

Inverse CIR and Semi-Affine Intensity-based Modeling on Credit Risk

Tom Hurd*, Chuang Yi †

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Abstract

In this paper, we propose a semi-affine intensity-based model on credit risk. The default-free interest rate, $r(t)$, is modeled as an inverse CIR (iCIR) process, which was firstly studied by Ahn & Gao (1999). The firm's short spread, $h_j(t)$, for firm j , is modeled as a linear combination of the default-free interest rate, the inverse of the interest rate and another idiosyncratic CIR factor. This model extends the Ahn & Gao (1999) one factor iCIR model on the interest rate to a multi-factor model on credit risk. This parsimonious model allows us to form rich correlation structures between the short spreads of different firms, while the non-negativity conditions for interest rates and short spreads are satisfied. The pricing formulae for default-free and defaultable bonds are non-affine in the common factor, but are affine in the idiosyncratic factor. These bond pricing formulae are obtained in explicit forms using the recent findings of stochastic integrals by Hurd & Kuznetsov (2006).

Keywords: iCIR, Intensity, Semi-Affine, Factor Models, Credit Spreads.

*Professor, Dept. of Mathematics and Statistics, McMaster University

†PH.D candidate in Financial Math, McMaster University. Email: yichuang@math.mcmaster.ca

1 Introduction

The CIR process was first introduced and used to model the default-free interest rate by Cox, Ingersoll & Ross (1985). Since then, it has received great attention by both the academic researchers and the industrial practitioners. The CIR process has become so popular because it is the simplest nonnegative process which admits affine term structure of bond pricing formulae. However, the empirical study by Pearson & Sun (1994) rejects CIR as a good candidate for describing the Treasury market. Duan & Simonato (1999) also provided support in favor of rejection. In order to capture the general shapes of the yield curves, additional two CIR factors may be needed as suggested by Chen & Scott (2003). Although multi-factor models are generally better than single factor models for the default-free interest rate, the expenses are the addition of too many parameters.

Chan, Karolyi, Longstaff & Sanders (1992) assumed that the volatility of the short rate is a power function of the form $\sigma r(t)^\beta$. Their estimation of the volatility is very accurate and their results strongly suggest that $\beta = 1.5$ instead of simply 0.5 (i.e. the square root). Without specifying the parametric form of the short rate process $r(t)$, econometricians, such as Ait-Sahalia (1996*a*), Ait-Sahalia (1996*b*) and Stanton (1997), applied nonparametric techniques to estimate the drift and diffusion as a function of the short rate. Their findings suggest that the drift is non-linear in $r(t)$. Moreover, they also find that the volatility is proportional to $r(t)^{1.5}$, which is similar to that estimated by Chan et al. (1992). Inspired by these findings, Ahn & Gao (1999) proposed a non-affine model that the diffusion of the short rate is proportional to $r(t)^{1.5}$ and that the drift is quadratic in $r(t)$. It turns out that their model is the same as an inverse of the CIR process. Their empirical studies also show that the iCIR model outperforms the one-factor affine type models in fitting the default-free yield curves.

The CIR process has been widely used in the literature of credit risk as well, ever since Lando (1998) and Duffie & Singleton (1999) built the foundations for intensity-based approach to credit risk. Duffie (1999) empirically estimated a model with two

CIR factors for the short rate process $r(t)$ and three CIR factors for the hazard rate process $h_j(t)$ for firm j , i.e.

$$\begin{aligned} r(t) &= -1 + X_1(t) + X_2(t), \\ h_j(t) &= \alpha_j + \beta_{1,j}X_1(t) + \beta_{2,j}X_2(t) + Y_j(t), \end{aligned}$$

where $X_i(t)$, $i = 1, 2$ and $Y_j(t)$, $j = 1, \dots, n$ are independent CIR factors, and α_j , $\beta_{i,j}$ are constants. He found that most $\beta_{i,j}$ are negative. As a consequence, the interest rate and the hazard rate are found to be negatively correlated, which is consistent with most empirical findings, such as Longstaff & Schwartz (1995). However, negative coefficients are problematic. First, there is a positive probability that the hazard rate process $h_j(t)$ will go below zero, which is counterfactual. Second, if all the $\beta_{i,j}$ are negative, the correlations between any two firms' hazard rates can only be positive, which is too restrictive. In order to have nonnegativity property of hazard rates for this model, we have to impose nonnegative coefficients. But, if all $\beta_{i,j}$ are positive, neither negative correlation between interest rates and hazard rates are captured, nor can this model produce rich correlation structure among hazard rates. This well known dilemma is not only for this model, but it is faced by all classical factor models, as discussed by Schönbucher (2003) and Duffie & Singleton (2003).

