In Search of Hybrid Models for Credit Risk: from Leland-Toft to Carr-Linetsky*

Tom Hurd,† Chuang Yi, ‡

First Version: April, 2007. This Version: April, 2008

Abstract

In this paper, we derive several forms of the equity volatility as a function of the equity value, from the structural credit risk literature. We then propose a new jump to default model by taking the equity volatility to be of the form implied by the models of Leland (1994) and Leland & Toft (1996). This model involves a process we call the Dual-Jacobi process and which has explicit formulae for its moments. Gram-Charlier expansions are then applied to approximate bond and call prices. Our model generalizes Linetsky (2006) by incorporating a local volatility which is bounded below by a positive constant. This local volatility will decrease to a positive constant for increasing stock prices, making the stock process asymptotic to Geometric Brownian Motion (GBM). In this sence, our model is more realistic than the Constant Elasticity of Variance (CEV) models.

Keywords: Hybrid Models, Constant Elasticity of Variance, Endogenous Bankruptcy, Credit Spreads, Default Intensity, Implied Volatility Skew, Dual-Jacobi Process.

---

*The authors thank Alexey Kuznetsov for very helpful discussions and comments.

†Professor, Department of Mathematics and Statistics, McMaster University

‡PH.D candidate in Financial Math, McMaster University. Email: yichuang@math.mcmaster.ca

1 Introduction

Classical structural models of credit risk start from modeling the dynamics of firm asset value as a discounted martingale under the risk-neutral measure, such as Merton (1974), Black & Cox (1976) and Leland (1994). The equity and defaultable bonds are then priced under this discounted martingale assumption on the asset value. In reality, common stocks rather than the firm's assets are publicly traded and hence should be considered to be a discounted martingale under the risk-neutral measure. This alternative approach, recently taken by Linetsky (2006) and Carr & Linetsky (2006), is to directly model the stock price as a risk-neutral discounted martingale. Credit risk is incorporated in this equity modeling approach by assuming that the stock price S_t at time t can jump to zero with an intensity $h(S_t)$, which is assumed to be a function of S_t.

Linetsky (2006) considers a constant volatility model with the hazard rate chosen to be a negative power function of the stock price. The pre-default stock price S_t under the risk-neutral measure is thus assumed to follow:[1]

$$dS_t = (r + h(S_t))S_t dt + \sigma S_t dW_t, \tag{1}$$

$$h(S_t) = \alpha S_t^{-p}, \tag{2}$$

with initial value S_0 at time $t = 0$, where W_t denotes the standard Brownian motion. The constant r denotes the default-free interest rate, and α, p and σ are positive constants.

Let Δ denote the bankruptcy state when the firm defaults at time τ. Then we can also write the dynamics for the stock price subject to bankruptcy S_t^Δ as follows:

$$dS_t^\Delta = S_{t-}^\Delta [rdt + \sigma dW_t - dM_t],$$

where

$$M_t = 1_{\{\tau \le t\}} - \int_0^{t\wedge\tau} h(S_u)du.$$

[1] Linetsky (2006) and Carr & Linetsky (2006) also assume a dividend yield q, which we set to zero here for simplicity.

is a martingale.

Note that the hazard rate function goes to infinity as the stock price approaches zero. Unbounded intensity kills the pre-default process almost surely by a jump-to-default instead of diffusing down to zero. The valuations of corporate liabilities and equity derivatives are then reduced to evaluation of the following risk-neutral expectations

$$F_\psi(S_0, T) := \mathbb{E}[e^{-\int_0^T r+h(S_t)dt}\psi(S_T)], \tag{3}$$

where ψ is any payoff function. Proposition 2.1 in Linetsky (2006) states that

$$F_\psi(S_0, T) = S_0\hat{\mathbb{E}}[S_T^{-1}\psi(S_T)], \tag{4}$$

where $\hat{\mathbb{E}}$ is the expectation with respect to a new probability measure $\hat{P}$ under which $\hat{W}_t := W_t - \sigma t$ is a standard Brownian motion. Under the new measure $\hat{P}$, the dynamics of the stock prices S_t has the same form as in equation (1), with r replaced by $r + \sigma^2$. The Stochastic Differential Equation (SDE) of the stock price has an explicit transition density representation solved by a change of variables $Z := S^p$. This closed form transition density is closely related to the Bessel process as discussed in Appendix B of Linetsky (2006). The pricing formulae for defaultable bonds, European puts, and European calls can thus be derived explicitly based on the known distribution of S_t under the $\hat{P}$ measure. The firm's asset value is out of interest and is therefore not discussed in this paper.

Carr & Linetsky (2006) extended Linetsky's model by combining a jump-to-default model with CEV models. The pre-default stock price is assumed to have the following dynamics

$$dS_t = (r + h(S_t))S_t dt + \sigma(S_t)S_t dW_t, \tag{5}$$

$$\sigma(S_t) = aS_t^{-\beta}, \tag{6}$$

$$h(S_t) = b + c\sigma^2(S_t), \tag{7}$$

where a, b, c and β are nonnegative constants. It was found by Carr & Linetsky (2006) that the process specified in equation (5) can be represented as a re-scaled and time-changed power of a Bessel process. The valuations of corporate liabilities, credit derivatives, and equity derivatives are then reduced to calculating expectations of a known function of a standard Bessel process evaluated at a changed time.

The choices of functions $h(S_t)$ and $\sigma(S_t)$ specified in equations (6) and (7) capture a positive relationship between the default intensity and the equity volatility, and the leverage effect between volatility and stock prices. A good property of their choice is that analytical tractability of the formulae is obtained. Theoretically speaking, one can generalize Carr & Linetsky (2006) by assuming different functional forms of $h(S_t)$ and $\sigma(S_t)$. However, not all functional forms of $h(S_t)$ and $\sigma(S_t)$ will be economic appealling and mathematical attractive. A disadvantage of CEV models is that the equity volatility will vanish to zero as the stock price approaches infinity. As a result, CEV models may not be appropriate to describe the stock dynamics of companies with high equity but low debt (i.e. low financial leverage). This fact motivates us to search for better alternatives.

The rest of this paper is organized as follows. In Section 2, we derive some functional forms of equity volatility from the structural credit risk modeling literature. Our new jump-to-default model is then described in Section 3. In Section 4, we discuss pricing of both credit and equity derivatives. Section 5 studies the Dual-Jacobi process, which is related to the pre-default stock price. In Section 6, the Gram-Charlier expansion is used to approximate call and bond prices. Some numerical analyses are provided in Section 7. In Section 8, we discuss an extension of our model. Section 9 summarizes this paper. All proofs are given in the appendix.

2 Implied Equity Volatility from Structural Credit Literature

We explore some implied functional forms of $\sigma(S_t)$ from structural credit literature in this section. Particularly, we are interested in the form implied by Leland (1994) and Leland & Toft (1996). Structural modeling approach on credit risk goes back to Merton (1974). Before we move on to Leland (1994), let us first look at Merton's implied equity volatility.

2.1 Merton (1974)

Merton (1974) assumes that the firm's asset value V follows GBM under the risk-neutral measure with interest rate r as its drift and volatility σ. Under Merton's assumption, the equity of the firm E can be evaluated as a call option on the asset value V, with the debt value K as the strike price and the debt maturity T as the maturity of the call option. The equity value is hence given by the celebrated Black-Scholes' formula

$$E_t = V_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-), \tag{8}$$

where $\Phi(\cdot)$ stands for the cumulative distribution function (cdf) of a standard normal random variable and $d_\pm$ are given by

$$d_\pm = \frac{\log \frac{V}{K} + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Black-Scholes-Merton equation can also be written in terms of asset, equity and debt as follows

$$E = V\frac{\partial E}{\partial V} + K\frac{\partial E}{\partial K}.$$

If we assume that equity follows the dynamics given by the following equation

$$\frac{dE}{E} = \mu_E dt + \sigma_E dW_t,$$

then the equity drift μ_E and the equity volatility σ_E can be recovered by applying Ito's lemma to equation (8). It is not surprising that μ_E turns out to be the default-free interest rate r. The equity volatility σ_E satisfies the following equation,

$$\sigma_E = \sigma E^{-1} V \Phi(d_+), \tag{9}$$

which is also derived in Hull, Nelken & White (2004). The equity volatility σ_E can thus be regarded as a function of E implicitly solved from equation (8) and (9). As seen in Figure 1, these two equations imply that the equity volatility σ_E goes to infinity as the equity E approaches zero, and tends to the asset volatility σ as E goes to infinity. The Figure 1 demonstrates that equity volatility σ_E is bounded below by the asset volatility σ. The implied equity volatility is found to be a decreasing function of maturity T. This implied local volatility should not be confused with Dupire's implied local volatility. Dupire (1994) derived the famous implied local volatility equation from arbitrage arguments on pricing options on the stock without considering default risk. The local volatility obtained here is derived from Merton (1974) credit structural model, without considering pricing options on the equity.

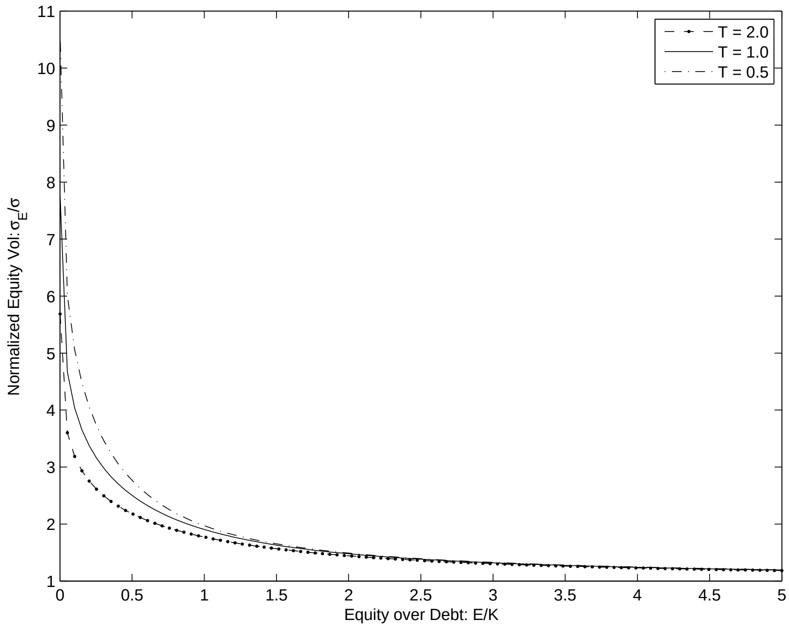


Figure 1: Local equity volatility implied from Merton (1974) as a function of equity. The x-axis denotes the equity-debt ratio, E/K. The y-axis denotes the normalized equity volatility, σ_E/σ.

