On the first passage time distribution of an Ornstein-Uhlenbeck process

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Abstract

In this short note, we study the relationship between Merton's probability of defaut (MPD) and the first passage probability of default (FPD) defined on an Ornstein-Uhlenbeck (OU) process. We find that the FPD is twice that of MPD if the default barrier is the asymptotic mean of the OU process. Three different proofs are given for this result. The first proof uses the reflection principle of a Brownian motion; the second approach invokes the Fortet's lemma, and the third proof is from elementary integration.

1 Introduction

The first passage time density of an OU process has not been studied until quite recently. Its explicit expression was first derived by Leblanc and Scaillet (1998) published in the Journal of Finance and Stochastics. Two years later, in the same journal, the same authors Leblanc, Renault and Scaillet (2000) published a correction note of their previous result. However, this correction itself contains an erroneous usage of the spatial homogeneity property for the 3-dimensional Bessel bridge. The final corrected version of the first passage time density did not appear until three years later. In the same journal, Going-Jaeschke and Yor (2003) published a clarification note on the correction note by Leblanc, Renault and Scaillet (2000). Different expressions of this density can also be found in Linetsky (2004) and Alili, Patie and Pedersen (2004).

This problem is also of great interest in credit risk literature. The OU process has been used to model the credit quality of a company such as Collin-Dufresne and Goldstein (2001) and Coculescu, Geman and Jeanblanc (2006). A credit risk manager is more concerned with the cumulative distribution function (cdf) or probability of default of the first passage time rather than the probability density function. Collin-Dufresne and Goldstein (2001) provided an efficient numerical approximation of the cumulative distribution function of the first passage time based on Fortet's (1943) lemma.

In this short note, we study the relationship between Merton's probability of defaut and the first passage probability of default defined on an OU process. We find that the FPD is twice that of

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MPD if the default barrier is the asymptotic mean of the OU process. Three different proofs are given for this result. The first proof uses the reflection principle of a Brownian motion; the second approach invokes the Fortet's lemma, and the third proof is from elementary integration.

2 The Problem

Let $\{X_t, t \geq 0\}$ be an OU process parameterized by two positive real numbers α and σ with the following stochastic differential equation (SDE):

$$dX_t = -\alpha X_t dt + \sigma\sqrt{2\alpha} dB_t, \qquad X_0 = x_0 > 0; \tag{1}$$

where B_t is a standard Brownian motion. For fixed t, X_t specified above is Gaussian distribution:

$$X_t \sim N(x_0 e^{-\alpha t}, \sigma^2(1 - e^{-2\alpha t})).$$

Let $T > 0$ be a fixed future time. In credit risk modeling, this can be thought of as the maturity of a defaultable bond. Typically in credit risk literature, the OU process is used to model the solvency ratio (log of asset over debt). The first probability we are interested in is the probability of $X_T \leq 0$, which we will refer to as MPD. The reason being is that, in the momentous work of Merton (1974), he modeled the default probability in a similar fashion.

Since X_T is Gaussian, the MPD, denoted as P_M, can be calculated as

$$P_M := P(X_T \leq 0) = \Phi\left(-\frac{x_0 e^{-\alpha T}}{\sigma\sqrt{1 - e^{-2\alpha T}}}\right); \tag{2}$$

where Φ is the cdf of standard normal distribution.

Let us define the first passage time τ as

$$\tau = \inf\{t \geq 0; X_t = 0\}. \tag{3}$$

The second probability we are interested in is the probability of $\tau \leq T$. We will refer to this probability as FPD, which was used in Black and Cox (1976). Let $f(t)$ denote the probability density function of τ. The first passage probability of default, denoted by P_F is thus given by:

$$P_F := P(\tau \leq T) = \int_0^T f(t)dt. \tag{4}$$

The problem is now to study how P_F and P_M are related to each other.

3 The Main Result

Our main result is given in the following theorem.

Theorem 3.1 *For an OU process given by equation (1), the FPD P_F is twice that of the MPD P_M, i.e. $P_F = 2P_M$, where P_M and P_F are given by equations (2) and (4) respectively.*

In the following, we provide three proofs of this result. The first proof uses the reflection principle of a Brownian motion.

