Randomized Structural Models of Credit Spreads

Chuang Yi   Alexander Tchernitser   Tom Hurd*

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*Chuang (corresponding author, email: chuang.yi@live.ca) and Alexander are from Market Risk, Bank of Montreal. This article presents the personal views of the authors and does not reflect any views or policies of Bank of Montreal. Tom is a Professor of Mathematics and Statistics at McMaster University.

Abstract

We propose to randomize the initial condition of a generalized structural model, where the solvency ratio instead of the asset value is modeled explicitly. This initial randomization assumption is motivated by the fact that market players cannot observe the solvency ratio accurately. We find that positive short spreads can be produced due to imperfect observation on the risk factor. The two models we have considered, the Randomized Merton (RM) and the Randomized Black-Cox (RBC), both have explicit expressions for Probability of Default (PD) and Credit Spreads (CS). In the RM model, both PD and LGD are found to be of order of $\sqrt{T}$, as the maturity T approaches zero. It therefore provides an example that has no well-defined default intensity but still admits positive short spreads. In the RBC model, the positive short spread is generated through the positive default intensity of the model. Because explicit formulas are available, these two Randomized Structure (RS) models are easily implemented and calibrated to the market data. This is illustrated by a calibration exercise on Credit Default Swap (CDS) spread data of Ford Motor Corporation.

Keywords: Randomized Merton, Randomized Black-Cox, Incomplete Information, Solvency Ratio, Credit Spreads.

1 Introduction

Quantitative modeling of credit risk is becoming an essential tool to assess and control credit exposures for banks and other financial institutions. The first approach to assess credit risk, known as structural models, was introduced by Merton (1974). In this model, the firm's asset value V_t is specified to be a Geometric Brownian motion (GBM), and is perfectly observed by investors. The debt of the firm is assumed to be a constant, K, maturing at a future time, T. Default time τ is defined to equal T if the firm's asset value is insufficient to repay the promised debt K. It is well documented in the literature that the term structure of credit spreads (TSCS) generated by Merton's model are too low, especially for short maturities (see Black & Cox (1976) and Giesecke & Goldberg (2004)). The short spreads in Merton's model are zero, which is counterfactual to empirical data (see Giesecke & Goldberg (2004)). Generalizations include Black & Cox (1976) first passage time model, Longstaff & Schwartz (1995) treatment of stochastic interest rates, Leland & Toft (1996) endogenous default and optimal capital structure, Collin-Dufresne & Goldstein (2001) mean-reverting leverage ratio model and Fouque, Sircar & Solna (2006) consideration of stochastic volatility. However, none of the models mentioned above can produce positive short spreads.

Zhou (2001) and Chen & Kou (2006) introduce jumps to the dynamics of the firm asset value in first passage time setting and they successfully obtain positive short spreads. A more promising method, known as the incomplete information approach, is introduced by Duffie & Lando (2001), Collin-Dufresne, Goldstein & Helwege (2003), Giesecke (2006) and Guo, Jarrow & Zeng (2009). Duffie & Lando (2001) notice that the observations of the asset value may not be exact but are realized with some random noise. Collin-Dufresne et al. (2003) provide an information updating perspective on credit risk. Giesecke (2006) argues that the both default barrier and asset process could be stochastic. Guo et al. (2009) unifies the noisy information in Duffie and Lando (2001) and the partial information in Collin-Dufresne et al. (2004) through the notion of delayed filtration. Their efforts succeed in raising the short spreads above zero. Without modeling the firm asset value or debt value, Coculescu, Geman & Jeanblanc (2006) define the default time to be the first passage time of some fundamental process through the default barrier. The real fundamental process is not observable and the observed process is modeled to be a Stochastic Differential Equation (SDE). Although both jump diffusion models and incomplete information models can produce positive short spreads, these models in the literature are too mathematically complicated to be implemented in practice. In general, neither jump diffusion models nor first passage models with incomplete information admit explicit expressions of credit spreads. These models then have to rely on numerical methods, which utilize either Laplace transforms, see Chen & Kou (2006), or Fortet's lemma, see Coculescu et al. (2006). However, calibration becomes problematic when it comes to implementation of these models. On one hand, the difficulty comes from the mathematical complexity of the models. On the other hand, the expenses to raise short spreads bring too many extra parameters. For example, Chen & Kou (2006) add extra four parameters in order to introduce double exponential jumps. Hence, most of the papers on these models are silent on the calibration issue.

In this paper, we propose to randomize the initial condition of a generalized structural model, where the solvency ratio instead of the asset value is modeled explicitly. This initial randomization assumption is motivated by the fact that market players cannot observe the solvency ratio accurately. The two models we have considered, the Randomized Merton (RM) and the Randomized Black-Cox (RBC), both have explicit expressions for Probability of Default (PD) and Credit Spreads (CS). Because explicit formulas are available, these two Randomized Structure (RS) models are easily implemented and calibrated to the market data.

The rest of this paper is organized as follows. In Section 2 and 3, the RM model and the RBC model are introduced respectively. Section 4 provides a delayed information perspective on the RS model. In Section 5, a calibration exercise is conducted. We summarize this article in Section 6. All proofs are given in the appendix.

2 Randomized Merton Model

In the classical structural models, the solvency ratio (log of asset over debt) X_t has a constant initial value $X_0 = x_0$. This means that we can fully observe the solvency ratio at current time. However, in reality, the current solvency ratio cannot be exactly observed by the market players. It is thus reasonable to randomize the initial value X_0.

Assume that the solvency ratio X_t follows a drifted Brownian motion (DBM) under the risk-neutral measure

$$X_t = X_0 + \mu t + \sigma W_t, \tag{1}$$

with a random initial value X_0. At time zero, we cannot observe the initial value X_0 accurately, but instead, we observe X_0 plus some random noise. We also assume that X_0 and W_t are independent for all $t > 0$. This is a reasonable assumption, since the noise should not affect the evolution of the solvency ratio process. However, it does contaminate the information observed by market players.

In this RM model, we assume the following distribution for X_0, which has no mass on $(-\infty, 0)$.

- RM Assumption on X_0: its Probability Density Function (pdf) $f(x_0; y_0, \sigma_0)$ is given by

$$f(x_0; y_0, \sigma_0) = \begin{cases} \phi(x_0; y_0, \sigma_0)/\Phi(y_0/\sigma_0) & if \quad x_0 \geq 0 \\ 0 & if \quad x_0 < 0. \end{cases} \tag{2}$$

  where function $\phi_(x; \mu, \sigma)$ denotes the pdf of $N(\mu, \sigma^2)$ given by

$$\phi(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{3}$$

This initial randomization will ensure zero default probability at time zero, namely $P(X_0 < 0) = 0$.