2.2 Leland (1994) and Leland-Toft (1996)

This section shows that the implied equity local volatility from Leland (1994) behaves like a power decay function of equity, while bounded below by the asset volatility.

Leland (1994) introduced the concept of endogenous bankruptcy by maximizing the equity value of the firm, whose debt promises a perpetual coupon payment C. As in Merton (1974), he started with the assumption that the asset value of the firm V is a GBM with drift of interest rate r and volatility σ under the risk-neutral measure. Tax benefits of debt financing grant the firm to deduct a fraction τ of the coupon payments as long as it is solvent. The firm will suffer from a bankruptcy cost αV_B when its asset value first hits the bankruptcy level V_B. The value of the equity, E, is then derived by the formula that the equity equals the asset value V plus tax benefits minus bankruptcy cost and minus the debt.[2] This yields equation (13) in Leland (1994), which is

$$E(V) = V - b + (b - V_B)(V/V_B)^{-X}, \tag{10}$$

where $X = 2r/\sigma^2$, $b = (1-\tau)C/r$. The optimal bankruptcy level V_B is then determined endogenously by solving the smooth-pasting condition: $dE(V)/dV\ |_{V=V_B} = 0$, which gives

$$V_B = \frac{(1-\tau)C}{r + 0.5\sigma^2}. \tag{11}$$

The equity $E(V)$ as a function of the asset value is thus obtained explicitly by plugging the optimal V_B into equation (10), which yields

$$E(V) = V + aV^{-X} - b, \tag{12}$$

where a is given by

$$a = \frac{b^{X+1}X^X}{(X+1)^{X+1}}. \tag{13}$$

However, Leland (1994) did not address the dynamics of the equity E and left the unanswered question of whether the firm's asset is a traded asset.[3] We answer this question here by analysing the stochastic dynamics of the equity. Applying Ito's lemma to $E(V)$, we obtain the following SDE for E

$$\frac{dE}{E} = \mu_E dt + \sigma_E dW_t,$$

[2]We will assume constant total number of shares, N. The equity E equals total number of shares times the stock prices, i.e. $E = NS$.

[3]See footnote 11 on page 1217 of Leland (1994)

where μ_E and σ_E denote the drift and volatility respectively, which solve the following equations

$$\begin{aligned} \mu_E &= r(1 + \frac{b}{E}), && (14) \\ \hat{E} + 1 &= \hat{E}\hat{\sigma}_E + (\hat{E}\hat{\sigma}_E/X + \hat{E} + 1)^{-X}, && (15) \end{aligned}$$

where $\hat{E}$ and $\hat{\sigma}_E$ are normalized equity and normalized volatility of the equity respectively, which are defined as $\hat{E} = E/b$, $\hat{\sigma}_E := \sigma_E/\sigma$. Note that the drift μ_E is always bigger than the interest rate r unless the firm has full tax benefits, namely $\tau = 1$. It turns out that $E + b$ is a discounted martingale under the risk-neutral measure, but E is not. This contradicts the fact that stocks are commonly traded in the market. This non-martingale property mainly comes from the assumption of the constant perpetual debt services. Similarly, we can show that the equity is not a discounted martingale in Leland & Toft (1996), where they assume stationary debt structure with finite maturity.

Nevertheless, we are interested in the implied equity volatility σ_E as a function of equity itself implicitly solved from eqution (15). Particularly, when $X = 1$ (same as $r = 0.5\sigma^2$), we can obtain an explicit formula for σ_E given by

$$\sigma_E = \sigma\sqrt{1 + \frac{2b}{E}}. \qquad (16)$$

The above local volatility function has very broad financial implications. First, the equity volatility is always bigger than the asset volatility, which echoes the arguments in Merton (1974) that the equity of a levered firm must be at least as risky as the firm's asset. Second, the equity volatility is a decreasing function of the equity value, which is known as leverage effect discussed in Black (1976). More precisely, the equity volatility is a decreasing function of the equity/debt ratio, considering that b denotes the total debt service deducted by tax benefits if there is no default. We notice that the equity volatility specified in equation (16) is a power decay function of the equity value, which coincides with the power law of CEV models. For CEV models, the local volatility is specified as $\alpha E^{-\beta}$, where α and β are positive numbers. The local volatility of CEV models and the implied local volatility in Leland (1994) both go to infinity when the equity tends to zero. For small E, the equity volatility specified in equation (16) is a $O(E^{-0.5})$. Finally, when equity E tends to infinity, the equity volatility σ_E converges to the asset volatility σ. As a consequence, the equity will behave much like GBM for large E. This is different from CEV models, because CEV local volatility will vanish to zero when equity goes to infinity. The above properties are not only true for the special case of σ_E, but are also true for all the σ_E implicitly solved from equation (15) as seen from numerical illustration. Figure 2 plots $\hat{\sigma}_E$ as a function of $\hat{E}$ for varying X. The graph demontrates that the equity volatility is an increasing function of X, when holding other parameters constant.

We have derived implicit equity local volatility function implied by Leland (1994) in this section. This local volatility is found to be bounded below by a positive constant which differs from the specification in Carr & Linetsky (2006).

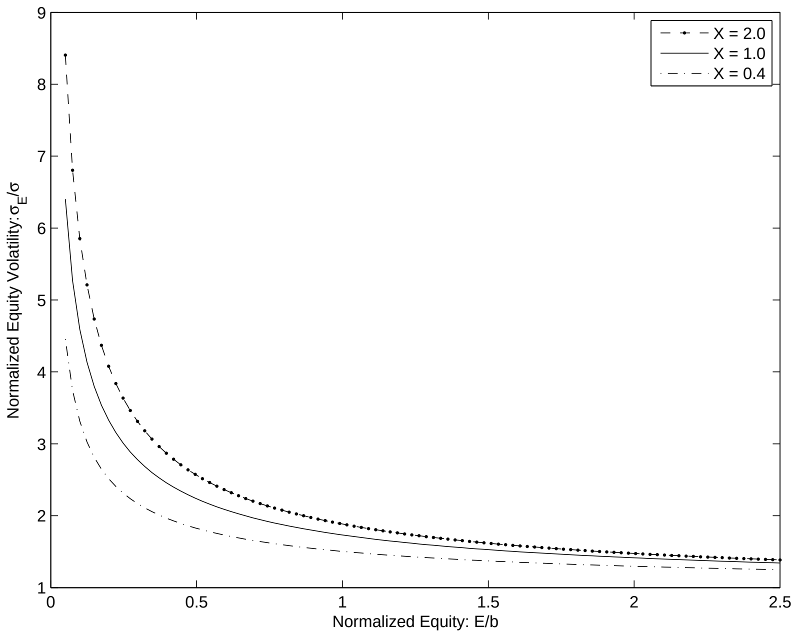


Figure 2: Local equity volatility implied from Leland (1994) as a function of equity. The x-axis denotes the normalized equity, E/b. The y-axis denotes the normalized equity volatility, σ_E/σ.

3 A New Jump to Default Model

Motivated by Carr & Linetsky (2006), we have derived some implied functional forms of equity volatility from structural credit literature. In CEV models, when the equity is large, the equity volatility approaches zero. The CEV specification of the equity volatility is consequently inappropriate for many existing firms with large equity and small debt (i.e. low financial leverage). These models predict that firms' equity values will grow linearly with almost no volatility for large equity companies. However, most high-tech firms, such as the Internet and biotechnology companies, are highly volatile, while having almost no debt as pointed out by Chen & Kou (2006). Consequently, Carr-Linetsky model might under price deep in-the-money calls or deep out-of-money puts. The implied equity volatility function of Leland (1994) is more realistic, because it captures the fact that equity should be at least as risky as the asset value of the firm, of which the asset volatility should not be regarded as zero. In this section, we propose a new jump-to-default model with equity volatility bounded below by a positive constant, which can be regarded as the asset volatility.

Assume that the pre-default stock price S_t has the following risk-neutral dynamics

$$dS_t = (r + aS_t^{-p})S_t dt + cS_t\sqrt{1 + bS_t^{-p}}dW_t, \tag{17}$$

where the constant r denotes the default-free interest rate; a, c, b, and p are positive constants. This specification of the pre-default stock price has the form of equation (5) with volatility and hazard rate functions given by

$$\sigma(S_t) = c\sqrt{1 + bS_t^{-p}}, \tag{18}$$

$$h(S_t) = aS_t^{-p}. \tag{19}$$

The volatility function $\sigma(S_t)$ specified in equation (18) is bounded below by a positive constant c. When S_t is large, $\sigma(S_t)$ is almost a constant, becoming asymptotically the case of constant volatility studied in Linetsky (2006). When S_t is close to zero, $\sqrt{1 + bS_t^{-p}}$ can be approximated as $\sqrt{b}S_t^{-p/2}$. Consequently, our model becomes asymptotically CEV case studied in Carr & Linetsky (2006) when the company is close to default. If we allow b to be zero, then this model is reduced to a constant volatility case studied in Linetsky (2006).

The hazard rate function $h(S_t)$ is related to the volatility function $\sigma(S_t)$ through the following equation

$$h(S_t) = \frac{a}{b}(\frac{\sigma^2(S_t)}{c^2} - 1).$$

As the stock price increases, the equity volatility declines to constant c and the hazard rate approaches zero, making the stock prices asymptotically GBM. The default time is defined to be the first jump time of a Cox process with intensity given by the hazard rate $h(S_t)$. The pricing of equity derivatives can therefore be performed using standard reduced-form intensity-based credit risk framework as discussed in Duffie & Singleton (1999).

