

The Potential Future Exposure of Path-dependent Instruments: Comment on Lomibao-Zhu's Approach

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Abstract

Lomibao & Zhu (2005) proposed a conditional valuation approach for simulating the Expected Exposure (EE) profile of path-dependent instruments. Since their approach simulates Value at Future (VaF)¹ instead of future Mark-to-Market (MtM) value, it is inappropriate to use for generating the Potential Future Exposure (PFE) profile. In this article, we propose a new and simple methodology to correctly simulate the PFE profile of path-dependent instruments. We also suggest a *modified* Lomibao-Zhu (LZ) approach to improve their simulation algorithm. For the numerical example on an up-and-out call option, we find that the LZ approach underestimates the upper quantiles and overestimates the lower quantiles of the PFE.

Keywords: Potential Future Exposure; Expected Exposure; Path-dependent Instruments; Counterparty Credit Risk; First Passage Time.

1 Introduction

Counterparty credit risk (CCR) is the loss due to counterparty default on a pool of over-the-counter (OTC) contracts. Typical risk management of CCR involves quantifying the PFE profile of the portfolio at counterparty level. The PFE profile characterises what the maximum loss might incur due to counterparty default at some future time. Let 0 denote current time and T denote the maximum maturity of the portfolio. Let MtM_t denote the MtM at future time $t \in (0, T]$. The PFE at time t , denote as PFE_t , is defined as the maximum of zero and the MtM value.

$$PFE_t = \max\{0, MtM_t\}. \quad (1)$$

When the MtM at t is negative, no loss will incur if the counterparty default at this time. We are therefore only concerned with the positive part of the MtM at future. If the MtM is non-negative for all $t \in (0, T]$, then the PFE will be just MtM itself, such as longing options.

Various risk measures can then be defined on the PFE. Two important profile statistics are the Expected Exposure (EE), denoted as EE_t , and the α -percentile of the PFE, denoted as PFE_t^α which are defined as follows respectively

$$PFE_t^\alpha \quad s.t. \quad P(PFE_t \geq PFE_t^\alpha) \leq 1 - \alpha; \quad (2)$$

$$EE_t = \mathbb{E}[PFE_t]; \quad (3)$$

where the expectation is taken on all the scenarios of the PFE at time t . For high percentile, $\alpha = 95\%$ for instance, PFE_t^α is also known as Peak Exposure (PE). For a review of CCR and various other risk measures, the readers are directed to Prisco & Rosen (2005).

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¹Please see section 2 for Lomibao-Zhu's definition of VaF.

In order to compute these risk measures, it is important to correctly simulate the risk factors into the future and then reevaluate the portfolio for each scenarios. For path-dependent instruments, ideally, one should simulate the whole path of the underlying risk factors. However, this is practically impossible due to apparent computation limitations. Lomibao & Zhu (2005) therefore proposed a conditional valuation approach for simulating the Expected Exposure (EE) profile of path-dependent instruments. However, their approach is inappropriate to use for generating the PFE profile. This is because that their approach simulates VaF based on Brownian bridge instead of future MtM value.

In this article, we propose a new and simple methodology to correctly simulate the PFE profile of path-dependent instruments. We also suggest a *modified* LZ approach to improve their simulation algorithm. For the numerical example on an up-and-out call option, we find that the LZ approach underestimates the upper quantiles and overestimates the lower quantiles of the PFE.

The rest of this article is organized as follows. Section 2 reviews the LZ approach and discusses the modified LZ approach. Section 3 proposes our new and simple simulation algorithm. A numerical illustration is given in Section 4. We conclude this article in Section 5.

2 LZ Approach and Modified LZ Approach

For narrative simplification, we will use an up-and-out call option for our discussion for the rest of this article. This is also used as an example in Lomibao & Zhu (2005). However, the simulation approaches that will be discussed are general and can be applied for other path-dependent instruments as well.