Remark 2.1 *As time progresses from zero to $t \in (0, T)$, the solvency ratio X_t can be negative without triggering a default in Merton's model. Therefore, the nonnegative assumption on X_0 is not a consistent assumption for a dynamic model. Nevertheless, this assumption is reasonable for a static model which can be used to price the current short spread.*

The default probability $PD(T)$, recovery rate $RR(T)$ and the credit spread $CS(T)$ can be calculated by conditioning

$$\begin{aligned} PD(T) &:= \mathbb{E}[P(X_T < 0|X_0)], && (4) \\ RR(T) &:= \frac{\mathbb{E}[\mathbb{E}[e^{X_T} 1_{\{X_T<0\}}|X_0]]}{\mathbb{E}[P(X_T < 0|X_0)]}, && (5) \\ CS(T) &:= -\frac{1}{T} \log\left(1 - PD(T)LGD(T)\right), && (6) \end{aligned}$$

where $LGD(T) := 1 - RR(T)$. The following Proposition gives explicit formulas for $PD(T)$, $RR(T)$ and $CS(T)$, as well as their asymptotics when $T \to +0$.

Proposition 2.1 *In the RM model, the default probability $PD(T)$, the recovery rate $RR(T)$ and the credit spreads $CS(T)$ have the following representations*[1]

$$\begin{aligned} PD(T) &= \frac{A}{\Phi(y_0/\sigma_0)}, && (7) \\ RR(T) &= \frac{Be^{y_0+\mu T+\frac{1}{2}\sigma^2 T+\frac{1}{2}\sigma_0^2}}{A}, && (8) \\ CS(T) &= -\frac{1}{T} \log\left(\frac{\Phi(y_0/\sigma_0) - A + Be^{y_0+\mu T+\frac{1}{2}\sigma^2 T+\frac{1}{2}\sigma_0^2}}{\Phi(y_0/\sigma_0)}\right), && (9) \end{aligned}$$

where the function $\Phi_2(x_1, x_2, \rho)$ denotes the cdf of a bivariate normal distribution with marginal distributions being standard normal and correlation coefficent ρ and

$$\begin{aligned} A &= \Phi_2\left(-\frac{y_0 + \mu T}{\sqrt{\sigma_0^2 + \sigma^2 T}}, \frac{y_0}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2 + \sigma^2 T}}\right), \\ B &= \Phi_2\left(-\frac{y_0 + \mu T + \sigma_0^2 + \sigma^2 T}{\sqrt{\sigma_0^2 + \sigma^2 T}}, \frac{y_0}{\sigma_0} + \sigma_0, -\frac{\sigma_0}{\sqrt{\sigma_0^2 + \sigma^2 T}}\right). \end{aligned}$$

A series expansion of these functions are given by Vasicek (1998).[2] *Moreover, we have*

$$\begin{aligned} \lim_{T \to +0} \frac{PD(T)}{\sqrt{T}} &= \frac{\sigma f(0; y_0, \sigma_0)}{\sqrt{2\pi}}, && (10) \\ \lim_{T \to +0} \frac{LGD(T)}{\sqrt{T}} &= \frac{\sigma\sqrt{2\pi}}{4}, && (11) \\ \lim_{T \to +0} CS(T) &= \frac{\sigma^2 f(0; y_0, \sigma_0)}{4}. && (12) \end{aligned}$$

[1] Pykhtin (2003) obtained a similar expression for the recovery rate in his recovery risk model.

[2] We thank Michael Gordy for pointing out this.

From the above proposition, we can see that $PD(T)$ indeed vanishes to zero as maturity T approaches zero. When there is no random noise of the initial observation, i.e. $\sigma_0 = 0$, the RM model reduces to Merton's model. Both $PD(T)$ and $LGD(T)$ are found to have an order of $\sqrt{T}$, as $T \to +0$. As a result, the default intensity does not exist in the RM model, but it can still generate positive short spread. This positive short spread has an explicit formula given by Equation (12).

The short spread only depends on σ, y_0 and σ_0, and it does not depend on the drift μ. However, if we allow y_0 to be a function of μ, the short spread may depend on μ too, as we can see in section 4. Equation (12) implies that the short spread increases when σ increases while holding other parameters constant. If we fix σ and σ_0, the short spread is a decreasing function of y_0/σ_0. The ratio y_0/σ_0 can be regarded as Distance to Default (DD). More uncertainty about the observed solvency ratio indicates higher risk and hence the short spread should be higher. This uncertainty should be measured by DD instead of σ_0. This result may also imply that a firm's credit spreads will fall after its annual report. This awaits empirical results from testing the model. The situation for σ_0 is more complicated. When σ_0 increases from zero, the short spread first increases to a maximum and then decreases. The maximum is reached at a σ_0^{max}, which solves the following equation

$$(y_0^2\sigma_0^2 - 1)\Phi(y_0/\sigma_0) + y_0\phi_0 = 0.$$

This equation is obtained by setting the first order derivative of $CS(+0)$ with respect to σ_0 to be zero.

Figures 1 and 2 show term structure of credit spreads for varying σ_0 and μ respectively, while holding other parameters constant. The short spread of the RM model is clearly above zero as seen from both figures. Figure 1 also shows that $CS(T)$ may decrease when σ_0 increases. Some people may argue that this is counter-intuitive, since more uncertainty about the current observation should require to pay more for the protection of default. Hence the credit spread should be higher for larger σ_0. This argument is only correct if we replace the risk measure σ_0 by DD (i.e. y_0/σ_0). The credit spread is indeed a monotone decreasing function of DD. Figure 2 also demonstrates that the RM model is capable of generating upwarding term structure of credit spreads by choosing sufficient negative μ. Merton's model cannot produce upward increasing term structure of credit spreads because of the nonnegativity restriction on the constant interest rate. In the RM, however, we do not specify the dynamics of the asset value V_t. We allow $\mu + \frac{1}{2}\sigma^2$ to be negative in the RM model, because $\mu + \frac{1}{2}\sigma^2$ does not necessarily represent the interest rate. As a result, the RM model is able to generate varying shapes of term structure of credit spreads.

Figure 3 plots the short spread defined in Equation (12) as a function of y_0, σ_0 and DD. The middle picture in Figure 3 shows a hump-shaped curve of the short spread as a function of σ_0. The maximum short spread is achieved at $\sigma_0^{max} = 0.4167$, in the case when $\sigma = 0.12$ and $y_0 = 0.35$.