A good model should be able to reflect the realistic properties as much as possible, while maintaining analytical tractability. For a good intensity-based factor model, we would like the following realistic properties to be satisfied.

- Nonnegativity: interest rates and short spreads should be nonnegative.
- Rich Correlation Structure: first, interest rates and short spreads should be negatively correlated in most cases as suggested in the empirical literature; second, both negative and positive correlations among different firms should be allowed.
- Tractability: the pricing formulae should admit explicit expression for both default-free and defaultable bonds.

In this paper, we propose a parsimonious model which is flexible enough to produce rich correlation structure among short spreads of different firms, while nonnegativ-

ity conditions are also satisfied. Following Ahn & Gao (1999), we model the interest rate $r(t)$ as an iCIR process. We then extend Ahn & Gao (1999) to a two-factor model for the short spread. More specifically, we model the short spread as a linear combination of the interest rate, the inverse of the interest rate and another idiosyncratic CIR factor. The coefficients in the model of the interest rate and the short spread are imposed to be nonnegative, in order to maintain the nonnegativity property. Our model is able to generate rich correlation structure among short spreads of different firms, while capturing the empirical fact that interest rates and short spreads are negatively correlated in most cases. This model remedies the efficacy for affine intensity-based factor models, which can only produce very restrictive correlation structure among short spreads of different firms. The pricing formulae of default-free and defaultable bonds are non-affine, but are obtained in explicit forms using the recent findings of stochastic integrals by Hurd & Kuznetsov (2006).

The rest of the paper is organized as follows. In Section 2, we study CIR and iCIR processes and recall some elementary results from Ahn & Gao (1999). In Section 3, we model the interest rate as an iCIR process, and assume a two-factor model for the short spread. In Section 4, both default-free and defaultable zero coupon bond prices are given in explicit formulae. In Section 5, we focus on the correlation structure among short spreads of different firms, and the correlation structure between each firm's short spreads and the default-free interest rates. In Section 6, some numerical analysis is provided. Calibration issues are discussed in Section 7. In Section 8, we discuss an extension of our model. Section 9 summarizes this chapter. All proofs are given in the appendix of this chapter.

2 CIR and iCIR

We start with some simple facts about the CIR process and its inverse. Let $X(t)$ be a CIR process, starting at $X(0) = x_0 > 0$, specified as

$$dX(t) = (a - bX(t))dt + c\sqrt{X(t)}dW(t), \quad (1)$$

where a, b, c are positive constants and $W(t)$ is standard Brownian motion. We denote this CIR process as $CIR(a, b, c, x_0)$. The conditional and unconditional means and variances are well known and are given by

$$\mathbb{E}[X(t)] = \frac{a}{b} + (x_0 - \frac{a}{b})e^{-bt}, \quad (2)$$

$$\mathbb{E}[X] = \frac{a}{b}, \quad (3)$$

$$Var[X(t)] = \frac{ac^2}{2b^2} + \frac{c^2}{b}(x_0 - \frac{a}{b})e^{-bt} + \frac{c^2}{b}(\frac{a}{2b} - x_0)e^{-2bt}, \quad (4)$$

$$Var[X] = \frac{ac^2}{2b^2}. \quad (5)$$

The CIR process $X(t)$ has unattainable boundaries if and only if $a \geq \frac{1}{2}c^2$. Given this condition, the iCIR process, $Y(t)$ will be well defined through

$$Y(t) = \frac{1}{X(t)}. \quad (6)$$

It's formula implies that $Y(t)$, starting from $Y(0) := y_0 = \frac{1}{x_0}$, has the following dynamics

$$dY(t) = [b - (a - c^2)Y(t)]Y(t)dt - cY(t)^{1.5}dW(t). \quad (7)$$

Ahn & Gao (1999) used this iCIR process to model the default-free interest rates. They provide the necessary and sufficient conditions for iCIR to have stationarity and unattainability of the boundaries. These conditions require that $a > c^2$ and $b > 0$, which in turn implies that the original CIR has unattainable boundaries. Ahn & Gao (1999) also derived the iCIR's conditional density as well as all the conditional and unconditional moments. We write the conditional and unconditional means and variances as follows, which will be needed later.