4 Pricing

Similarly, as in Linetsky (2006), pricing equity derivatives can be reduced to computing the expectations of the form in equation (3). This simplification is because of the choice of the hazard rate function. Since the hazard rate will go to infinity when the stock price vanishes to zero, the pre-default process will be almost surely killed by a jump-to-default instead of diffusing to zero. Proposition 2.1 in Linetsky (2006) can be generalized as follows:

Proposition 4.1 *Assume that the pre-default stock price S_t has risk-neutral dynamics*

$$dS_t = (r + h(S_t))S_t dt + \sigma(S_t)S_t dW_t,$$

starting from $S_0 > 0$ at time $t = 0$. Then the following identity holds:

$$F_\psi(S_0, T) := \mathbb{E}[e^{-\int_0^T r+h(S_t)dt}\psi(S_T)] = S_0\hat{\mathbb{E}}[S_T^{-1}\psi(S_T)], \quad (20)$$

where $\hat{\mathbb{E}}$ is the expectation with respect to a new probability measure $\hat{P}$ under which $\hat{W}_t := W_t - \int_0^t \sigma(S_u)du$ is a standard Brownian motion and

$$dS_t = (r + h(S_t) + \sigma^2(S_t))S_t dt + \sigma(S_t)S_t d\hat{W}_t. \quad (21)$$

Particularly, for the equity volatility and hazard rate functions given by equations (18) and (19), the new dynamics of the pre-default stock price S_t can be written as

$$dS_t = [(r + c^2) + (a + bc^2)S_t^{-p}]S_t dt + cS_t\sqrt{1 + S_t^{-p}}d\hat{W}_t, \tag{22}$$

which has the same form as in the original measure with r replaced by $r + c^2$ and a replaced by $a + bc^2$.

Proposition 4.1 can be applied immediately to obtain the pricing formulae for the defaultable zero-coupon bond, European calls and European puts. For a zero-coupon bond with unit face value, let R denote the constant recovery rate paid at the maturity T. A European call option with strike price K has payoff $(S_T - K)^+$ at expiration date T if there is no default and it has no recovery if the firm defaults before T. A European put option promises a payoff $(K - S_T)^+$ at expiration if no default happens and a recovery payment K at expiration in the event of bankruptcy before expiration date T. Then, the values of a zero coupon bond, $B_R(S_0, T)$, the call, $C_K(S_0, T)$ and the put, $P_K(S_0, T)$, take the following forms respectively

$$B_R(S_0, T) = e^{-rT}R + (1 - R)S_0\hat{\mathbb{E}}[S_T^{-1}], \tag{23}$$

$$C_K(S_0, T) = S_0\hat{\mathbb{E}}[(1 - KS_T^{-1})^+], \tag{24}$$

$$P_K(S_0, T) = S_0\hat{\mathbb{E}}[(KS_T^{-1} - 1)^+] + K(e^{-rT} - S_0\hat{\mathbb{E}}[S_T^{-1}]). \tag{25}$$

Note that the put-call parity still holds here

$$C_K(S_0, T) + Ke^{-rT} = P_K(S_0, T) + S_0. \tag{26}$$

When $a = 0$, the bond becomes a default-free bond which only depends on the interest rate r and maturity T. When S_0 is close to zero, the company is almost in default. As a result, the call price and the zero bond with zero recovery will worth almost nothing. From the put-call parity, the put price will be close to the discounted value of strike price when S_0 is close to zero. For very large S_0, the hazard rate will be very small and the equity volatility will be almost a constant. Consequently, the bond price will be close to default-free bond and the call price will be close to Black-Scholes price with parameters of interest rate r and volatility c. In order to obtain general explicit expressions for these formulae, we need to study the distribution of the pre-default stock price S_T under the hat measure.

5 The Dual-Jacobi Process

Since the distribution of the pre-default stock price S_T under the hat measure and the risk-neutral measure are in the same distribution family, we will consider the format in equation (17) for the sake of simplicity. Let $Y_t := S_t^p$, where S_t is specified as in equation (17), the SDE for Y_t can be written as follows using Ito's lemma

$$\frac{dY_t}{Y_t} = [(rp + \frac{1}{2}p(p-1)c^2) + (ap + \frac{1}{2}p(p-1)bc^2)Y_t^{-1}]dt + pc\sqrt{1 + bY_t^{-1}}dW_t. \quad (27)$$

This SDE takes the form of equation (17). Therefore, in order to obtain the distribution of S_T for a general p, it is sufficient to study the case when $p = 1$.

Considering the case when $p = 1$, equation (17) is simplified into the following SDE (we use notation X_t instead of S_t here for this particular SDE)

$$dX_t = (a + rX_t)dt + c\sqrt{X_t^2 + bX_t}dW_t, \quad (28)$$

with initial value $X_0 := x_0 > 0$ at time $t = 0$. Note that this process has the form resembling a Jacobi process, but in fact it is not. For a Jacobi process, its volatility term typically looks like $c\sqrt{bX_t - X_t^2}$, which restrict the process itself to lie in the finite interval $(0, b)$.[4] In our model, however, the volatility term is $c\sqrt{bX_t + X_t^2}$ and the process lies in the half positive domain $(0, +\infty)$. We will refer to this SDE as the *Dual-Jacobi* process.

The strong unique solution of the Dual-Jacobi process is guaranteed by the Yamada & Watanabe (1971) theorem.[5] This solution is a nonnegative diffusion which has unattainable boundaries on both zero and infinity. When X_t is large, the volatility of the Dual-Jacobi process is approximately cX_t. Therefore, for large X_t, the Dual-Jacobi can be approximated as

$$dX_t = (a + rX_t)dt + cX_t dW_t.$$

[4]It is known as Wright-Fisher diffusion in probability literature, see Feng & Wang (2007)

[5]See Karatzas & Shreve (1991), page 291.

This process has a well-known explicit solution.[6] This result allows Linetsky (2006) to write a spectral representation of the transition density of their stock dynamics. The transition density has both discrete and continuous spectrums which are related to Whittaker functions and Laguerre polynomials. For sufficient large X_t when the effect of a can be neglected, we can also think of GBM as a proxy of the Dual-Jacobi process. When X_t is small, the volatility of the Dual-Jacobi is approximately $c\sqrt{bX_t}$. Consequently the Dual-Jacobi can be approximated by a CIR process which follows non-central chi-square distribution. However, to the author's best knowledge, no explicit formula for the transition density has been found for the Dual-Jacobi process.

One may resort to PDE approach and try to solve numerically. Let $f(t, x)$ denote the probability density function (pdf) of X_t at time t. This pdf satisfies the Kolmogorov Forward Equation, or Focker-Planck Equation given by[7]

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}[(a + rx)f] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[c^2(x^2 + bx)f], \tag{29}$$

for $x > 0$ with initial condition and boundary conditions given by

$$\begin{aligned} f(0, x) &= \delta(x - x_0), \\ f(t, 0) &= f(t, +\infty) = 0, \quad \forall t \geq 0. \end{aligned}$$

Since $f(t, x)$ is a pdf, it is nonnegative and it also satisfies the integrability condition

$$\int_0^\infty f(t, x)dx = 1, \quad \forall t \geq 0.$$

One nice property about Dual-Jacobi SDE is that all its moments can be explicitly calculated. The following proposition gives a recursive equation for the moments of the Dual-Jacobi process.

Proposition 5.1 *Let X_t be a Dual-Jacobi process given by (28) with initial value $X_0 = x_0$. Let $\alpha_m(t) := \mathbb{E}[X_t^m]$, $m = 0, 1, 2, ...,$ with $\alpha_0(t) = 1$. Then $\alpha_m(t)$ for*

[6]See Karatzas & Shreve (1991), page 360.

[7]This PDE is closely related to *singular* Sturm-Liouville problem with *semi-infinite* domain.

$m = 1, 2, ...,$ *satisfies the following iterative equation*

$$\begin{aligned} \alpha_m(t) &= x_0^m e^{r_m t} + a_m \int_0^t e^{r_m(t-s)} \alpha_{m-1}(s) ds, \\ r_m &:= m[r + \frac{1}{2}c^2(m-1)], \\ a_m &:= m[a + \frac{1}{2}bc^2(m-1)]. \end{aligned} \tag{30}$$

Explicit formulae for all positive moments can therefore be calculated by direct integration or applying Laplace transforms to equation (30). Particularly, the first four moments are given by

$$\begin{aligned} \alpha_1(t) &= (x_0 + \frac{a}{r})e^{rt} - \frac{a}{r}, \\ \alpha_2(t) &= (x_0^2 + A_2 - B_2)e^{r_2 t} - A_2 e^{rt} + B_2, \\ \alpha_3(t) &= (x_0^3 + A_3 - B_3 + C_3)e^{r_3 t} - A_3 e^{r_2 t} + B_3 e^{rt} - C_3, \\ \alpha_4(t) &= (x_0^4 + A_4 - B_4 + C_4 - D_4)e^{r_4 t} - A_4 e^{r_3 t} + B_4 e^{r_2 t} - C_4 e^{rt} + D_4. \end{aligned}$$

where

$$\begin{aligned} A_2 &= \frac{a_2(x_0 + \frac{a}{r})}{r_2 - r}, \ B_2 = \frac{aa_2}{rr_2}, \\ A_3 &= \frac{a_3(x_0^2 + A_2 - B_2)}{r_3 - r_2}, \ B_3 = \frac{a_3 A_2}{r_3 - r}, \ C_3 = \frac{a_3 B_2}{r_3}, \\ A_4 &= \frac{a_4(x_0^3 + A_3 - B_3 + C_3)}{r_4 - r_3}, \ B_4 = \frac{a_4 A_3}{r_4 - r_2}, \ C_4 = \frac{a_4 B_3}{r_4 - r}, \ D_4 = \frac{a_4 C_3}{r_4}. \end{aligned}$$

Proposition 5.1 motivates us to approximate the pdf of a Dual-Jacobi process by using these explicit moments.

6 The Gram-Charlier Approximation

In this section, we first transform our original stock price dynamics to a Dual-Jacobi process. Gram-Charlier Expansions are then used to obtain explicit asymptotic expansions of both zero coupon bond prices and European call prices.