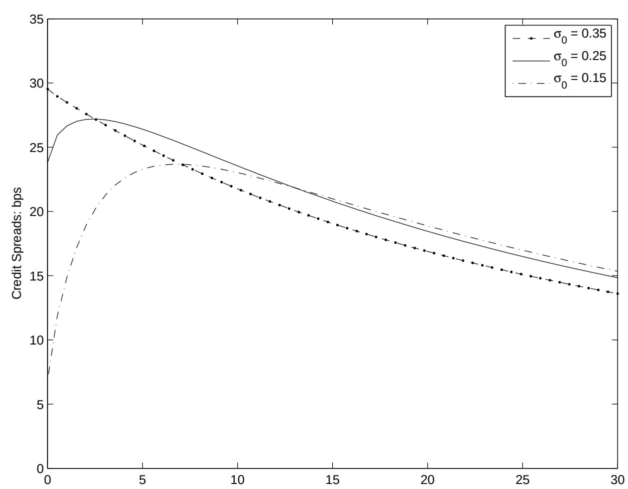

Figure 1: Term structure of credit spreads of RM model for varying σ_0: $\mu = 0.01$, $\sigma = 0.12$, $y_0 = 0.35$.

In order to prove Proposition 2.1, the following two lemmas are needed.

Lemma 2.2 *Let (X, Y) be a bivariate normal with correlation coefficient ρ and marginals $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$. Then the following equation holds*

$$\Phi_2\left(-\frac{\mu_x}{\sigma_x}, -\frac{\mu_y}{\sigma_y}, \rho\right) = \int_0^{+\infty} \Phi\left(\frac{y\rho\sigma_x/\sigma_y + \mu_y\rho\sigma_x/\sigma_y - \mu_x}{\sigma_x\sqrt{1-\rho^2}}\right)\phi(y; -\mu_y, \sigma_x)dy. \tag{13}$$

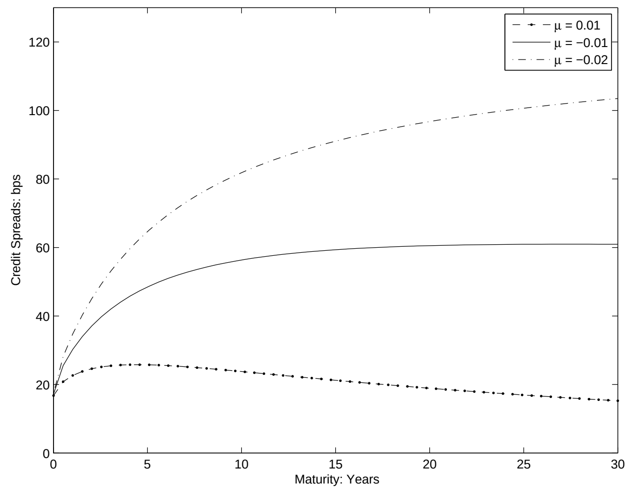

Figure 2: Term structure of credit spreads of RM model for varying μ: $\sigma = 0.12$, $y_0 = 0.35$, $\sigma_0 = 0.20$.

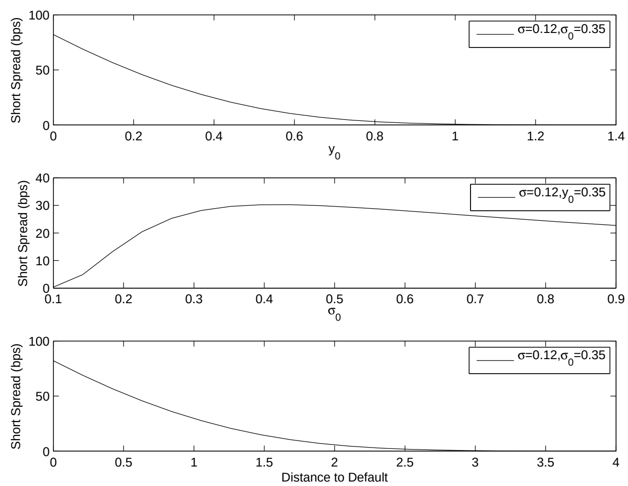

Figure 3: RM short spread as a function of y_0, σ_0 and DD.

Lemma 2.3 *As $T \to +0$ the following expansion holds*

$$\Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}, \frac{-\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right) = \frac{\sigma\phi(0;y_0,\sigma_0)\sqrt{T}}{\sqrt{2\pi}} + \frac{y_0\sigma^2\phi(0;y_0,\sigma_0)T}{4\sigma_0^2} + O(T^{3/2}). \tag{14}$$

3 Randomized Black-Cox (RBC) Model

As mentioned in the Remark 2.1, the assumption that $X_0 > 0$ is inconsistant for a dynamic model in Merton's definition of default. However, it is natural to make this assumption in the Black-Cox setting.

In this section, we apply the randomization technique to the Black-Cox model. The solvency ratio is still assumed to follow Equation (1). The default time is assumed to be the first passage time defined by the following equation

$$\tau = \inf\{t > 0 : X_t \leq 0\}. \tag{15}$$

The interest rate and the expected recovery rate are assumed to be constant and the credit spreads are calculated using Equation (6).

In this RBC model, we propose the following distribution for X_0:

- RBC Assumption on X_0: its pdf $f(x_0; a, v_0, \sigma_0)$ is assumed to be

$$f(x_0; a, v_0, \sigma_0) = \begin{cases} \frac{\phi(x_0;a+v_0,\sigma_0)-e^{-2av_0/\sigma_0^2}\phi(x_0;v_0-a,\sigma_0)}{\Phi\left(\frac{a+v_0}{\sigma_0}\right)-e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)} & if \quad x_0 \geq 0 \\ 0 & if \quad x_0 < 0 \end{cases} \tag{16}$$

where $\sigma_0 > 0$ and $a > |v_0|$.

Direct integration shows that $f(x_0; a, v_0, \sigma_0)$ is indeed a valid pdf. Figure 4 shows an example of this pdf.