$$\mathbb{E}[Y(t)] = \frac{\zeta_t e^{-ut}}{q} M(q, 1 + q, u_t), \quad (8)$$

$$\mathbb{E}[Y] = \frac{2b}{2a - c^2}, \quad (9)$$

$$Var[Y(t)] = \frac{\zeta_t^2 e^{-2ut}}{q} \left[\frac{M(q - 1, 1 + q, u_t)}{q - 1} - \frac{e^{-ut} M(q, 1 + q, u_t)^2}{q} \right], \quad (10)$$

$$Var[Y] = \frac{2b^2 c^2}{(a - c^2)(2a - c^2)^2}, \quad (11)$$

where

$$\begin{aligned}\zeta_t &= \frac{2b}{c^2(1 - e^{-bt})}, \\ u_t &= \zeta_t x_0 e^{-bt}, \\ q &= \frac{2a}{c^2} - 1,\end{aligned}$$

and $M(\cdot, \cdot, \cdot)$ is the confluent hypergeometric function, which can be represented as hypergeometric series or integral form

$$\begin{aligned}M(a, b, z) &= 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \dots \\ &= \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt.\end{aligned}$$

Following Ahn & Gao (1999), we will model the default-free interest rate as an iCIR process, as seen in next section.

3 Semi-affine Intensity-based Model

In this section, all the dynamics specified are under the physical measure. The default-free interest rate, $r(t)$, and the short spread $h_j(t)$, for firm j , $j = 1, \dots, n$, are specified as follows:¹

$$r(t) = Y(t), \tag{12}$$

$$h_j(t) = \alpha_j X(t) + \beta_j Y(t) + X_j(t), \tag{13}$$

where $\alpha_j, \beta_j, j = 1, \dots, n$, are nonnegative constants, and $X_j(t)$ are independent $CIR(a_j, b_j, c_j, x_0^j)$, $j = 1, \dots, n$, i.e.

$$dX_j(t) = (a_j - b_j X_j(t))dt + c_j \sqrt{X_j(t)} dW_j(t). \tag{14}$$

¹Alternatively, one can assume constant recovery of the market value as in Duffie & Singleton (1999), and then model the hazard rate. Here, we model the product of loss given default and the hazard rate directly.

where a_j, b_j, c_j are nonnegative constants, and $W_j(t)$ are standard Brownian motions. We also assume pair-wise independence between $X(t)$ and $X_j(t)$, for all j . Therefore, $X_j(t)$ denotes the idiosyncratic factor for firm j . Since all the coefficients are nonnegative and the CIR processes are nonnegative, this implies that the interest rate $r(t)$ and the short spread $h_j(t)$ are all nonnegative.

If $\alpha_j > 0$ and $\beta_j > 0$, from the definition of $h_j(t)$, we see that $h_j(t) \geq 2\sqrt{\alpha_j\beta_j}$, which puts a lower bound for the short spreads. Considering that the interest rates are small, say around 5%, $X(t)$ will be very big, comparatively speaking, say magnitude of 20. The short spread $h_j(t)$ is also very small in reality, say 10 basis points for a BBB company. This reality implies that α_j will be very small. For considerable β_j , the lower bound of $h_j(t)$ will be a very small positive number. Lower bounded short spreads implied from this model are not unrealistic at all. The fact that any company might default in the next infinitesimal time requires a premium in order to compensate for this risk.

Note that, the short spreads specified in our model have three components, although only two CIR factors are used for each of the firm. The only common factor that is shared by every company is the economic systematic factor $X(t)$, which is the inverse of the interest rate process. Our model is significantly different from others in factor-model literature, because the common factors are shared in different ways. In traditional factor models, the short spreads only linearly depend on the common factor, such as Duffie & Singleton (1999), Duffee (1999), Duffie & Liu (2001), Schönbucher (2003), to mention a few. In our model, the short spreads not only linearly depend on the common factor, but also linearly depend on the inverse of the common factor. As a consequence, an increase of the common factor $X(t)$, will have two folded influences on the short spreads, given positive coefficients β_j and α_j . Increasing $X(t)$ will increase the first component of $h_j(t)$, but the second component will be decreased. This feature adds more flexibility to generate rich correlation structure among the short spreads of different firms. We will save this discussion for Section 5.