Proposition 6.1 *Assume that the stock price S_t has risk-neutral dynamics specified in equation (17). Let $Y_t := S_t^p$ with $Y_0 := S_0^p$. Then under the hat measure, Y_t*

satisfies the following Dual-Jacobi SDE

$$
\begin{aligned}
dY_t &= (\hat{a} + \hat{r}Y_t)dt + \hat{c}\sqrt{Y_t^2 + bY_t}d\hat{W}_t, && (31)\\
\hat{a} &= p[a + \frac{1}{2}bc^2(p+1)], && (32)\\
\hat{r} &= p[r + \frac{1}{2}c^2(p+1)], && (33)\\
\hat{c} &= pc. && (34)
\end{aligned}
$$

This proposition can be shown by applying the hat measure change to SDE (27). From Proposition 5.1, all the moments of Y_T under the hat measure are finite and have explicit expressions. This fact motivates us to use these finite explicit moments to approximate bond and call prices. The zero coupon bond with zero recovery and the European call price can thus be expressed as

$$
\begin{aligned}
B_0(S_0, T) &= S_0\hat{\mathbb{E}}[Y_T^{-1/p}], && (35)\\
C_K(S_0, T) &= S_0\hat{\mathbb{E}}[(1 - KY_T^{-1/p})^+]. && (36)
\end{aligned}
$$

Jarrow & Rudd (1982) were the first to introduce the Gram-Charlier expansion to quantitative finance literature.[8] Ever since, the Gram-Charlier expansion has been widely used in option pricing, such as Madan & Milne (1994) and Jondeau & Rockinger (2001) etc. These expansion theories have very old history in statistics literature and they arise from approximating the distribution of a sum of independent random variables, see Cramer (1946).

Let $f(y)$ denote our unknown target pdf with unknown cdf F. Let $g(y)$ denote our known base pdf with known cdf G. Let α_i^F and α_i^G denote the i-th moments of distribution F and G respectively, for $i = 1, 2, 3, ...$ Assume all these moments are finite and have explicit expressions. Let κ_i^F and κ_i^G denote the i-th cumulants of distribution F and G respectively. All κ_i^F and κ_i^G will be known explicitly, since all moments are known. The following gives the algebraic relationships between

[8] Jarrow & Rudd (1982) named Edgeworth series in their paper, but in fact, the series expansions they used are Gram-Charlier.

cumulants and moments up to the fourth order

$$\begin{aligned}
\kappa_1 &= \alpha_1, \\
\kappa_2 &= \alpha_2 - \alpha_1^2, \\
\kappa_3 &= \alpha_3 - 3\alpha_2\alpha_1 + 2\alpha_1^3, \\
\kappa_4 &= \alpha_4 - 4\alpha_3\alpha_1 - 3\alpha_2^2 + 12\alpha_2\alpha_1^2 - 6\alpha_1^4. \\
& \quad ....
\end{aligned} \tag{37}$$

Let $\epsilon_i := \kappa_i^F - \kappa_i^G$. The Gram-Charlier expansion considers the following expression as a candidate to approximate the true pdf $f(y)$.

$$f(y) = g(y) - \frac{\eta_1 g^{'}(y)}{1!} + \frac{\eta_2 g^{''}(y)}{2!} - \frac{\eta_3 g^{'''}(y)}{3!} + ..., \tag{38}$$

where $\eta_i, i = 1, 2, ...$ are given by

$$\begin{aligned}
\eta_1 &= \epsilon_1, \\
\eta_2 &= \epsilon_2 + \epsilon_1^2, \\
\eta_3 &= \epsilon_3 + 3\epsilon_2\epsilon_1 + \epsilon_1^3, \\
\eta_4 &= \epsilon_4 + 4\epsilon_3\epsilon_1 + 3\epsilon_2^2 + 6\epsilon_2\epsilon_1^2 + \epsilon_1^4, \\
& \quad ....
\end{aligned} \tag{39}$$

Jarrow & Rudd (1982) and Cramer (1946) provided detailed derivation of the above algebraic relationships.

In our case, the target pdf is the pdf of Y_T under the hat measure. For a given function $\psi(y)$, the expectation $\hat{\mathbb{E}}[\psi(Y_T)]$ can be expanded as

$$\begin{aligned}
\hat{\mathbb{E}}[\psi(Y_T)] &:= \int_0^{+\infty} \psi(y) f(y) dy, \\
&= \int_0^{+\infty} \psi(y) g(y) dy - \frac{\eta_1}{1!} \int_0^{+\infty} \psi(y) g^{'}(y) dy \\
& \quad + \frac{\eta_2}{2!} \int_0^{+\infty} \psi(y) g^{''}(y) dy - \frac{\eta_3}{3!} \int_0^{+\infty} \psi(y) g^{'''}(y) dy + ...
\end{aligned} \tag{40}$$

The zero coupon bond and the European call can be priced using these expansions. Different choices of the base pdf will yield different expansions. The base pdf is usually chosen to be normal or log-normal distribution. Considering that Y_T is nonnegative, it is more appropriate to choose log-normal as its base pdf.

Proposition 6.2 *Assume the base pdf $g(y)$ is log-normal with parameters $\hat{\mu}$ and $\hat{\sigma}^2$. Then the prices of a zero coupon bond with zero recovery and the European call can be expanded as*

$$
\begin{aligned}
B_0(S_0,T) &= z_0 - \frac{\eta_1 z_1}{p} + \frac{\eta_2(1+p)z_2}{2!p^2} - \frac{\eta_3(1+p)(1+2p)z_3}{3!p^3} \\
&\quad + \frac{\eta_4(1+p)(1+2p)(1+3p)z_4}{4!p^4} + \ldots \\
C_K(S_0,T) &= [S_0\Phi(d_1) - K\theta_0] + \frac{\eta_1 K\theta_1}{p} - \frac{\eta_2}{2!}\left[\frac{S_0}{pK^p}g(K^p) + \frac{K(1+p)\theta_2}{p^2}\right] \quad (41)\\
&\quad + \frac{\eta_3}{3!}\left[\frac{S_0}{pK^p}g^{'}(K^p) + \frac{S_0(1+p)}{(pK^p)^2}g(K^p) + \frac{K(1+p)(1+2p)\theta_3}{p^3}\right] \\
&\quad - \frac{\eta_4}{4!}\left[\frac{S_0}{pK^p}g^{''}(K^p) + \frac{S_0(1+p)}{(pK^p)^2}g'(K^p) + \frac{S_0(1+p)(1+2p)}{(pK^p)^3}g(K^p)\right. \\
&\quad \left. + \frac{K(1+p)(1+2p)(1+3p)\theta_4}{p^4}\right] + \ldots
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{-p\log K + \hat{u}}{\hat{\sigma}}, \\
z_n &= S_0 e^{-\frac{(1+np)}{p}\hat{\mu} + \frac{(1+np)^2}{2p^2}\hat{\sigma}^2}, \\
\theta_n &= z_n\Phi(d_1 - \frac{1+np}{p}\hat{\sigma}).
\end{aligned}
$$

For log-normal base pdf, the m-th moments are all finite and given by

$$\alpha_m^G = e^{m\hat{\mu} + \frac{1}{2}m^2\hat{\sigma}^2}.$$

With explicit expressions of α_m^F and α_m^G, the coefficients η_m in the above proposition can be computed by equations (37) and (39). These coefficents depends on the model parameters a, r, c, b, p as well as initial stock price S_0. The following lemma will be needed in order to prove the above proposition.

Lemma 6.3 *Assume that random variable X is log normal with parameters μ and σ^2. Let $g(x)$ denote its pdf. Then, for any real number α and $K > 0$, the following equation holds*

$$\int_K^{+\infty} x^\alpha g(x)dx = e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}\Phi(\frac{-\log K + \mu + \alpha\sigma^2}{\sigma}).$$

Log-normal distribution is determined by two parameters. There are many possible ways to choose these two parameters, such as matching the first two moments of the target distribution. We can also match instantaneous drift and volatility. The following are some examples.

Example 6.1 *Asymptotic match.*

The base pdf is chosen to be a GBM with drift $\hat{r}$ and volatility $\hat{c}$. The log-normal parameters for the base pdf are given by

$$\begin{aligned}\hat{\mu} &= p[\log S_0 + (r + \frac{1}{2}c^2)T], \\ \hat{\sigma}^2 &= p^2 c^2 T.\end{aligned}$$

The first term for $B_0(S_0, T)$ becomes the default-free bond price. The first term for $C_K(S_0, T)$ becomes the Black-Scholes price with no default parameters. The rest of the terms are corrections accounting for default possibilties and local volatility effects.

Example 6.2 *Matching the first two moments.*

Parameters $\hat{\mu}$ and $\hat{\sigma}^2$ are chosen to match the first two moments of the target distribution. More specifically, these parameters are given by

$$\begin{aligned}\hat{\mu} &= 2\log[\alpha_1^F] - \frac{1}{2}\log[\alpha_2^F], \\ \hat{\sigma}^2 &= \log[\alpha_2^F] - 2\log[\alpha_1^F].\end{aligned}$$

This specification allows standard simplified Gram-Charlier expansion as follows

$$f(y) = g(y) - \frac{\epsilon_3}{3!}\frac{d^3 g(y)}{dy^3} + \frac{\epsilon_4}{4!}\frac{d^4 g(y)}{dy^4} - \frac{\epsilon_5}{5!}\frac{d^5 g(y)}{dy^5} + ...$$

Example 6.3 *Matching instantaneous drift and volatility.*

The base pdf is chosen to be a GBM with drift $\hat{r} + \hat{a}S_0^{-p}$ and volatility $\hat{c}\sqrt{1 + bS_0^{-p}}$. The log-normal parameters for the base pdf are then given by

$$\begin{aligned}\hat{\mu} &= p[\log S_0 + (r + \frac{1}{2}c^2)T + (a + \frac{1}{2}bc^2)TS_0^{-p}], \\ \hat{\sigma}^2 &= p^2 c^2 T(1 + bS_0^{-p}).\end{aligned}$$

Remark 6.1 *It is known that the Gram-Charlier expansion is not guaranteed to converge. Cramer (1946) has studied the Gram-Charlier expansion when the base pdf is standard normal. He showed that if the target pdf* $f(y)$ *is of bounded vairation in* $(-\infty, +\infty)$ *and* $\int_{-\infty}^{+\infty} e^{y^2/4} f(y)dy$ *is convergent, then the Gram-Charlier expansion will converge to* $f(x)$ *in every continuity point of* $f(y)$*. However, in reality, there is only a small class of distributions that will validate the Gram-Charlier expansion. In practice, most of these expansions are of the asymptotic type that they will converge for a small number of terms and then diverge, see Barndorff-Nielsen & Cox (1989). But, this does not mean these expansions will not be useful. Because if a small number of terms (usually not more than two or three) suffice to give a good approximation, it does not concern us much whether the infinite series is convergent or divergent. On the other hand, a convergent series is of little practical value if a large number of terms are required to be calculated in order to have a reasonable approximation, see Cramer (1946) for a discussion.*

7 Numerical Analysis

As mentioned in the preceding remark, our expansions are not necessarily convergent. However, we will numerically show that a small number of terms suffice to provide very good approximations. Numerical examples are used to study the residual errors and the price sensitivity to our model parameters. The Gram-Charlier approximation is much faster than the finite difference method for pricing securites. We also use numerical examples to illustrate implied volatility surface/skew and credit spreads.