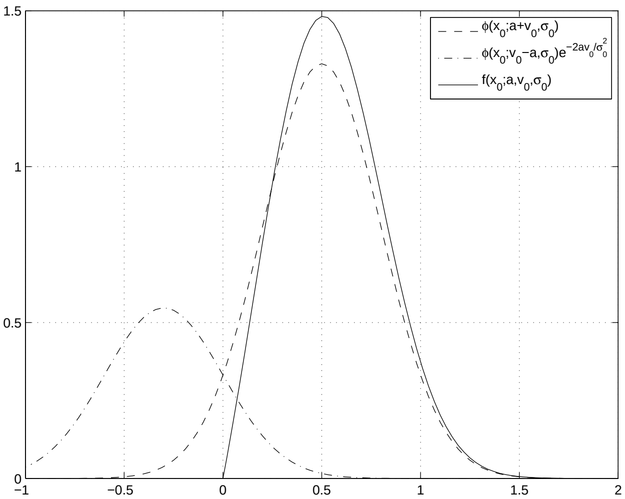

Figure 4: Probability Density Function $f(x_0; a, v_0, \sigma_0)$ with parameters $a = 0.4$, $v_0 = 0.1$ and $\sigma_0 = 0.3$.

The motivation of this distribution is that the conditional first passage probability $P(t < \tau < T | \tau > t)$ can be written as an integral of a pdf which has a form of $f(x_0; a, v_0, \sigma_0)$ defined above. Using Lemmas 2.2 and 2.3, the explicit formula for $P(\tau < T)$ and its asymptotics (as $T \to +0$) are given by the following Proposition.

Proposition 3.1 *For the RBC model, the default probability has the following expression*

$$P(\tau < T) = \frac{A + B - C - D}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)},$$

where

$$\begin{aligned}
A &= \Phi_2\left(-\frac{a+v_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{a+v_0}{\sigma_0}, \rho\right), \\
B &= \Phi_2\left(-\frac{a+v_0-2\mu\sigma_0^2/\sigma^2-\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{a+v_0-2\mu\sigma_0^2/\sigma^2}{\sigma_0}, \rho\right) e^{2\mu^2\sigma_0^2/\sigma^4 - 2\mu(a+v_0)/\sigma^2}, \\
C &= \Phi_2\left(-\frac{v_0-a+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{v_0-a}{\sigma_0}, \rho\right) e^{-2av_0/\sigma_0^2}, \\
D &= \Phi_2\left(-\frac{v_0-a-2\mu\sigma_0^2/\sigma^2-\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{v_0-a-2\mu\sigma_0^2/\sigma^2}{\sigma_0}, \rho\right) e^{2\mu^2\sigma_0^2/\sigma^4 - 2av_0/\sigma_0^2 - 2\mu(v_0-a)/\sigma^2}, \\
\rho &= \frac{-\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\lim_{T\to+0} \frac{P(\tau < T)}{T} &= \frac{a\sigma^2\phi(0; a+v_0, \sigma_0)/\sigma_0^2}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)} \\
&= \frac{\sigma^2}{2}\frac{\partial f(x_0; a, v_0, \sigma_0)}{\partial x_0}\Big|_{x_0=0}.
\end{aligned}$$

The above Proposition ensures positive intensity for the RBC model. Consequently, positive short spreads are generated in the RBC model. Note that we have obtained an equivalent expression of the intensity as in Duffie & Lando (2001). In their model, the log of asset value follows DBM with a constant initial value z_0: $Z_t = z_0 + mt + \sigma W_t$. The default time τ is defined to be $\tau := \inf\{s > 0 : Z_s = 0\}$. Consider fixed time $t > 0$, assume the only information available is $\mathcal{H}_t := \{1_{\{\tau > s\}} : s \le t\}$. Conditional on $\tau > t$,

Z_t has a conditional density $f(\cdot)$ which is bounded and has bounded derivative with $f(0) = 0$ and $f'(0)$ is defined from the right. Duffie & Lando (2001) stated that, the default intensity $\lambda := \lim_{h \to +0} P(t < \tau \leq t + h | \tau > t)/h$, is given by $\frac{1}{2}\sigma^2 f'(0)$. However, in their model, there is perfect information at time zero and they can only establish the existence of an intensity for $t > 0$. By randomizing the initial value of the solvency ratio, the RBC model avoids this difficulty.

Example 3.1 *When $\mu/\sigma^2 = v_0/\sigma_0^2$.*
In this case, the probability of default is reduced to

$$P(\tau < T) = \frac{\Phi\left(-\frac{a+v_0+\mu T}{\sqrt{\sigma^2 T + \sigma_0^2}}\right) + e^{-2a\mu/\sigma^2}\Phi\left(-\frac{a-v_0-\mu T}{\sqrt{\sigma^2 T+\sigma_0^2}}\right) - \Phi\left(-\frac{a+v_0}{\sigma_0}\right) - e^{-2a\mu/\sigma^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)},$$

where Φ_2 functions disappear and only Φ functions are involved. The reduction of dimensionalities of the integrals can be proved either by Vasicek's expansions, or by separation of integration techniques.

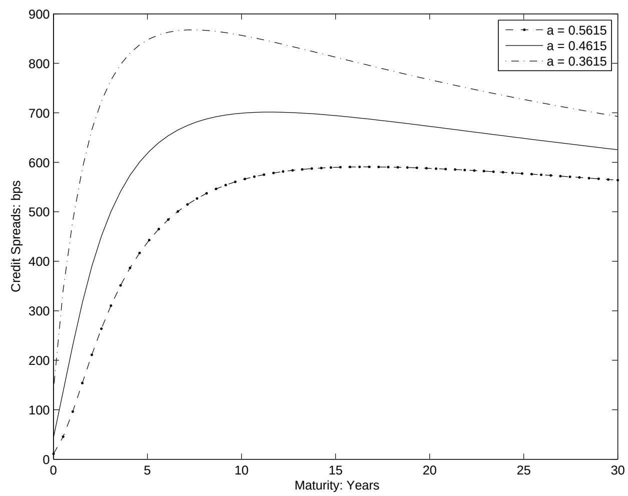

Figure 5: Term structure of credit spreads of RBC model for varying a: $\mu = -0.0417$, $\sigma = 0.2030$, $v_0 = 0.2402$, $\sigma_0 = 0.2162$ and $l = 1$.

