorem 7.1 and refer to [HaKr92$_2$] for the details. Let G act on M with fixed point set either a 2-sphere and semi-free action (Theorem 7.1, ii)) or with finite singular set with prescribed fixed point data (Theorem 7.2). Then we denote by M_0 the complement of an open equivariant tubular neighborhood around the fixed point set resp. singular set. Given another action choose a homeomorphism between the boundaries of $X = M_0/G$. We have to show that the homeomorphism type of M_0/G rel. boundary is determined by the data. For this one first proves that the homeomorphism extends stably. This is an application of a relative version of Theorem 2.3. This relative version says that a homeomorphism between two compact topological 4-manifolds M_0 and M_1 with the same algebraic normal 1-type $[\pi, w_1, w_2]$ extends to a stable homeomorphism, if and only if they have the same Euler characteristic and if they admit normal 1-smoothings $\bar{\nu}_0$ and $\bar{\nu}_1$ resp., which are compatible with the homeomorphism between the boundaries and such that the union of $(M_0, \bar{\nu}_0)$ and $(M_1, \bar{\nu}_1)$ via the homeomorphism along the boundaries represent zero in $\Omega_4(B^{Top}[\pi, w_1, w_2])$ ([Kr85], Theorem 2.1). By assumption there exist compatible normal 1-smoothings. In our situation, the Atiyah-Hirzebruch spectral sequence implies that this bordism group is determined by the signature and Kirby-Siebenmann obstruction. Then one uses a relative version of Theorem 3.1 to cancel ([HaKr92$_2$], Corollary 3.6). For this one has to show that one can split off $S^2 \times S^2$ from M_0/G, something which is assumed in Theorem 7.2, and which follows from the inequalities in Theorem 7.1, ii) and the existence result in Theorem 19 i). This finishes the proof of Theorem 7.2. For Theorem 7.1, ii) one has to show that the resulting homeomorphism of N mapping the two embedded 2-spheres into each other is isotopic to Id. For this one carries the program above out with more care to control the induced map on homology which has to be the identity. Then one applies a Theorem from [Kr79] which says that a self-homeomorphism on a 1-connected 4-manifold inducing Id on homology is pseudo-isotopic to the identity. By a theorem of Perron [Pe86], this implies the existence of an isotopy.

To prove Theorem 7.1 i) one uses again a stabilization argument. The point will be to construct an embedding of S^2 into $N' = N \# r(S^2 \times S^2)$ for some r representing $x + 0$. Now, consider the ramified covering M' over N', ramified over the embedded 2-sphere. One has to carry out the construction of N' and the embedding in such a way that $H_2(M'; \mathbb{Z})$, considered as module over $\mathbb{Z}[G]$ with equivariant intersection form splits off a hyperbolic summand of rank r, such that the fixed point set under the G-action on this orthogonal complement is isomorphic to $H_2(N; \mathbb{Z})$ and the homology class represented by the embedded 2-sphere is x. This will follow from some purely algebraic arguments. Then it is not difficult to cancel the hyperbolic summand geometrically using Freedman's techniques, to realize the homology class x by an embedded 2-sphere in the original manifold N.