7.1 Residual Error and Sensitivity

In this numerical analysis, we study two Gram-Charlier approximations which are decribed in Examples 6.2 and 6.3. We use three terms for the first approximation and four terms for the second approximation. The parameters in Table 1 is the base case parameters used for comparison study. Since no explicit pricing formulae are available, Monte-Carlo prices are used as the benchmark. Prices calculated using numerical PDE approach (implicit finite difference method) are also provided.

a	r	c	b	p	S_0	T	K	R
3.6421	0.0518	0.2923	23.5930	1.8751	7.55	0.5	7.55	0.3228

Table 1: Base Case Parameters for Numerical Analysis.

Table 2 illustrates the bond prices for varying parameters. Around 500 points per path and totally 10000 paths are used for one Monte-Carlo price simulation with variance reduction techniques. For the finite difference method, the step sizes for the stock price and the maturity are \$0.5 and 5/2400-year respectively. The first row of prices are calculated with the base case parameters given in Table 1. The rest of the rows are obtained by changing one parameter, while holding the other parameters constant as in the base case. Comparing with Monte-Carlo prices, our approximations are very good as seen in "E1" and "E2" columns: there are only a few basis points (bps) difference across all our considerations. The Mean Absolute

Relative Error (MARE), calculated based on the 15 samples, are around 0.314% and 0.24% for the two approximations respectively.

	DF	MC	FD	A1	A2	E1	E2
base case	0.9744	0.9468	0.9472	0.9440	0.9454	-0.2957	-0.1479
$a = 4.6421$	0.9744	0.9404	0.9404	0.9374	0.9348	-0.3190	-0.5955
$a = 2.6421$	0.9744	0.9543	0.9543	0.9508	0.9562	-0.3668	0.1991
$r = 0.0618$	0.9696	0.9425	0.9426	0.9394	0.9407	-0.3289	-0.1910
$r = 0.0418$	0.9793	0.9513	0.9518	0.9486	0.9502	-0.2838	-0.1156
$c = 0.3923$	0.9744	0.9446	0.9451	0.9366	0.9437	-0.8469	-0.0953
$c = 0.1923$	0.9744	0.9485	0.9485	0.9478	0.9496	-0.0738	0.1160
$b = 28.593$	0.9744	0.9468	0.9470	0.9429	0.9437	-0.4119	-0.3274
$b = 18.593$	0.9744	0.9477	0.9474	0.9451	0.9457	-0.2743	-0.2110
$p = 2.0751$	0.9744	0.9558	0.9559	0.9553	0.9550	-0.0523	-0.0837
$p = 1.6751$	0.9744	0.9344	0.9346	0.9307	0.9313	-0.3960	-0.3318
$T = 1.00$	0.9495	0.8968	0.8968	0.8951	0.8938	-0.1896	-0.3345
$T = 0.25$	0.9871	0.9732	0.9734	0.9722	0.9723	-0.1028	-0.0925
$S_0 = 8.55$	0.9744	0.9526	0.9527	0.9502	0.9514	-0.2519	-0.1260
$S_0 = 6.55$	0.9744	0.9394	0.9393	0.9351	0.9367	-0.4577	-0.2874
$R = 0.4228$	0.9744	0.9513	0.9512	0.9485	0.9568	-0.2943	0.5782
$R = 0.2228$	0.9744	0.9432	0.9432	0.9395	0.9407	-0.3923	-0.2651
					MARE	0.3140	0.2411

Table 2: Zero coupon bond prices. "DF" denotes the default-free price. "MC" denotes the Monte-Carlo price. "FD" denotes the finite difference price. "A1" denotes the first approximation price. "A2" denotes the second approximation price. The relative error (in %) "E1" is defined as E1:=(A1-MC)/MC. The relative error (in %) "E2" is defined as E2:=(A2-MC)/MC. "MARE" denotes the mean absolute relative error (in %).

Note that our model prices are consistantly smaller than the default-free bond prices. This is largely due to none-zero default parameters a and p. When a increases or p decreases, the credit deteriorates and hence the bond price drops. Our model prices will go down as default-free bond does when interest rate r or maturity T increases. When the volatility parameter c increases, the bond price will go down to reflect more risk. For a larger initial stock price S_0, the bond price will be higher since the company is less likely to default. We also find that the bond price is relatively insensitive to the parameter b.

Table 3 shows the European call prices for various scenarios. The Black-Scholes price in Table 3 is calculated by imposing $a = b = 0$. Our approximations are reasonably good compared with the Monte-Carlo price. The MAREs for the two approximation approaches are around 0.3885% and 1.2290% respectively. The call prices are found to be under valued by Black-Scholes' formula, compared with our model prices. The call prices will go up as interest rate r rises, or volatility c increases, or maturity T increases, or the initial stock price S_0 becomes higher, or the strike price K becomes smaller. This is consistant with Black-Scholes' formula. When parameters a increases or p decreases, or b increases, the call prices will go up to compensate for more risks.

	BS	MC	FD	A1	A2	E1	E2
base case	0.7148	0.9881	0.9884	0.9856	0.9836	-0.2530	-0.4554
$a = 4.6421$	0.7148	1.0287	1.0249	1.0232	1.0253	-0.5347	-0.3305
$a = 2.6421$	0.7148	0.9542	0.9522	0.9484	0.9388	-0.6078	-1.6139
$r = 0.0618$	0.7336	1.0100	1.0075	1.0048	1.0008	-0.5149	-0.9109
$r = 0.0418$	0.6962	0.9673	0.9694	0.9665	0.9625	-0.0827	-0.4962
$c = 0.3923$	0.9223	1.2351	1.2351	1.2274	1.1998	-0.6234	-2.8581
$c = 0.1923$	0.5078	0.7530	0.7479	0.7540	0.7624	0.1328	1.2483
$b = 28.593$	0.7148	1.0143	1.0143	1.0084	0.9987	-0.5817	-1.5380
$b = 18.593$	0.7148	0.9670	0.9615	0.9620	0.9637	-0.5171	-0.3413
$p = 2.0751$	0.7148	0.9025	0.9001	0.9012	0.8997	-0.1440	-0.3102
$p = 1.6751$	0.7148	1.1167	1.1152	1.1079	1.0992	-0.7880	-1.5671
$T = 1.00$	1.0590	1.4985	1.4979	1.4964	1.4858	-0.1401	-0.8475
$T = 0.25$	0.4873	0.6591	0.6567	0.6605	0.6576	-0.2124	-0.2276
$S0 = 8.55$	1.4160	1.6794	1.6781	1.6794	1.6487	-0.0000	-1.8280
$S0 = 6.55$	0.2565	0.4874	0.4845	0.4833	0.4972	-0.8412	2.0107
$K = 8.55$	0.3354	0.5456	0.5488	0.5485	0.5571	-0.5315	2.1078
$K = 6.55$	1.3356	1.6221	1.6210	1.6237	1.5864	-0.0986	-2.2009
					MARE	0.3885	1.2290

Table 3: European call prices. “BS” denotes the Black-Scholes price. “MC” denotes the Monte-Carlo price. “FD” denotes the finite difference price. “A1” denotes the first approximation price. “A2” denotes the second approximation price. The relative error (in %) “E1” is defined as E1:=(A1-MC)/MC. The relative error (in %) “E2” is defined as E2:=A2-MC. “MARE” denotes the mean absolute relative error (in %).

7.2 Accuracy versus Speed

For the finite difference method in Tables 2 and 3, the average computational time is 0.1442 second per call price and 0.1353 second per bond price. For the first approximation scheme in Tables 2 and 3, the average computational time is 0.002 second per call price and 0.0012 second per bond price. The Gram-Charlier approximation scheme is more than 70 times faster than the implicit finite difference method on pricing calls, and more than 100 times faster on pricing bonds.

Table 4 illustrate a comparison of the computaional time for using finite difference method versus the Gram-Charlier approximation to price bonds. As maturity T increases, while holding other parameters constant, the Gram-Charlier approximation becomes less accurate than the finite difference method. However, the finite difference method becomes much slower than the Gram-Charlier approximation. The computational time for the second approximation is constant 0.0012 second, which does not depend on maturity T. For the finite difference method, the computational time increases almost 10 times, when T increases from 1-year to 10-year. There is a trade off between the accuracy and the computation speed. The accuracies of the approximations are not universally the same. The expansions may diverge for certain parameters if inappropriate terms are used. Therefore, we recommend using Monte-Carlo or PDE approach to check the numerical residual errors before using the asymptotic expansions.