We list some simple facts about the mean and variances of $h_j(t)$ and $X_j(t)$ as

follows for later use.

$$\mathbb{E}[h_j(t)] = \alpha_j \mathbb{E}[X(t)] + \beta_j \mathbb{E}[Y(t)] + \mathbb{E}[X_j(t)], \quad (15)$$

$$\mathbb{E}[h_j] = \alpha_j \mathbb{E}[X] + \beta_j \mathbb{E}[Y] + \mathbb{E}[X_j], \quad (16)$$

$$\text{Var}[h_j(t)] = \alpha_j^2 \text{Var}[X(t)] + 2\alpha_j \beta_j (1 - \mathbb{E}[X(t)]\mathbb{E}[Y(t)]) \quad (17)$$

$$+ \beta_j^2 \text{Var}[Y(t)] + \text{Var}[X_j(t)], \quad (18)$$

$$\text{Var}[h_j] = \alpha_j^2 \text{Var}[X] + 2\alpha_j \beta_j (1 - \mathbb{E}[X]\mathbb{E}[Y]) + \beta_j^2 \text{Var}[Y] + \text{Var}[X_j], \quad (19)$$

where $E[X(t)], \mathbb{E}[X], \text{Var}[X(t)], \text{Var}[X], \mathbb{E}[Y(t)], \mathbb{E}[Y], \text{Var}[Y(t)], \text{Var}[Y]$ are given in equations (2-5) and equations (8-11) respectively, and $\mathbb{E}[X_j(t)]$ and $\text{Var}[X_j(t)]$ are the mean and variance of $CIR(a_j, b_j, c_j, x_0^j)$.

4 Bond Prices

The previous section completes our specification of a two-factor model for a multi-firm setup, under the physical measure. In this section, we provide the formulae for computing zero coupon bond prices for both default-free and defaultable bonds. In order to do so, we need to change the measure to the risk-neutral measure Q .

Assume that, under the risk-neutral measure Q , there exists some constants λ_1, λ_2 , and $\lambda_1^j, \lambda_2^j, j = 1, \dots, n$, such that $\widetilde{W}(t)$ and $\widetilde{W}_j(t)$, for $j = 1, \dots, n$ are Q -Brownian motions, given by

$$\widetilde{W}(t) = W(t) + \int_0^t \left(\frac{\lambda_1}{c\sqrt{X(s)}} + \frac{\lambda_2\sqrt{X(s)}}{c} \right) ds,$$

$$\widetilde{W}_j(t) = W_j(t) + \int_0^t \left(\frac{\lambda_1^j}{c_j\sqrt{X_j(s)}} + \frac{\lambda_2^j\sqrt{X_j(s)}}{c_j} \right) ds.$$

This implies that, under the risk-neutral measure Q , the dynamics of $X(t)$ and $X_j(t)$ can be written as

$$dX(t) = [(a - \lambda_1) - (b + \lambda_2)X(t)]dt + c\sqrt{X(t)}d\widetilde{W}(t), \quad (20)$$

$$dX_j(t) = [(a_j - \lambda_1^j) - (b_j + \lambda_2^j)X_j(t)]dt + c_j\sqrt{X_j(t)}d\widetilde{W}_j(t). \quad (21)$$

In order to procure the equivalence of the two measures, the following conditions are required: $\lambda_1 < a - \frac{1}{2}c^2$, $\lambda_1^j < a_j - \frac{1}{2}c_j^2$, $\lambda_2 > -b$ and $\lambda_2^j > -b_j$. Our market

price of risk specification precludes arbitrage opportunities, see Cheridito, Filipovic & Kimmel (2007) for a discussion.

Using Ito's formula, the Q -dynamics for the iCIR process, $Y(t)$, can be written as

$$dY(t) = [(b + \lambda_2) - (a - \lambda_1 - c^2)Y(t)]Y(t)dt - cY(t)^{1.5}d\widetilde{W}(t). \quad (22)$$

In order for this to be well defined, the unattainable boundary conditions require that: $\lambda_1 < a - c^2$.

Recall from Duffie & Singleton (1999), the price for a default-free zero coupon bond, $P(t, T)$, and the price for a defaultable zero coupon bond, $P_j(t, T)$, for firm j , can be calculated through

$$\begin{aligned} P(t, T) &= E_t^Q \left[\exp \left(- \int_t^T r(s) ds \right) \right], \\ P_j(t, T) &= E_t^Q \left[\exp \left(- \int_t^T r(s) + h_j(s) ds \right) \right]. \end{aligned}$$

The following propositions give explicit formulae for computing these bond prices.