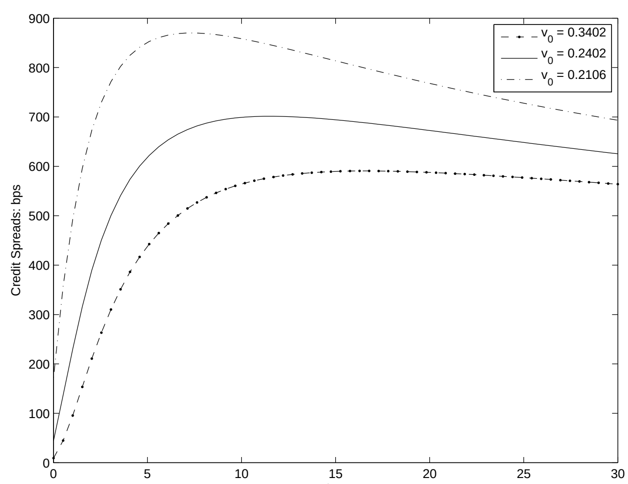

Figure 6: Term structure of credit spreads of RBC model for varying v_0: $\mu = -0.0417$, $\sigma = 0.2030$, $a = 0.4615$, $\sigma_0 = 0.2162$ and $l = 1$.

Figures 5 and 6 show that the TSCS increases as a or v_0 decreases. Since for smaller a or v_0, the initial distribution of X_0 has more mass close to zero. It is therefore more likely to default which in

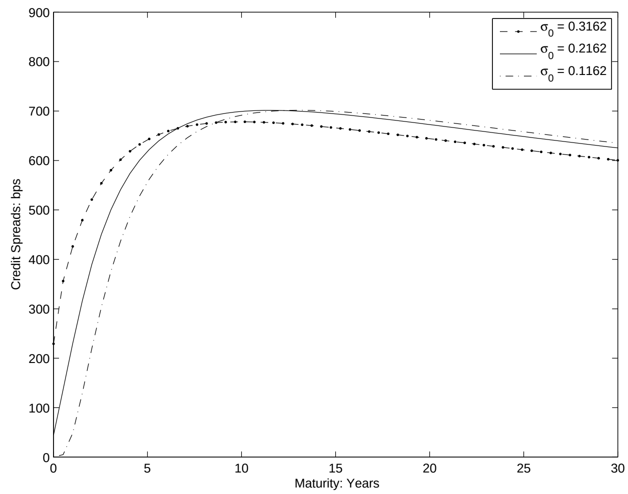

Figure 7: Term structure of credit spreads of RBC model for varying σ_0: $\mu = -0.0417$, $\sigma = 0.2030$, $v_0 = 0.2402$, $a = 0.4615$ and $l = 1$.

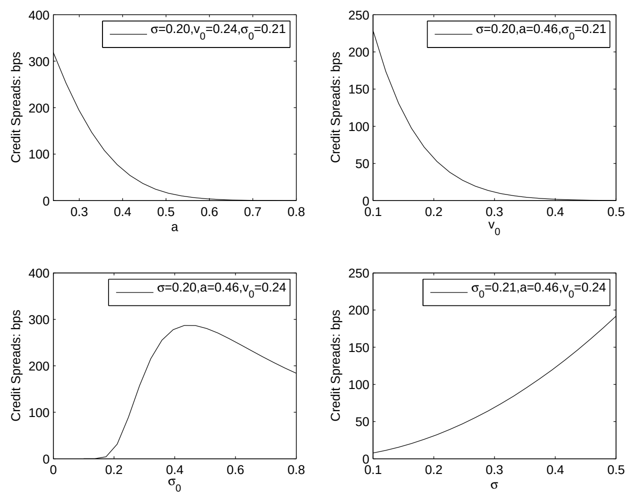

Figure 8: RBC short spread as a function of a, v_0, σ_0 and σ.

turn implies higher credit spreads. Figure 7 shows that the credit spread does not depend on σ_0 in a monotonic way. For some maturities the credit spread increases with σ_0 while it may decrease with σ_0 for other maturities. Figure 8 shows the short spread as a function of parameters a, v_0, σ_0 and σ. The short spread increases as a decreases, or v_0 decreases or σ increases. As similar to the RM model, the short spread of the RBC model first increases to a maximum and then decreases, as σ_0 increases from zero. This maximum achieved at some σ_0 which can be solved by setting $\partial CS(+0)/\partial\sigma_0 = 0$.

4 Delayed Information vs. Randomized Structural Models

Randomization of the initial value in the RS model may look awkward at first glance. This section gives a natural construction of the RS model through delayed information.

4.1 Delayed Information vs. RM

Assume the solvency ratio X_t is given by Equation (1) with constant initial value $X_0 = x_0$. Let time t be the current time and T be a future time. Let ϵ be a small positive number. At the current time t, we assume two sets of information available. First, we assume $X_t > 0$. Second, X_t is not observed, but the delayed solvency ratio $X_{t-\epsilon} = a$ is realized at time t. Then, conditional on $X_{t-\epsilon} = a$, we have

$X_t \sim N(a + \mu\epsilon, \sigma^2\epsilon)$. This is because

$$X_t = X_{t-\epsilon} + \mu\epsilon + \sigma(W_t - W_{t-\epsilon}).$$

The conditional default probability can be calculated through

$$\begin{aligned} P(X_T < 0 | X_t > 0, X_{t-\epsilon} = a) &= \frac{P(X_T < 0, -X_t \leq 0 | X_{t-\epsilon} = a)}{P(-X_t < 0 | X_{t-\epsilon} = a)} \\ &= \frac{\Phi_2\left(-\frac{a+\mu(T-t+\epsilon)}{\sigma\sqrt{T-t+\epsilon}}, \frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}, -\frac{\sqrt{\epsilon}}{\sqrt{T-t+\epsilon}}\right)}{\Phi\left(\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right)}. \end{aligned}$$

This is equivalent to the RM model if we set $y_0 = a + \mu\epsilon$ and $\sigma_0 = \sigma\sqrt{\epsilon}$. This construction suggests that uncertainty about the current solvency ratio may come from the delayed realization of the solvency ratio. The longer the observation is delayed (i.e. for larger ϵ), the more uncertainty is the current solvency ratio (i.e. larger σ_0). The short spread derived from this construction becomes

$$\frac{\frac{\sigma^2}{4\sqrt{2\pi\sigma^2\epsilon}} e^{-\frac{(a+\mu\epsilon)^2}{2\sigma^2\epsilon}}}{\Phi\left(\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right)}.$$

The short spread derived from this delayed information depends on the drift parameter μ through y_0. This is because y_0 is a linear function of μ. Figure 9 shows that the short spread increases as a or μ decreases, or σ or ϵ increases. This indicates that the longer the information is delayed, the higher the short spread.