Let us consider an up-and-out call with barrier H , strike K ($K < H$) and maturity T . Assume that the underlying stock price S_t follows Geometric Brownian Motion (GBM) under historical measure given by²

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}. \quad (4)$$

Given the path of the stock price from zero to a future time $t \in (0, T]$, the MtM of this option at time t is zero if this path has exceeded the barrier H before t (knocked out). Otherwise, the MtM of this option at time t is non-zero and can be computed from the formula in Appendix B. Let τ denote the knock-out and c_t^{uo} denote this non-negative value. Then the future MtM at time t can be written as

$$\text{MtM}_t = c_t^{uo} \mathbf{1}_{\{\tau > t\}}. \quad (5)$$

It is clear that MtM_t is a mixture distribution which has a positive mass at zero with a continuous density from zero to a positive number.

Lomibao & Zhu (2005) did not simulate the future MtM value. Instead, they simulate so called Value at Future (VaF) which they defined as:

$$\text{VaF}_t = c_t^{uo} P(\tau > t | S_t), \quad (6)$$

where $P(\tau > t | S_t)$ has an explicit expression given by

$$P(\tau > t | S_t) = \begin{cases} 1 - (S_t/H)^{\frac{2 \ln(H/S_0)}{\sigma^2 t}} & \text{if } S_t < H; \\ 0 & \text{if } S_t \geq H. \end{cases} \quad (7)$$

Their approach can simply be summarised as follows for an up-and-out call.

- Simulate S_t as log normal distribution given by equation (4);
- Calculate $P(\tau > t | S_t)$ from Equation (7);
- Compute VaF_t from Equation (6).

Their exposure profiles are then calculated based on this simulated VaF. The EE profile based on this VaF is the same as the one based on future MtM value. Tower property of conditional expectation implies that

$$\mathbb{E}[\text{MtM}_t] = \mathbb{E}[\mathbb{E}[c_t^{uo} \mathbf{1}_{\{\tau > t\}} | S_t]] = \mathbb{E}[\text{VaF}_t].$$

²We assume zero drift and zero interest rate for simplicity.

However their quantiles will differ simply because VaF_t and MtM_t are two different random variables for a given future time t . The distribution of VaF_t differs from the distribution of MtM_t in two ways. First, the mass at zero. For VaF_t , the probability of taking zero is $P(S_t \geq H)$ while it is $P(\tau < t)$ for MtM_t . These two probabilities are related through³

$$P(\tau < t) = P(S_t \geq H) + P(S_t < H, \tau < t).$$

Therefore VaF_t has less mass at zero than MtM_t has. As a result, for small α , VaF_t is likely to overestimate the quantile of the exposure than MtM_t does. Second, continuous part of these two random variables are also different. The random variable $VaF_t|S_t < H$ has a smaller range than $MtM_t|\tau > t$ does. This is due to the fact that the continuous component of VaF_t is the product of c_t^{uo} and a probability, which is less than one. Consequently, for large α , VaF_t is likely to underestimate the quantile of the exposure than MtM_t does. The LZ approach can be corrected by simulating an extra Bernoulli variable as follows.⁴

- Simulate S_t as log normal distribution given by equation (4);
- If $S_t \geq H$ set $MtM_t=0$;
- If $S_t < H$, calculate conditional survival probability p ;

$$p := P(\tau > t|S_t) = 1 - (S_t/H)^{\frac{2 \ln(H/S_0)}{\sigma^2 t}}; \quad (8)$$

- Simulate $B \sim Bernoulli(p)$ set

$$MtM_t = \begin{cases} c_t^{uo} & \text{if } B=1; \\ 0 & \text{if } B=0. \end{cases} \quad (9)$$

It is clear that the modified LZ approach can correctly simulate MtM_t . The drawback of this approach is that it can not be easily generalized to multi-dimensional case. This is due to the difficulties of finding and simulating appropriate multi-variate Bernoulli random variables, see Chaganty & Joe (2006) for a discussion.

3 Our New Approach

The knock-out time τ can be mathematically written as

$$\tau = \inf\{t > 0 : S_t = H\} = \inf\{t > 0 : X_t = 0\}; \quad (10)$$

where

$$X_t = \ln(S_0/H) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t. \quad (11)$$

In order to simulate MtM_t , it is essential to simulate the distribution of drifted Brownian motion X_t conditional on survival $\tau > t$. Fortunately, this distribution $f_{X_t|\tau > t}(x)$ has an explicit expression given by Appendix A. The following is the simulation steps for simulating MtM_t .