7.3 The Implied Volatility Surface and Credit Spreads

In this section, we apply our model to some real observed data and exam its goodness of fit. Figure 3(a) plots the observed implied volatility surface against moneyness (strike over spot) and maturity of the option. The real data is taken from Ford Motor Corp. on March 16, 2007 from Bloomberg. Figure 3(b) shows the theoretical implied volatility surface computed by using the calibrated parameters. We first calculate the model call price using the first approximation for a given moneyness and maturity. Then the implied volatilities are obtained by implying the Black-Scholes implied volatility. The stock price of Ford on March 16, 2007 is \$7.55. The short rate r is taken to be 5.18%, which is the 1-month U.S Treasury yield on March 16, 2007. The parameters a, c, b, and p are calibrated by minimizing the differences of the observed and theoretical implied volatilites in the sense of the Root of Mean

	MC	FD	A1	TFD	TA1
T = 0.5	0.9806	0.9805	0.9806	0.1353	0.0012
T = 1	0.9602	0.9613	0.9613	0.2858	0.0012
T = 2	0.9207	0.9237	0.9238	0.5164	0.0012
T = 3	0.8880	0.8872	0.8880	0.8239	0.0012
T = 4	0.8478	0.8520	0.8540	1.0708	0.0012
T = 5	0.8165	0.8178	0.8218	1.3135	0.0012
T = 6	0.7811	0.7848	0.7914	1.5884	0.0012
T = 7	0.7487	0.7530	0.7627	1.8490	0.0012
T = 8	0.7121	0.7223	0.7355	2.1448	0.0012
T = 9	0.6874	0.6928	0.7098	2.3755	0.0012
T = 10	0.6564	0.6644	0.6855	2.6148	0.0012

Table 4: Acuracy versus Speed. "MC" denotes the Monte-Carlo price. "FD" denotes the finite difference price. "A1" denotes the second approximation price. "TFD" denotes the computaional time for FD method (in seconds). "TA1" denotes the computational time for A1 method (in seconds).

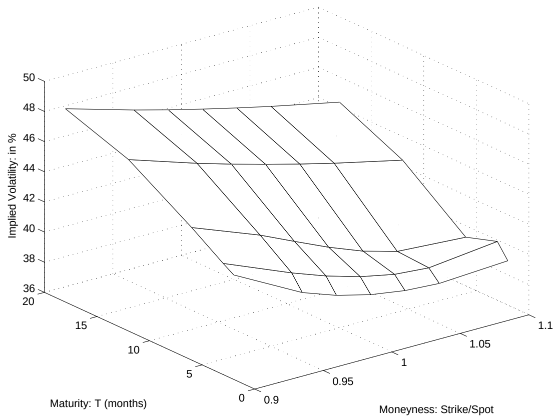
Square Error (RMSE).

As seen in Figures 3(a) and 3(b), the theoretical surface is able to capture the main two properties of real implied volatility surface: a) negative skew for fixed maturity; b) as maturity increases, the skew becomes less pronounced. Table 5 provides numerical details of the observed and model implied surfaces. The RMSE across all the data on the volatility surface is found to be 0.5472%, which is very small.

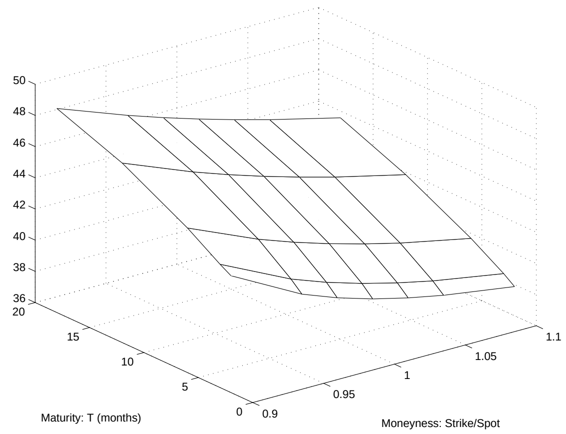
Figure 4 demonstrates implied volatility skews from the model. Both the level and the slope (in absolute value) of the implied volatility skew increase as the creditworthiness deteriorates (i.e. the default parameter a increases or p decreases). For large volatility parameters c and b, the level of the implied volatility will be higher but the slope of the skew is not affected very much. For higher moneyness, the implied volatility asymptotically goes down to volatility parameter c.

The term structure of CDS on March 16, 2007 from Ford is also taken from Bloomberg. We then estimate the recovery rate R, by minimizing the differences between the market CDS spreads and model CDS spreads, while holding the parameters a, c, b and p constant as calibrated from the volatility surface. The recovery rate is estimated at 0.3228. However, the CDS term structure, generated by the parameters calibrated from the volatility surface, is unable to match the real observed CDS term structure. As seen in Figure 5, the market CDS curve is upwarding, but the model CDS curve implied the from volatility surface is downward shaped. This discrepency may indicate either the model is wrong for Ford, or there exists arbitrage opportunities.

As an inverse exercise, we can first calibrate the parameters by fitting the CDS curve and then compute the implied volatility surface using these calibrated parameters. For this exercise, we calibrate R, a, c, b, and p using the CDS data. As seen in Figure 6, the calibrated CDS curve fits the market curve very well. The RMSE of the calibrated CDS curve is found to be 9.7122 bps, which is very small considering large credit spreads of Ford. However, the implied volatility surface, calculated using the parameters calibrated from the CDS data, does not provide a good fit to the observed volatility surface. As seen in Table 6, this model implied volatility surface is too flat and too high compared with the observed surface.



(a) Observed Implied Volatility Surface.



(b) Model Implied Volatility Surface.

Figure 3: Implied Volatility Surfaces: Observed vs. Model Implied. The observed surface is from Ford Corp. on March 16, 2007. The calibrated parameters are: $a = 3.6421$, $c = 0.2923$, $b = 23.5930$, and $p = 1.8751$.

Observed	90	95	97.5	100	102.5	105	110
T=2m	42.9223	40.5271	39.7240	39.1576	38.8114	38.6691	38.9445
T=3m	43.3745	41.5105	40.6955	40.0286	39.5712	39.3848	39.9058
T=6m	44.7892	43.0559	42.0286	41.0245	40.1506	39.5141	39.2246
T=12m	47.3736	45.8992	45.2300	44.5992	44.0032	43.4409	42.4144
T=18m	48.8218	47.5070	46.9075	46.3397	45.7997	45.2870	44.3412
Model	90	95	97.5	100	102.5	105	110
T=2m	43.4910	41.0786	40.2266	39.5444	38.9939	38.5458	37.8710
T=3m	43.9048	41.7269	40.9028	40.2144	39.6368	39.1495	38.3823
T=6m	45.2695	43.3181	42.5120	41.8007	41.1722	40.6158	39.6820
T=12m	47.5087	45.7109	44.9216	44.1976	43.5332	42.9232	41.8471
T=18m	49.0884	47.4113	46.6557	45.9506	45.2926	44.6784	43.5690

Table 5: Implied Volatility Surfaces: Observed vs. Model Implied (in %). The row with 90,...,110, denotes moneyness (in %). The root mean square error is 0.5472%.

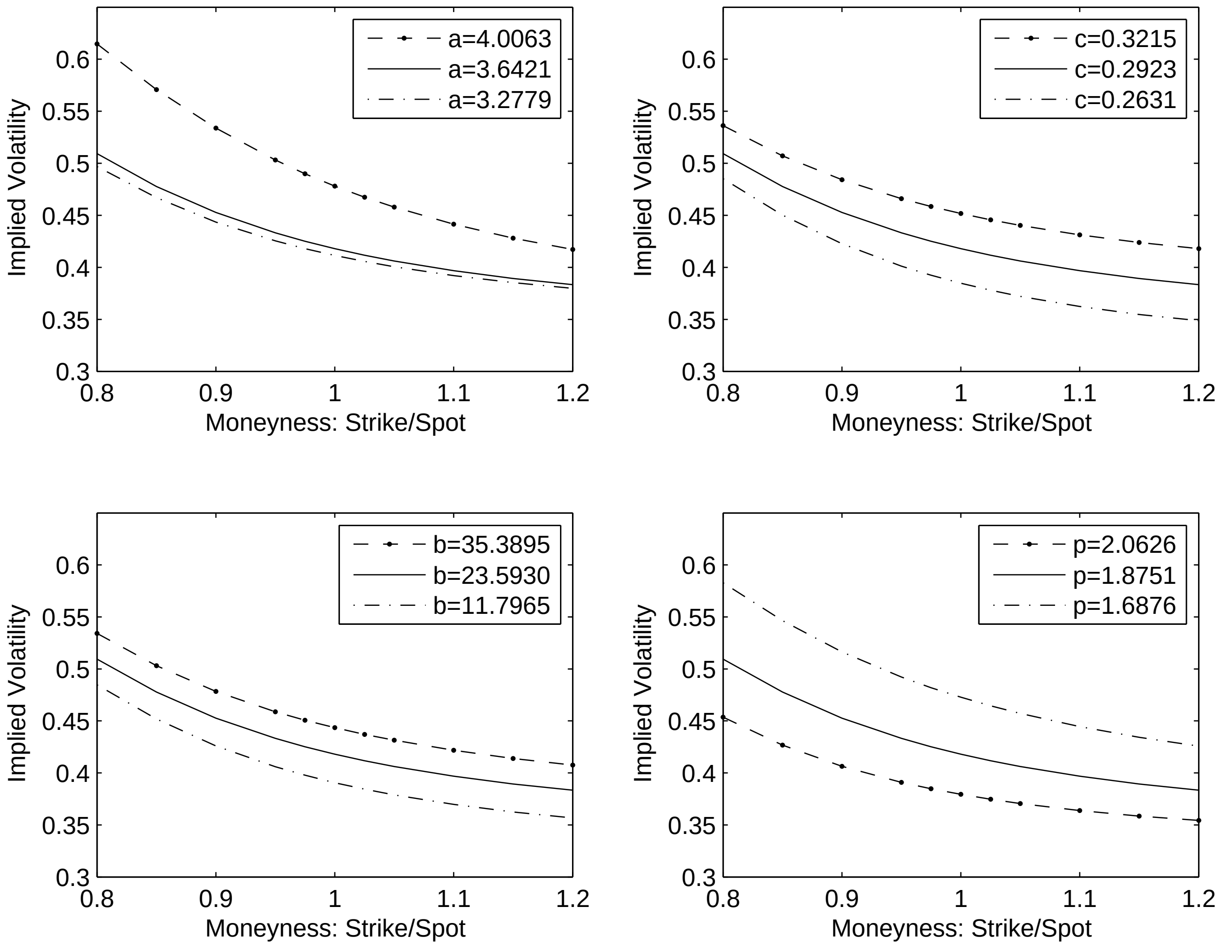


Figure 4: Model Implied Volatility Skew. Base parameters are chosen to be the parameters in Table 1.