Proof 1: Let us define the reflected OU process $\tilde{X}_t$ as follows:

$$\tilde{X}_t = \begin{cases} X_t & t \leq \tau \\ -X_t & t > \tau. \end{cases}$$

Note that $-X_t$ satisfies the following SDE,

$$d(-X_t) = -\alpha(-X_t)dt + \sigma\sqrt{2\alpha}d\tilde{B}_t;$$

where $\tilde{B}_t = -B_t$, which is also a standard Brownian motion. Therefore, for every fixed t, X_t and $\tilde{X}_t$ are equal in distribution. At time $t = \tau$, both X_t and $-X_t$ are zero. It follows that $\tilde{X}_t$ and X_t share the same law for every fixed time t. Hence

$$\begin{aligned} P(\tau \leq T) &= P(\tau \leq T \cap X_T \leq 0) + P(\tau \leq T \cap X_T > 0) \\ &= P(X_T \leq 0) + P(\tilde{X}_T \leq 0) \\ &= 2P(X_T \leq 0). \end{aligned}$$

The second proof invokes Fortet's lemma.

Proof 2: Let us recall Fortet's lemma:

Theorem 3.2 ***Fortet's Lemma****: For a one-factor continuous Markov process l_t, define $\pi(l_t, t|l_s, s)$ as the free transition density. Define $f(l_s = \bar{l}, s|l_0, 0)$ as the probability density that the first passage time through a constant boundary $\bar{l}$ occurs at date s. Then*

$$\pi(l_t, t|l_0, 0) = \int_0^t f(l_s = \bar{l}, s|l_0, 0)\pi(l_t, t|l_s = \bar{l}, s)ds;$$

for $\forall l_t > \bar{l} > l_0$ or $\forall l_t < \bar{l} < l_0$.

In our case, the Markov process l_t is the OU process X_t. The barrier $\bar{l}$ is 0 and the initial $l_0 = x_0$. Thus, the free transition density $\pi(X_t, t|X_s, s)$ is a Gaussian density. Let us denote that

$$\begin{aligned} M(t) &:= E[X_t|X_0 = x_0] = x_0 e^{-\alpha t}; \\ L(t-s) &:= E[X_t|X_s = 0] = 0; \\ S^2(t-s) &:= Var[X_t|X_s] = \sigma^2(1 - e^{-2\alpha(t-s)}). \end{aligned}$$

We integrate Fortet's equation by $\int_{-\infty}^0 dX_t$ to obtain

$$\Phi\left(-\frac{M(t)}{S(t)}\right) = \int_0^t f(X_s = x_s, s|x_0, 0)\Phi\left(-\frac{L(t-s)}{S(t-s)}\right) ds.$$

Noticing that $L(t-s) = 0$, we end up with

$$\Phi\left(-\frac{M(t)}{S(t)}\right) = \frac{1}{2}\int_0^t f(X_s = x_s, s|x_0, 0)ds;$$

whose left hand side is MPD and the right hand side is a half of the FPD.

Finally, we give an elementary proof which applies only integration.

Proof 3: In this proof, we need the following proposition.

Proposition 3.1 *For an OU process defined in equation (1), the first passage time τ defined in equation (3) has the following density:*

$$f(t) = \frac{x_0}{2\sigma\sqrt{\pi\alpha}} \exp\left\{-\frac{x_0^2 e^{-\alpha t}}{4\sigma^2\sinh(\alpha t)} + \frac{\alpha t}{2}\right\}\left(\frac{\alpha}{\sinh(\alpha t)}\right)^{\frac{3}{2}}.$$

This result can be found in Alili, Patie and Pedersen (2000) and Lachaud (2004), Pitman and Yor (1981).