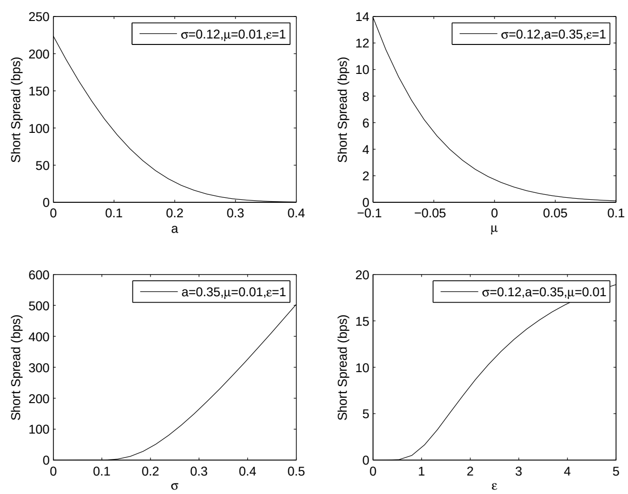

Figure 9: The short spread curve derived from delayed information as a function of a, μ, σ and ϵ.

4.2 Delayed Information vs. RBC

This delayed information approach can easily be extended to first passage models, where the default time τ is defined as in Equation (15). As before, X_t denotes the solvency ratio process which we assume staionary increments. At the current time t, we assume the following information are available. First, default has not happened up to now, namely $\tau > t$ is known. Second, at current time t, we can only observe the path of the solvency ratio up to a previous time $t - \epsilon$, particulary $X_{t-\epsilon} = a > 0$ is realized at time t but X_t is not. Let $\mathcal{F}_{t-\epsilon}$ denote the filtration generated by the solvency ratio process up to time $t - \epsilon$. Then, the default time for a given future time $T > t$ can be calculated through

$$\begin{aligned} P(\tau < T | \mathcal{F}_{t-\epsilon}, \tau > t) &= \frac{P(t < \tau < T | \mathcal{F}_{t-\epsilon})}{P(\tau > t | \mathcal{F}_{t-\epsilon})} \\ &= \frac{P(\epsilon < \tau < T - t + \epsilon | X_0 = a)}{P(\tau > \epsilon | X_0 = a)} \\ &= \frac{P(\tau < T - t + \epsilon | X_0 = a) - P(\tau < \epsilon | X_0 = a)}{1 - P(\tau < \epsilon | X_0 = a)}. \end{aligned} \quad (17)$$

The above formula can be calculated explicitly if $P(\tau < \epsilon | X_0 = a)$ has explicit expression. The default intensity is then given by

$$\lambda_t = \frac{\partial P(\tau < \eta + \epsilon | X_0 = a)/\partial \eta|_{\eta=0}}{P(\tau > \epsilon | X_0 = a)}. \tag{18}$$

Note that the numerator is exactly the pdf of the first passage time taking value at ϵ. We thus have obtained positive intensities through delayed information.

Consider the case when X_t is a DBM. We know that $P(\tau < T | X_0 = a)$ has explicit formula given by the following equation

$$P(\tau < T | X_0 = x_0) = \Phi\left(-\frac{x_0 + \mu T}{\sigma\sqrt{T}}\right) + e^{-2x_0\mu/\sigma^2}\Phi\left(-\frac{x_0 - \mu T}{\sigma\sqrt{T}}\right). \tag{19}$$

Then the default probability is given by

$$P(\tau < T | \mathcal{F}_{t-\epsilon}, \tau > t) = \frac{\Phi\left(-\frac{a+\mu(T-t+\epsilon)}{\sigma\sqrt{T-t+\epsilon}}\right) + e^{-2a\mu/\sigma^2}\Phi\left(-\frac{a-\mu(T-t+\epsilon)}{\sigma\sqrt{T-t+\epsilon}}\right) - \Phi\left(-\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right) - e^{-2a\mu/\sigma^2}\Phi\left(\frac{\mu\epsilon - a}{\sigma\sqrt{\epsilon}}\right)}{\Phi\left(\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right) - e^{-2a\mu/\sigma^2}\Phi\left(\frac{\mu\epsilon - a}{\sigma\sqrt{\epsilon}}\right)}.$$

This is equivalent to the RBC model discussed in Example 3.1 with $v_0 = \mu\epsilon$ and $\sigma_0^2 = \sigma^2\epsilon$. These parameter constrains satisfy $\mu/\sigma^2 = v_0/\sigma_0^2$. The default intensity λ_t has the following elegant expression:

$$\lambda_t = \frac{a e^{-\frac{(a+\mu\epsilon)^2}{2\sigma^2\epsilon}} / \sqrt{2\pi\sigma^2\epsilon^3}}{\Phi\left(\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right) - e^{-2a\mu/\sigma^2}\Phi\left(\frac{\mu\epsilon - a}{\sigma\sqrt{\epsilon}}\right)} \tag{20}$$

Note that this construction only makes sense when $t > \epsilon > 0$, since $\mathcal{F}_{-\epsilon}$ is not well-defined here. We can therefore only establish the existence of an intensity for $t > 0$ through the construction of delayed information. The Duffie-Lando approach is thus equivalent to the delayed information approach.

5 Calibration Excercise

In this section, we demonstrate the incomplete information effects on term structure of credit spreads (TSCS) by fitting the theoretical CS in both the RM model and the RBC model to the observed CDS spread curve.

In the RM model, the parameters of interest are μ, σ, y_0 and σ_0. In the delayed information version of the RBC model, the parameters of interest are μ, σ, a, ϵ. In this calibration exercise, we set the $LGD = 1$ for the RBC model and the Black-Cox model.

For comparison, we also fit the Merton's CS and the Black-Cox's CS to the market data. The parameters of interest in both the Merton and the Black-Cox models are μ, σ and y_0. We take the CDS curve of Ford Motor Corporation on May 28, 2009, from MarkIt Partner with 0.5-, 1-, 2-, 3-, 4-, 5-, 7-, 10-, 15-, 20-, and 30-year maturities. Parameters are calibrated by minimizing the cross-sectional Mean Absolute Error (MAE).

Figures 10 and 11 show the calibrated results. It can be seen from the picture that both RM and RBC models outperform the classical structural models in fitting the CDS spreads, especially for short maturities. We also find that the RM model fits slightly better than the RBC model. In fact, the MAE for the RBC, the RM, the Black-Cox and the Merton's model are 89 basis points (bps), 70 bps, 795 bps and 784 bps respectively. For the short end maturities, such as 6-month, both Merton's and Black-Cox's spreads are close to zero, but catering for incomplete information allows the RM and RBC to generate large positive short spreads. Our randomized models also fit much better for long maturities as well.

The calibrated parameters are illustrated in Table 1.