- Simulate $X_t|\tau > t$ according to $f_{X_t|\tau > t}(x)$ given by equation (17);
- Calculate $S_t|\tau > t = He^{X_t}|\tau > t$;
- Plug $S_t|\tau > t$ into c_t^{uo} ;
- Plot the histogram of c_t^{uo} together with mass of $P(\tau < t)$ at zero;

The simulation of $f_{X_t|\tau > t}(x)$ can easily be conducted from inverse cumulative distribution function (cdf) approach. This new approach has been extended to multi-name and multi-step simulation by Yi (2009b). One of its application is on credit value adjustment calculation for credit default swaps, see Yi (2009c).

³In credit risk literature, $P(S_t \geq H)$ is known as Merton's probability while $P(\tau < t)$ is known as first passage probability, see Yi (2009a) for a discussion of their relationships.

⁴Much of the contribution of this section is due to Patric Yang from Bank of Montreal

μ	r	σ	T	S_0	K	H
0	0	0.1	1	100	100	110

Table 1: Parameters used for simulation.

4 Numerical Illustration

We consider an up-and-out call with one year maturity and the barrier level at 10% above the at-the-money strike. This is the same numerical example that was discussed in Lomibao & Zhu (2005). The parameters are provided in Table 1. We compute exposure profiles at 95%, 50% and 5% confidence levels respectively using both LZ approach (simulate VaF) and our new approach (simulate PFE), see Figure 2. Comparing with PFE, the VaF underestimate the exposure at 95% confidence level while overestimate the exposure at 50% and 5% confidence level.

Figure 1(a) shows the histograms of VaF and PFE in 8 months. It can be seen that both the mass at zero and the range of VaF and PFE are quite different. Figure 1(b) plots their cdf respectively. They intersects approximately at 65th quantile. This figure shows that VaF overestimate lower quantiles when $\alpha < 65\%$, while underestimate higher quantiles when $\alpha > 65\%$.

5 Conclusions

In this article, we pointed out the drawback of the conditional valuation approach proposed by Lomibao and Zhu (2005) that their methodology is inappropriate to use for PFE calculation. The reason is because that they simulate VaF instead of future MtM value. We propose a new and simple simulation methodology that remedies this drawback by simulating the correct distribution of future MtM value. We also suggest a modified LZ approach to improve their simulation algorithm. For the numerical example on an up-and-out call option, we find that the LZ approach underestimates the upper quantiles and overestimates the lower quantiles of the PFE.

A First Passage Times of Drifted Brownian Motions

Let X_t be a drifted Brownian motion (DBM) with initial point x_0 given by

$$X_t = x_0 + \theta t + \sigma W_t. \quad (12)$$

Define the first passage time τ^* as follows

$$\tau^* = \inf\{t > 0 : X_t = 0\}. \quad (13)$$

We have the following propositions.

Proposition A.1. *For a given time $t > 0$, the probability that $\tau^* \leq t$ have the following explicit expression, if $x_0 > 0$, we have*

$$P(\tau^* \leq t) = \Phi\left(-\frac{x_0 + \theta t}{\sigma\sqrt{t}}\right) + e^{-2x_0\theta/\sigma^2} \Phi\left(-\frac{x_0 - \theta t}{\sigma\sqrt{t}}\right), \quad (14)$$

where Φ denotes the cumulative distribution function (cdf) of a standard normal variable. Further more the survival distribution of X_t , namely the distribution of X_t conditional on $\tau^* > t$, is given by

$$f_{X_t|\tau^*>t}(x) = \frac{\phi(x; x_0 + \theta t, \sigma\sqrt{t}) - e^{-2x_0\theta/\sigma^2} \phi(x; \theta t - x_0, \sigma\sqrt{t})}{\Phi\left(\frac{x_0 + \theta t}{\sigma\sqrt{t}}\right) - e^{-2x_0\theta/\sigma^2} \Phi\left(-\frac{x_0 - \theta t}{\sigma\sqrt{t}}\right)}, \quad \forall x > 0. \quad (15)$$

where $\phi(x; \mu, \sigma)$ denotes the probability density function (pdf) of a normal distribution with mean μ and volatility σ .