It is sufficient to show that $P_F = 2P_M$ for the case $\sigma = 1 = \alpha$. In this case, P_M can be written as

$$P_M = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi(1-e^{-2T})}} \exp\left\{-\frac{(x - x_0 e^{-T})^2}{2(1-e^{-2T})}\right\} dx; \tag{5}$$

and P_F is given by

$$P_F = \int_0^T \frac{x_0}{2\sqrt{\pi}} \exp\left\{-\frac{x_0^2 e^{-t}}{4\sinh(t)} + \frac{t}{2}\right\} \left(\frac{1}{\sinh(t)}\right)^{\frac{3}{2}} dt. \tag{6}$$

It is straightforward to show that $P_F = 2P_M$ by first changing variable $y = e^{-t}$ and $a := e^{-T}$ in equation (6); and then changing variable y^2 as follows

$$y^2 = \frac{(x - ax_0)^2}{(x - ax_0)^2 + x_0^2(1 - a^2)};$$

with limits chosen according to

$$\begin{aligned} x &= 0, \quad when \;\; y = a; \\ x &= -\infty, \quad when \;\; y = 1. \end{aligned}$$

4 Drifted Brownian Motion Case

Let $\{X_t, t \geq 0\}$ be a drifted Brownian motion given in the following SDE with drift $m > 0$ and volatility $\sigma > 0$,

$$dX_t = m dt + \sigma dB_t, \qquad X_0 = x_0 > 0.$$

Let $T > 0$ be a fixed future time. Then the MPD is given by

$$P_M := P(X_T < 0) = \Phi\left(-\frac{x_0 + mT}{\sigma\sqrt{T}}\right). \tag{7}$$

The FPD is defined in equation (3). This FPD can be calculated using the reflection principle of Brownian motion.

$$P_F := P(\tau \leq T) = \Phi\left(-\frac{x_0 + mT}{\sigma\sqrt{T}}\right) + e^{-2x_0 m/\sigma^2}\Phi\left(-\frac{x_0 - mT}{\sigma\sqrt{T}}\right). \tag{8}$$

Detailed derivation can be found in Steele (2000). Note that the first term is exactly the Merton's probability. The second term comes from possibilities of hitting 0 before T.

Proposition 4.1 *For pure Brownian motion, the first passage probability P_F is twice of the Merton's probability P_M, i.e. $P_F = 2P_M$.*

This result is immediate by setting $m = 0$ in equations (7) and (8). However, for drifted Brownian motion, we do not have this perfect relationship. A weaker result about the drifted Brownian motion is given in the following proposition.

Proposition 4.2 *For drifted Brownian motion given, the FPD P_F is twice of the MPD P_M in the following limit sense*

$$\lim_{T\to 0} \frac{P_F}{P_M} = 2. \tag{9}$$

Proof: It is equivalent to show that

$$\lim_{T\to 0} \frac{\Phi\left(-\frac{x_0 - mT}{\sigma\sqrt{T}}\right)}{\Phi\left(-\frac{x_0 + mT}{\sigma\sqrt{T}}\right)} = e^{2x_0 m/\sigma^2}.$$

Applying Hospital's rule to the left term of the above equation leads us to the end of the proof.

5 Generalization of The Main Result

Our main result could be extended to the negative initial case and non-centered OU processes.

Proposition 5.1 *For the OU process starting from x_0(which could be less than 0), the FPD P_F is twice that of the MPD P_M.*

If $x_0 < 0$, this theorem can be proved by invoking the main result and proposition 3.1 with x_0 replaced by $|x_0|$. A general non-centered OU process can be written as:

$$dX_t = \kappa(\theta - X_t)dt + \sigma dB_t, \qquad X_0 = x_0. \tag{10}$$

Proposition 5.2 *For a non-centered OU process given by equation (10), define the FPD time $\tau = \inf\{t : X_t = \theta\}$. Then the FPD probability P_F is twice that of the MPD P_M.*

This could be easily proved by change of variable $Y_t := X_t - \theta$, using our main result. Unfortunately, this will not hold if we define $\tau = \inf\{t : X_t = 0\}$. In short, this theorem holds only if we define the first passage time as the first time the OU process crosses its long run mean.