Model	μ	σ	y_0	σ_0	a	ϵ	MAE (bps)
Merton	0.0009	0.0029	0.0065				784
RM	-1.0084	2.2151	1.8637	0.8380			70
Black-Cox	-0.0089	0.0099	0.0119				795
RBC	-0.0025	0.0081			0.0145	0.4795	89

Table 1: Calibrated parameters for the four models.

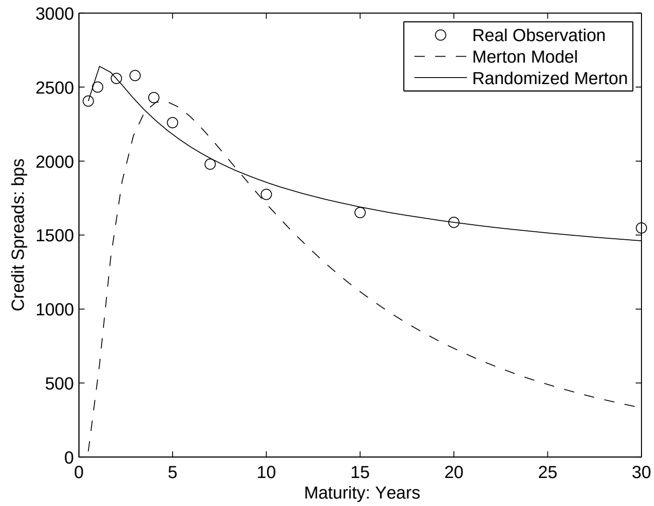

Figure 10: RM-II model vs. Merton model fit to Ford Motor CDS curve on March 16, 2007.

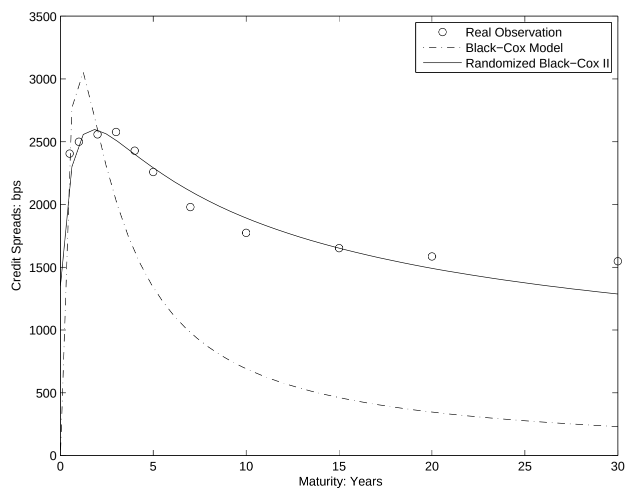

Figure 11: RBC model vs. Black-Cox model fit to Ford Motor CDS curve on March 16, 2007.

6 Summary

Motivated by structural models, we proposed to treat the solvency ratio directly as the risk factor. Based on the classical structural models, we then proposed to randomize the initial value of the solvency ratio to take account of imperfect information. We have mainly looked at two different RS models, which cover two types of definition of default and different assumptions on the initial randomization.

From the RM and the RBC models, we found that positive short spreads can be produced due to imperfect observation on the solvency ratio. We also found that various shapes of the term structure of credit spreads can be generated. The PD, the LGD and the CS are given in explicit formulae for both models, which have explicit expressions for their short spreads. In the RM model, we found that both $PD(T)/\sqrt{T}$ and $LGD(T)/\sqrt{T}$ converge to positive constants as $T \to +0$. The default intensity is thus not defined in the RM model, but positive short spreads can still be produced. In the RBC model, we found that $PD(\tau < T)/T$ converges to a positive constant. Therefore, the default intensity does exist in the RBC model and it generates positive short spreads under the assumption of constant LGD.

Merton's model becomes a special case of the RM model while the Black-Cox model is a special case of the RBC model. Numerical analysis and a calibration exercise illustrate that the randomized structural models outperforms the classical structural model in fitting the term structure of credit spreads, especially for short maturities. These two RS models generalize the classical structural models in two folds. First, instead of modeling the asset value and debt separately, we modeled the solvency ratio directly as a

DBM. Second, imperfect information is considered and positive short spreads are generated.

We next provided a delayed information perspective on the RS models. The models constructed through delayed information are special cases in our general RS models in two ways. First, number of parameters are reduced due to some parameter constrains. Second, positive short spreads can be generated for only $t > 0$.

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Appendices

- **Proof of Proposition 2.1:**
From Equations (2-6), we have

$$
\begin{aligned}
PD(T) &= \frac{1}{\Phi(y_0/\sigma_0)}\int_0^{+\infty} P(X_T<0|X_0=x_0)\phi(x_0;y_0,\sigma_0)dx_0 \\
&= \frac{1}{\Phi(y_0/\sigma_0)}\int_0^{+\infty} \Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)\phi(x_0;y_0,\sigma_0)dx_0, \\
&= \frac{\Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}},\frac{y_0}{\sigma_0},-\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right)}{\Phi(y_0/\sigma_0)}, \\
RR(T) &= \frac{\int_0^{+\infty} e^{x_0+\mu T+\frac{1}{2}\sigma^2 T}\Phi\left(-\frac{x_0+\mu T+\sigma^2 T}{\sigma\sqrt{T}}\right)\phi(x_0;y_0,\sigma_0)dx_0}{\Phi(y_0/\sigma_0)PD(T)} \\
&= \frac{e^{y_0+\mu T+\frac{1}{2}(\sigma_0^2+\sigma^2 T)}\int_0^{+\infty}\Phi\left(-\frac{x_0+\mu T+\sigma^2 T}{\sigma\sqrt{T}}\right)\phi(x_0;y_0+\sigma_0^2,\sigma_0)dx_0}{\Phi(y_0/\sigma_0)PD(T)} \\
&= \frac{\Phi_2\left(-\frac{y_0+\mu T+\sigma_0^2+\sigma^2 T}{\sqrt{\sigma_0^2+\sigma^2 T}},\frac{y_0}{\sigma_0}+\sigma_0,-\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right)e^{y_0+\mu T+\frac{1}{2}\sigma^2 T+\frac{1}{2}\sigma_0^2}}{\Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}},\frac{y_0}{\sigma_0},-\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right)}.
\end{aligned}
$$

We have used Lemma 2.2 for the last steps of each calculation. Then the formula for $CS(T)$ comes in handy. For the asymptotics, Equation (10) is a direct result from Lemma 2.3. Note that $LGD(T) = 1 - RR(T)$, Equation (11) is obtained by applying Lemma 2.3 to $1 - RR(T)$. Equation (12) is a direct result of Equations (10) and (11).