Table 7 illustrates the two sets of parameters calibrated using separate dataset of implied volatility and CDS data. The two sets of parameters are very different, except for p. One may wants to calibrate the model using implied volatility and CDS data simultaneously. A more advanced calibration scheme is needed to do this, since the naive minimizing RMSE is not feasible here. First, these two datasets have different denomination. The implied volatility data is denominated in percentage, while the CDS data is usually expressed in bps. Even though we can express CDS in percentage, it is not clear how to compare 50% implied volatility to 5% CDS. This is similar as asking if an orange is more beautiful than an apple. Second, there is no

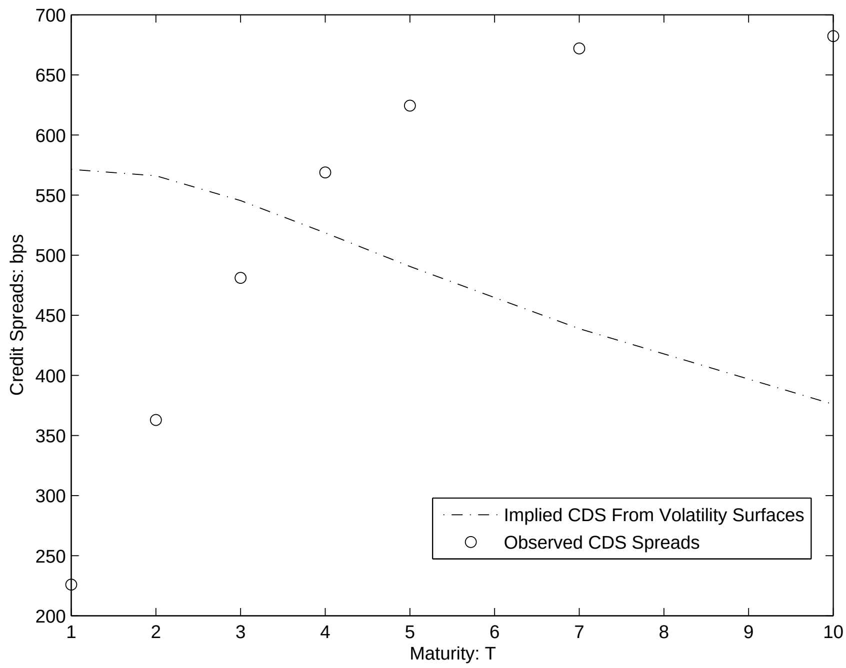


Figure 5: Term Structure of CDS Spreads. The dashed curve is calculated using the parameters calibrated from the implied volatility surface on March 16, 2007. The circle-dot is observed CDS term structure of Ford on March 16, 2007. The recovery rate is estimated at 0.3228.

Model	90	95	97.5	100	102.5	105	110
T=2m	64.0690	63.8675	63.7885	63.7205	63.6614	63.6097	63.5241
T=3m	64.1805	63.9865	63.9071	63.8371	63.7749	63.7195	63.6250
T=6m	64.4496	64.2677	64.1893	64.1179	64.0527	63.9929	63.8871
T=12m	64.6959	64.5432	64.4750	64.4115	64.3523	64.2970	64.1965
T=18m	64.6928	64.5694	64.5135	64.4610	64.4116	64.3650	64.2793

Table 6: Implied Volatility Surfaces (in %): using model parameters calibated from CDS spreads. The row with 90,...,110 denotes moneyness (in %). The root mean square error is 9.7122.

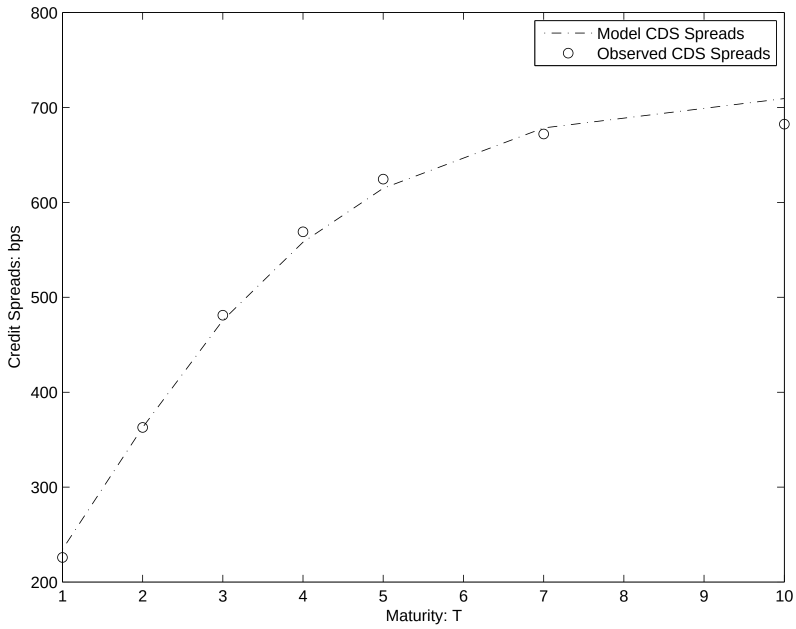


Figure 6: Fitting The Term Structure of CDS Spreads. The circle-dot is observed CDS term structure of Ford on March 16, 2007. The dashed-curve is model predicted CDS curve. Calibrated parameters are: $a = 0.6417$, $c = 0.6252$, $b = 1.0071$, $p = 1.8865$ and $R = 0.0094$.

consensus way how to assign weights to different datasets. The calibrated parameters will be different for arbitrary choice of weights. The weights we use here for this exercise are 0 for one dataset and 1 for the other.

	a	c	b	p	R
VolPara	3.6421	0.2923	23.5930	1.8751	0.3228
CDSPara	0.6417	0.6252	1.0071	1.8865	0.0094

Table 7: Comparison of Calibrated Parameters. "VolPara" denotes the parameters calibrated from the implied volatility data of Ford Motor Corp. on March 16, 2007. "CDSPara" denotes the parameters calibrated from the CDS data of the same company on the same day.

Table 8 shows calibrated parameters from daily observed implied volatility surface data in March 2007. The calibrated parameters have large variations. The mean of the calibrated parameters are $a = 1.1105$, $c = 0.1937$, $b = 47.6545$ and $p = 1.2973$. We observe that parameter b is significantly larger than zero. This indicates that the component bS_t^{-p} in the the volatility function $\sigma(S_t)$ has large contributions to the observed implied volatility.

In this calibration exercise, we find that our model is able to fit very well the market implied volatility surface and the market CDS curve separately on Ford's data. However, it is unsuccessful to fit these two datasets jointly. Similar results have been obtained by Hull et al. (2004), who used implied volatility data to calibrate default parameters in Merton's model. They find that the model CDS spreads, calculated using the parameters calibrated from the implied volatility surface, is about 95 bps higher than the observed CDS spreads on average. Carr & Wu (2006) considered a two factor model with stochastic volatilities and stochastic hazard rates. Carr & Wu (2006) applied the Extended Kalman Filter (EKF) and calibrated their model using combined datasets of implied volatilities and CDS spreads. However, their calibartion does not provide satisfactory results either, especially for fitting CDS spreads. About 50 percent of the variation in the CDS spreads on General Motors and only 30 percent on Altria Group can be explained by their model.

8 Extension of The Model

In our model, the local equity volatity is chosen to be of the form implied from the Leland-Toft structural model. A natural extention of the model is to choose the hazard rate function to be of the form implied from other structural models with jumps.

One shortcoming of Leland-Toft model is that the short spreads are zero. A natural approach to raise the short spreads above zero is to add jumps. Hilberink & Rogers (2002) extended the Leland-Toft model by generalizing the firm value process to be an exponential Levy with only downward jumps. Chen & Kou (2006) introduced double exponential jumps to Leland-Toft and studied the implication of the credit spreads and implied volatility. We are interested in deriving the implied hazard rate in Hilberink & Rogers (2002) and Chen & Kou (2006) as a function of the equity.

Both of the papers obtain an implicit expression of the hazard rate, h, as follows

$$h = \lambda \left(\frac{V_B}{V} \right)^{\eta}, \tag{42}$$

where λ denotes the Poisson rate of downward jumps with exponential distribution of parameter η. As before, V_B denotes the default barrier determined endogenously and V denote the asset value. However, neither of the studies give an explicit formula for h as a function of the equity E.

When V is close to V_B, following Hilberink & Rogers (2002), we can approximate the equity as follows

$$E \approx V - V_B(1 + \log(V/V_B)). \tag{43}$$

Then applying Taylor expansion, to the first order approximation, we obtain

$$\frac{V}{V_B} \approx \sqrt{\frac{2E}{V_B}} + 1. \tag{44}$$

Therefore, equations (42) and (44) imply that the hazard rate, $h(E)$, as a function of the equity can be approximated by

$$h(E) \approx \lambda \left(\sqrt{\frac{2E}{V_B}} + 1 \right)^{-\eta}. \tag{45}$$

It is worth to note the following observations of the above implied hazard rate function. First, the implied hazard rate is a decreasing function of E/V_B. This captures a negative relationship of equity price and the hazard rate. Second, the implied hazard rate vanishes to zero as the equity E approaches infinity. This is reasonable since a company with infinite equity will not default. Third, the implied hazard rate is bounded above by the Poisson rate of downward jumps λ. The third observation differs significantly from the specification of hazard rate in Carr & Linetsky (2006) in that their hazard rate is unbounded, when the stock price goes to zero. An upper bounded hazard rate captures the fact that the firm value not only can jump-to-default, but also can diffuse down to the bankruptcy level. Figure 7 plots different curves of $h(E)$, for varying η.

Using the implied hazard rate function from Hilberink-Rogers, we are interested in the following model with the pre-default stock price S_t specified as

$$dS_t = [r + \lambda(a\sqrt{S_t} + 1)^{-\eta}]S_t dt + cS_t\sqrt{1 + bS_t^{-p}}dW_t.$$

Gram-Charlier approximation is not feasible for this model, since we do not have explicit expressions for its moments. Numerical PDE approach can be applied to study its properties and implications.