6 Collin-Dufresne-Goldstein Approximation

In Collin-Dufresne and Goldstein (2001), the solvency ratio is modeled as an OU process under the risk-neutral measure given by (10) with negative initial value $x_0 < 0$. Negative initial condition is from the assumption that the firm's asset value is above the debt value at time zero. The default time τ is defined as the first passage time that the solvency ratio crosses zero.

Knowing the difficulty of directly integrating the first passage time density, Collin-Dufresne and Goldstein (2001) gave a numerical approximation scheme to calculate the default probability by invoking Fortet's (1943) implicit formula for the first passage time density. This approximation scheme is one of their main results which can be summarized as follows:

Collin-Dufresne and Goldstein (2001) : *Discretize time into n equal intervals, and define date $t_j = jT/n := j\triangle t$, for $j \in (1, 2, ..., n)$. The default probability $P(\tau \leq T)$ can be calculated through:*

$$\begin{aligned}
P(\tau \leq t_j) &= \sum_{i=1}^{j} q_i, \quad j = 2, 3, ..., n; \\
q_1 &= \frac{\Phi(a_1)}{\Phi(b_{(1/2)})}, \qquad q_i = \left[\Phi(a_i) - \sum_{j=1}^{i-1} q_j \Phi(b_{i-j+1/2})\right] / \Phi(b_{(1/2)}); \\
a_i &= \frac{M(i\triangle t)}{S(i\triangle t)}, \qquad b_i = \frac{L(i\triangle t)}{S(i\triangle t)}, \quad i = 2, 3, ..., n; \\
M(t) &= x_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}), \quad L(t) = \theta(1 - e^{-\kappa t}), \quad S^2(t) = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}).
\end{aligned}$$

For detailed derivation, please refer to Appendix A in Collin-Dufresne and Goldstein (2001). Recent work by Coculescu, Geman and Jeanblanc (2006) also applied this approximation.

If $\theta = 0$, then $L(t) = 0$ in the above approximation. Consequently, we have

$$P(\tau < t_j) = \frac{1}{2}\Phi(a_j),$$

which is the exact value based on our main result. This indicates that Collin-Dufresne and Goldstein's approximation becomes exact when $\theta = 0$.

References

[1] Alili, L., Patie, P. and Pedersen, J.L. (2004), Representations of the First Hitting Time Density of an Ornstein-Uhlenbeck Process. Stochastic Models, 21(4), 967-980.

[2] Black, F. and Cox, J. (1976), Valuing corporate securities: Some effects of bond indenture provisions, Journal of Finance 31, 351-367.

[3] Coculescu D., Geman, H. and Jeanblanc, M. (2006), Valuation of Default Sensitive Claims Under Imperfect Information. Working paper, Universite Paris-Dauphine.

[4] Collin-Dufresne, P. and Goldstein, R. (2001), Do credit spreads reflect stationary leverage ratios? Journal of Finance 56(5), 1929-1957.

[5] Going-Jaeschke, A. and Yor, M. (2003), A clarification note about hitting times densities for Ornstein-Uhlenbeck processes. Finance and Stochastics 7, 413-415.

[6] Lachaud, B. (2004), Hitting times of Ornstein-Uhlenbeck processes. Available at: `http://www.math-info.univ-paris5.fr/~robin/Anglais/ou_english.pdf`

[7] Leblanc, B., and Scaillet, O. (1998), Path-dependent options on yield in the affine term structure model. Finance and Stochastics 2, 349-367.

[8] Leblanc, B., Renault, O., and Scaillet, O. (2000), A correction note on the first passage time of an Ornstein-Uhlenbeck process to a boundary. Finance and Stochastics 4, 109-111.

[9] Linetsky, V. (2004). Computing hitting time densities for CIR and OU diffusions: Applications to meanreverting models. Journal of Computational Finance 7, 1-22.

[10] Merton, R. (1974), On the pricing of corporate debt: The risk structure of interest rates, Journal of Finance 29, 449-470.

[11] Pitman, J.W. and Yor, M. (1981), Bessel process and infinitely divisible laws. Stochastics Integrals, Springer-verlag LNM 851, 285-370.

[12] Steele, J.M. (2000), Stochastic Calculus and Financial Applications, Springer.