The Equation (14) has appeared in numerous textbooks, such as Steele (2004). The Equation (15) has appeared in Yi, Tchernitser & Hurd (2010), where they term it mirrored normal distribution. Similary, we have the following for the case $x_0 < 0$.

Proposition A.2. For a given time $t > 0$, the probability that $\tau^* \leq t$ have the following explicit expression, **if** $x_0 < 0$, we have

$$P(\tau^* \leq t) = \Phi\left(\frac{x_0 + \theta t}{\sigma\sqrt{t}}\right) + e^{-2x_0\theta/\sigma^2} \Phi\left(\frac{x_0 - \theta t}{\sigma\sqrt{t}}\right), \quad (16)$$

where Φ denotes the cumulative distribution function (cdf) of a standard normal variable. Further more the survival distribution of X_t , namely the distribution of X_t conditional on $\tau^* > t$, is given by

$$f_{X_t|\tau^*>t}(x) = \frac{\phi(x; x_0 + \theta t, \sigma\sqrt{t}) - e^{-2x_0\theta/\sigma^2} \phi(x; \theta t - x_0, \sigma\sqrt{t})}{\Phi\left(-\frac{x_0 + \theta t}{\sigma\sqrt{t}}\right) - e^{-2x_0\theta/\sigma^2} \Phi\left(\frac{x_0 - \theta t}{\sigma\sqrt{t}}\right)}, \quad \forall x < 0. \quad (17)$$

where $\phi(x; \mu, \sigma)$ denotes the probability density function (pdf) of a normal distribution with mean μ and volatility σ .

The proposition A.2 can be proved using proposition A.1 with the following facts. For $x_0 < 0$, we have

$$\begin{aligned} P(\tau^* < t) &= P(\max_{0 \leq s \leq t} X_s > 0) = P(-\min_{0 \leq s \leq t} \{-X_s\} > 0) \\ &= P(\min_{0 \leq s \leq t} \{-x_0 - \theta t - \sigma W_t\} < 0) = P(\min_{0 \leq s \leq t} \{-x_0 - \theta t + \sigma W_t\} < 0). \end{aligned}$$

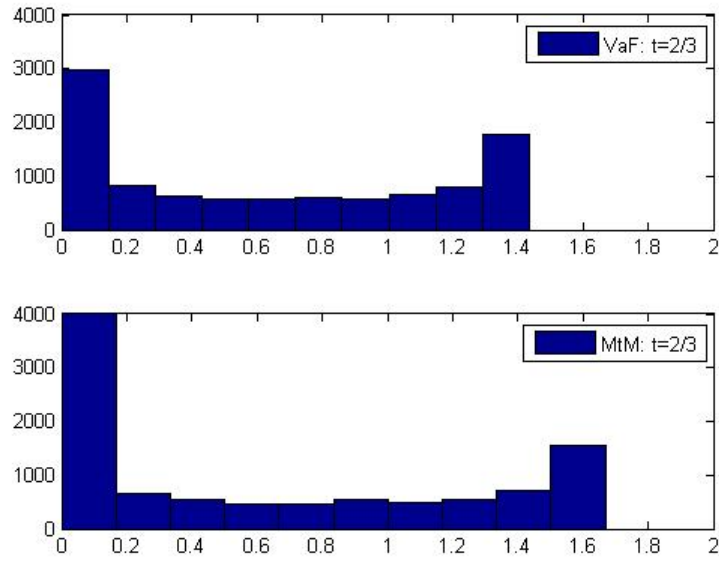
B Pricing Formulas for Barrier Options

If the stock price S_t is given by Equation (4) under risk-neutral measure (assuming zero interest rate). The price of a European call at time t with maturity date T , strike K is given by