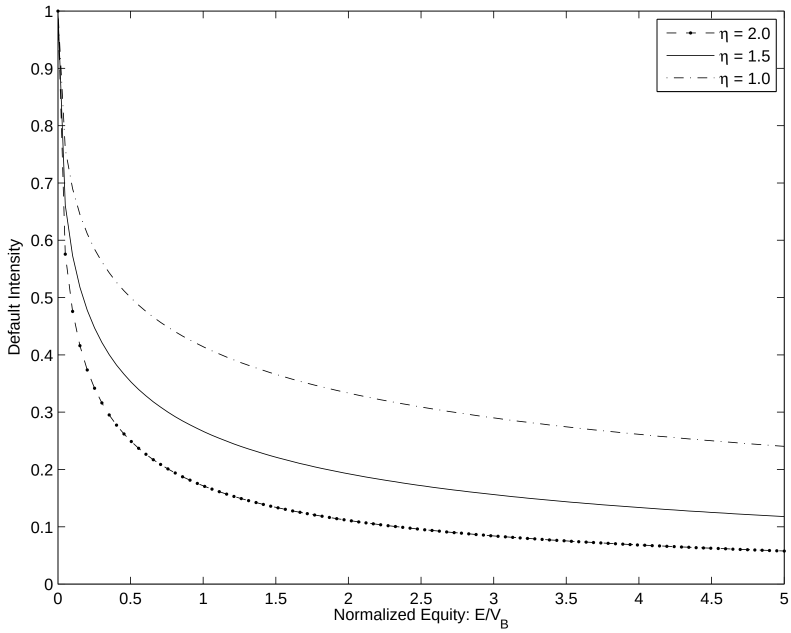


Figure 7: Hazard Rate implied from Hilberink & Rogers (2002) as a function of equity. The x-axis denotes the normalized equity, E/V_B. The y-axis denotes the hazard rate with maximum $\lambda = 1$.

9 Summary

Motivated by Linetsky (2006) and Carr & Linetsky (2006), we have analyzed the implied equity volatility as a function of the stock price, from the structural credit risk literature. We have derived that the implied equity volatility in Leland-Toft was a power decay function of the equity, bounded below by a positive constant, which can be thought of as the asset volatility. We then proposed a new jump-to-default model with the local volatity implied from Leland & Toft (1996). All the moments can be calculated explicitly for the transformed pre-default stock process. The Gram-Charlier expansions were then used to approximate bond and call prices.

Numerical examples showed that the asymptotic approximation is very accurate and more than 70 times faster than the implicit finite difference method. The model was calibrated separately using one day implied volatility data and CDS data on the same day for Ford Corp. We found that our model could fit the separate dataset very well respectively. However, it is unable to fit both the implied volatility surface and the CDS curve simultaneously. This indicates that either this model is not feasible for Ford or there exists arbitrage opportunities.

Our model is different from Carr & Linetsky (2006) in that our local volatility specification is bounded below by a positive constant, while the volatility of CEV model goes to zero when the stock price approaches infinity. Our model is more realistic for firms that have large equity but low debt. We also discussed an extention of the model, whose hazard rate function is implied from Hilberink & Rogers (2002) and Chen & Kou (2006).

10 Appendix III

- **Proof of Proposition 4.1:**

  From equation (5) we have

$$S_T = S_0 e^{\int_0^T r + h(S_t)dt + \int_0^T \sigma(S_t)dW_t - \frac{1}{2}\int_0^T \sigma^2(S_t)dt},$$

  and therefore

$$\mathbb{E}[e^{-\int_0^T r+h(S_t)dt}\psi(S_T)] = S_0\mathbb{E}[e^{\int_0^T \sigma(S_t)dW_t - \frac{1}{2}\int_0^T \sigma^2(S_t)dt} S_T^{-1}\psi(S_T)],$$

  The proof is completed by invoking Girsanov's theorem.

- **Proof of Proposition 5.1:**

  Ito's lemma implies that

$$d[X^m] = [l_a(m)X^{m-1} + l_r(m)X^m]dt + cmX^{m-1}\sqrt{X^2 + bX}dW.$$

Taking integral and then expectation will complete the proof.

- **Proof of Proposition 6.2:**

  We will only prove for $B_0(S_0, T)$ here. The proof for $C_K(S_0, T)$ is analogous. The Gram-Charlier expansion implies that

$$\begin{aligned} B_0(S_0,T) &= S_0 \int_0^{+\infty} y^{-1/p} g(y) dy - \frac{\eta_1 S_0}{1!} \int_0^{+\infty} y^{-1/p} g'(y) dy \\ &+ \frac{\eta_2 S_0}{2!} \int_0^{+\infty} y^{-1/p} g''(y) dy - \frac{\eta_3 S_0}{3!} \int_0^{+\infty} y^{-1/p} g'''(y) dy + ... \end{aligned}$$

  We then calculate term by term.

$$\begin{aligned} S_0 \int_0^{+\infty} y^{-1/p} g(y) dy &= S_0 e^{-\frac{1}{p}\hat{\mu} + \frac{1}{2p^2}\hat{\sigma}^2}. \\ S_0 \int_0^{+\infty} y^{-1/p} g'(y) dy &= S_0 \left[y^{-1/p} g(y)|_0^{+\infty} + \frac{1}{p} \int_0^{+\infty} y^{-\frac{1}{p}-1} dy \right] \\ &= \frac{S_0}{p} e^{-\frac{(1+p)}{p}\hat{\mu} + \frac{(1+p)^2}{2p^2}\hat{\sigma}^2}. \end{aligned}$$

  We have used the lemma 6.3 for the above calculations. The other terms can be calculated similarly.

References

Barndorff-Nielsen, O. & Cox, D. (1989), *Asymptotic Techniques for Use in Statistics*, Chapman and Hall.

Black, F. (1976), 'Studies of stock price volatility changes', *Proceedings of the 1976 American Statistical Association, Business and Economics Statistics Section* pp. 177–181.

Black, F. & Cox, J. C. (1976), 'Valuing corporate securities: Some effects of bond indenture provisions', *Journal of Finance* **31**(2), 351–367.

Carr, P. & Linetsky, V. (2006), 'A jump to default extended cev model: An application of bessel processes', *Finance and Stochastics* **10**(3), 303–330.

Carr, P. & Wu, L. (2006), Stock options and credit default swaps: A joint framework for valuation and estimation. Working paper, New York University.

Chen, N. & Kou, S. (2006), 'Credit spreads, optimal capital stucture, and implied volatility with endogenous default and jump risk', *Working Paper, Columbia University* .

Cramer, H. (1946), *Mathematical Methods of Statistics*, Princeton University Press.

Duffie, D. & Singleton, K. J. (1999), 'Modeling term structures of defaultable bonds', *Review of Financial Studies* **12**(4), 687–720.

Dupire, B. (1994), 'Pricing with a smile', *Risk* **7**, 18–20.

Feng, S. & Wang, F.-Y. (2007), 'A class of infinite dimensional diffusion processes with connection to population genetics', *Journal of Applied Probability* **44**(4), 938–949.

Hilberink, B. & Rogers, L. C. G. (2002), 'Optimal capital structure and endogeneous default', *Finance and Stochastics* **6**(2), 237–263.

Hull, J., Nelken, I. & White, A. (2004), 'Merton's model, credit risk, and volatility skews', *Journal of Credit Risk* **1**(1), 3–27.

Jarrow, R. & Rudd, A. (1982), 'Approximate option valuation for arbitrary stochastic processes', *Journal of Financial Economics* **10**, 347–369.

Jondeau, E. & Rockinger, M. (2001), 'Gram-charlier densities', *Journal of Economical Dynamics and Control* **25**, 1457–1483.

Karatzas, I. & Shreve, S. (1991), *Brownian Motion and Stochastic Calculus*, Springer.

Leland, H. (1994), 'Corporate debt value, bond covenants, and optimal capital structure', *Journal of Finance* **49**(4), 1213–1252.

Leland, H. & Toft, K. B. (1996), 'Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads', *Journal of Finance* **51**(3), 987–1019.

Linetsky, V. (2006), 'Pricing equity derivatives subject to bankruptcy', *Mathematical Finance* **16**(2), 255–282.

Madan, D. & Milne, F. (1994), 'Contingent claims valued and hedged by pricing and investing in a basis', *Mathematical Finance* **4**, 223–245.

Merton, R. (1974), 'On the pricing of corporate debt: The risk structure of interest rates', *Journal of Finance* **29**(2), 449–470.

Yamada, T. & Watanabe, S. (1971), 'On the uniqueness of solutions of stochastic differential equations', *J. Math. Kyoto Univ.* **11**, 155–167.

mm/dd/yy	a	c	b	p
03/02/07	0.1290	0.2253	53.8955	1.4321
03/05/07	0.4328	0.1970	89.9888	1.5359
03/06/07	0.9227	0.2019	53.7872	1.3986
03/07/07	1.2158	0.2597	64.0233	1.8103
03/08/07	1.0758	0.2002	86.7238	1.5823
03/09/07	0.9820	0.1648	73.4251	1.3491
03/12/07	0.2719	0.2007	80.0007	1.4426
03/13/07	0.3554	0.1445	29.8695	0.7924
03/14/07	1.3785	0.2562	30.0186	1.5021
03/15/07	1.2250	0.2501	30.0157	1.4726
03/16/07	3.6421	0.2923	23.5930	1.8751
03/19/07	1.0784	0.1675	24.9472	1.1088
03/20/07	0.2720	0.0913	29.9566	0.5109
03/21/07	1.4006	0.2397	15.0315	1.3369
03/22/07	1.8488	0.1903	51.6305	1.5006
03/23/07	2.0773	0.2529	57.2209	1.7950
03/26/07	1.5601	0.2069	44.2211	1.4781
03/27/07	1.5739	0.2003	44.2211	1.4389
03/28/07	1.5828	0.1913	44.2210	1.4144
03/29/07	0.1413	0.0578	44.1666	0.1973
03/30/07	0.1536	0.0771	29.7862	0.2684
Mean	1.1105	0.1937	47.6545	1.2973
	(0.8372)	(0.0611)	(21.5153)	(0.4688)

Table 8: Calibrated parameters for Ford Motor Corp. from daily implied volatility surface data in March, 2007. The numbers in the parenthesis are standard deviations.