- **Proof of Lemma 2.2:**
The Left Hand Side (LHS) of Equation 13 is the probability that both X and Y are less than zero, i.e. $P(X<0, Y<0)$. We can also calculate this probability by conditioning

$$
\begin{aligned}
P(X<0,Y<0) &= \int_{-\infty}^{0} P(X<0|Y=y)\phi(y;\mu_y,\sigma_y)dy \\
&= \int_{-\infty}^{0}\Phi\left(\frac{-y\rho\sigma_x/\sigma_y+\mu_y\rho\sigma_x/\sigma_y-\mu_x}{\sigma_x\sqrt{1-\rho^2}}\right)\phi(y;\mu_y,\sigma_y)dy \\
&= \int_{0}^{+\infty}\Phi\left(\frac{y\rho\sigma_x/\sigma_y+\mu_y\rho\sigma_x/\sigma_y-\mu_x}{\sigma_x\sqrt{1-\rho^2}}\right)\phi(y;-\mu_y,\sigma_y)dy
\end{aligned}
$$

For the second step, we have used the fact that X is still normally distributed conditional on $Y = y$, i.e. $X|Y = y \sim N(y\rho\sigma_x/\sigma_y - \mu_y\rho\sigma_x/\sigma_y + \mu_x, \sigma_x\sqrt{1-\rho^2})$.

- **Proof of Lemma 2.3:**
From Lemma 2.2, the LHS of Equation (14) can be written as

$$
\begin{aligned}
LHS &= \frac{1}{\sqrt{2\pi\sigma_0^2}}\int_0^{+\infty}\Phi\left(-\frac{x+\mu T}{\sigma\sqrt{T}}\right)e^{-\frac{(x-y_0)^2}{2\sigma_0^2}}dx \\
&= \frac{\sigma\sqrt{T}}{\sqrt{2\pi\sigma_0^2}}\int_{\mu\sqrt{T}/\sigma}^{+\infty}\Phi(-z)e^{-\frac{(z\sigma\sqrt{T}-\mu T-y_0)^2}{2\sigma_0^2}}dz \\
&= \sigma\sqrt{T}\phi(0;y_0,\sigma_0)\int_{\mu\sqrt{T}/\sigma}^{+\infty}\Phi(-z)e^{-\frac{-2y_0 z\sigma\sqrt{T}+O(T)}{2\sigma_0^2}}dz \\
&= \sigma\sqrt{T}\phi(0;y_0,\sigma_0)\int_{\mu\sqrt{T}/\sigma}^{+\infty}\Phi(-z)[1+y_0 z\sigma\sqrt{T}/\sigma_0^2+O(T)]dz \\
&\to \frac{\sigma\phi(0;y_0,\sigma_0)\sqrt{T}}{\sqrt{2\pi}}+\frac{y_0\sigma^2\phi(0;y_0,\sigma_0)T}{4\sigma_0^2}+O(T^{3/2}).
\end{aligned}
$$

For the last step, we have used the following two equalities

$$\int_0^{+\infty}\Phi(-z)dz = \frac{1}{\sqrt{2\pi}},\quad \int_0^{+\infty}z\Phi(-z)dz = \frac{1}{4}.$$

- **Proof of Proposition 3.1**

$$
\begin{aligned}
P(\tau < T) &= \int_{-\infty}^{+\infty} P(\tau < T | X_0 = x_0) f(x_0; a, v_0, \sigma_0) dx_0 \\
&= \frac{A + B - C - D}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)},
\end{aligned}
$$

where A, B, C and D are given by

$$
\begin{aligned}
A &= \int_0^{+\infty} \Phi\left(-\frac{x_0 + \mu T}{\sigma\sqrt{T}}\right) \phi(x_0; a + v_0, \sigma_0) dx_0, \\
B &= \int_0^{+\infty} e^{-2x_0\mu/\sigma^2} \Phi\left(-\frac{x_0 - \mu T}{\sigma\sqrt{T}}\right) \phi(x_0; a + v_0, \sigma_0) dx_0, \\
C &= \int_0^{+\infty} e^{-2av_0/\sigma_0^2} \Phi\left(-\frac{x_0 + \mu T}{\sigma\sqrt{T}}\right) \phi(x_0; v_0 - a, \sigma_0) dx_0, \\
D &= \int_0^{+\infty} e^{-2av_0/\sigma_0^2 - 2x_0\mu/\sigma^2} \Phi\left(-\frac{x_0 - \mu T}{\sigma\sqrt{T}}\right) \phi(x_0; v_0 - a, \sigma_0) dx_0.
\end{aligned}
$$

Note that C and D can also be written as

$$
\begin{aligned}
C &= e^{-2\mu(a+v_0)/\sigma^2 + 2\mu^2\sigma_0^2/\sigma^4} \\
&\quad * \int_0^{+\infty} \Phi\left(-\frac{x_0 - \mu T}{\sigma\sqrt{T}}\right) \phi(x_0; a + v_0 - 2\mu\sigma_0^2/\sigma^2, \sigma_0) dx_0, \\
D &= e^{-2av_0/\sigma_0^2 - 2\mu(v_0-a)/\sigma^2 + 2\mu^2\sigma_0^2/\sigma^4} \\
&\quad * \int_0^{+\infty} \Phi\left(-\frac{x_0 - \mu T}{\sigma\sqrt{T}}\right) \phi(x_0; v_0 - a - 2\mu\sigma_0^2/\sigma^2, \sigma_0) dx_0.
\end{aligned}
$$

Explicit expressions for A, B, C and D are finally obtained by invoking Lemma 2.2. The asymptotic equation of $P(\tau < T)$ is obtained by using Lemma 2.3 and the following identities:

$$
\begin{aligned}
\phi(0; a + v_0, \sigma_0) &= \phi(0; v_0 - a, \sigma_0) e^{-2av_0/\sigma_0^2} \\
&= e^{2\mu^2\sigma_0^2/\sigma^4 - 2\mu(a+v_0)/\sigma^2} \phi(0; a + v_0 - 2\mu\sigma_0^2/\sigma^2, \sigma_0) \\
&= e^{2\mu^2\sigma_0^2/\sigma^4 - 2av_0/\sigma_0^2 - 2\mu(v_0-a)/\sigma^2} \phi(0; v_0 - a - 2\mu\sigma_0^2/\sigma^2, \sigma_0).
\end{aligned}
$$