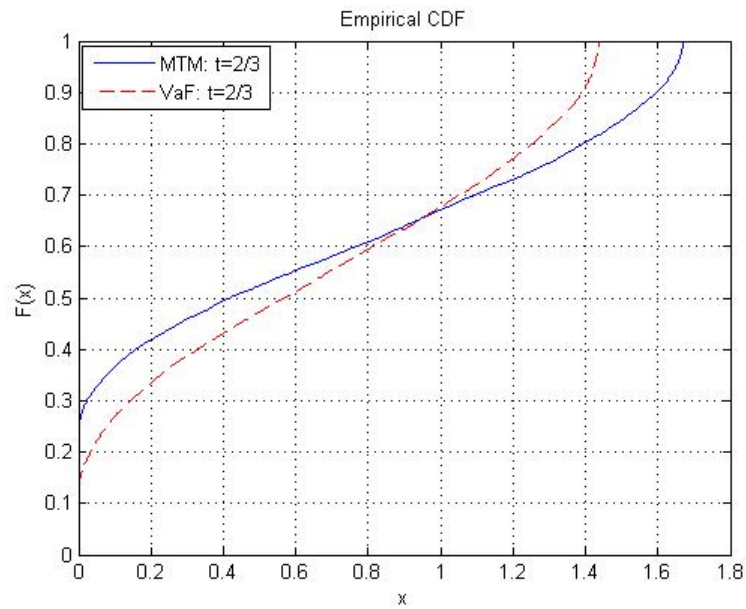
$$\begin{aligned} c_t &= S_t \Phi(d_1) - K \Phi(d_2), \\ d_1 &= \frac{\ln(S_t/K)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}, \\ d_2 &= \frac{\ln(S_t/K)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}. \end{aligned}$$

Let H denote the up barrier which is bigger than S_0 . When $H \leq K$, the value of the up-and-out call, c_t^{uo} , is zero and the value of the up-and-in call, c_t^{ui} , is c_t . When $H > K$, from Hull (2006), we have

$$\begin{aligned} c_t^{uo} &= c_t - c_t^{ui}, \\ c_t^{ui} &= S_t \Phi(a_1) - K \Phi(a_2) - H[\Phi(b_2) - \Phi(a_2)] + \frac{KS_t}{H}[\Phi(b_1) - \Phi(a_1)], \\ a_1 &= \frac{\ln(S_t/H)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}, \quad a_2 = \frac{\ln(S_t/H)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}, \\ b_1 &= \frac{\ln(S_t K/H^2)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}, \quad b_2 = \frac{\ln(S_t K/H^2)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}. \end{aligned}$$



(a) Histograms.



(b) Empirical CDF.

Figure 1: Distribution of Potential Future Exposure and Value at Future for an up-and-out: $t = 2/3$.

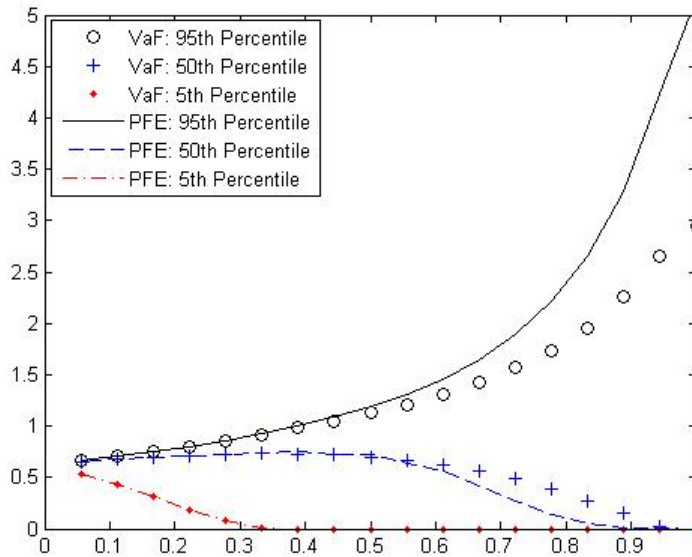


Figure 2: Profiles of Potential Future Exposure and Value at Future.

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