Proposition 4.1 *Assume that the default-free interest rate $r(t)$ is specified as in equation (12). The risk-neutral dynamics for factor $X(t)$ is specified in equation (20), such that $a > c^2$, $b > 0$, $c > 0$, $\lambda_1 < a - c^2$ and $\lambda_2 > -b$. At time t , assume $X(t) = x_t = 1/r_t$. The time t value, $P(t, T)$, for a default-free T -bond is then given by*

$$P(t, T) = G^{CIR(a-\lambda_1, b+\lambda_2, c)}(t, T, x_t, 0, 1), \quad (23)$$

where the function $G^{CIR(a,b,c)}(t, T, x_t, l_1, l_2)$ associated with $X_t \sim CIR(a, b, c)$ is given by

$$G^{CIR(a,b,c)}(t, T, x_t, l_1, l_2) = E_t \left[\exp \left(- \int_t^T l_1 X(s) + \frac{l_2}{X(s)} ds \right) \right]. \quad (24)$$

The explicit formula for the G function is given in the appendix. This formula is also derived in Ahn & Gao (1999) using the Partial Differential Equation (PDE) approach.

Proposition 4.2 *Assume that the short spreads $h_j(t)$ for firm j are modeled as in equation (13). The risk-neutral dynamics of factors $X(t)$, $X_j(t)$, are specified in equations (20-21), such that $a > c^2$, $b > 0$, $c > 0$, $a_j > 0$, $b_j > 0$, $c_j > 0$, $\lambda_1 < a - c^2$,*

$\lambda_2 > -b$, $\lambda_1^j < a_j - \frac{1}{2}c_j^2$ and $\lambda_2^j > -b_j$, $\alpha_j \geq 0$, $\beta_j \geq 0$, for $j = 1, \dots, n$. At time t , assume $X(t) = x_t$ and $X_j(t) = x_t^j$. The time t value, $P_j(t, T)$, for a defaultable T -bond of firm j is then given by

$$P_j(t, T) = G^{CIR(a-\lambda_1, b+\lambda_2, c)}(t, T, x_t, \alpha_j, 1 + \beta_j) \times G^{CIR(a_j-\lambda_1^j, b_j+\lambda_2^j, c_j)}(t, T, x_t^j, 1, 0) \quad (25)$$

Proposition 4.2 states that the defaultable bond prices can be evaluated as a product of two $G^{CIR(\cdot, \cdot, \cdot, \cdot, \cdot)}(\cdot, \cdot, \cdot, \cdot, \cdot)$ functions. The first one is a function of the common factor x_t and it does not depend on the idiosyncratic factor x_t^j . The second one is a function of the idiosyncratic factor x_t^j and it is independent of the common factor x_t . From the proof of Proposition 4.2 in the appendix, it is easy to see that the last $G^{CIR(\cdot, \cdot, \cdot, \cdot, \cdot)}(\cdot, \cdot, \cdot, \cdot, \cdot)$ function in equation (25) has a well known affine term structure in x_t^j ,

$$G^{CIR(a_j-\lambda_1^j, b_j+\lambda_2^j, c_j)}(t, T, x_t^j, 1, 0) = A_j(t, T)e^{-B_j(t, T)x_t^j}, \quad (26)$$

where

$$\begin{aligned} A_j(t, T) &= \left(\frac{2\eta_j \exp\{(T-t)(b_j + \lambda_2^j + \eta_j)/2\}}{(b_j + \lambda_2^j + \eta_j)(\exp\{(T-t)\eta_j\} - 1) + 2\eta_j} \right)^{2a_j/c_j^2}, \\ B_j(t, T) &= \frac{2(\exp\{(T-t)\eta_j\} - 1)}{(b_j + \lambda_2^j + \eta_j)(\exp\{(T-t)\eta_j\} - 1) + 2\eta_j}, \\ \eta_j &= \sqrt{(b_j + \lambda_2^j)^2 + 2c_j^2}. \end{aligned}$$

Proposition 4.2 provides an explicit formula for computing defaultable zero coupon bond prices. Since coupon bonds can be expressed as linear combinations of zero coupon bonds, we can also obtain an explicit formula for computing defaultable coupon bond prices. Calculating bond prices is reduced to computing $G^{CIR(\cdot, \cdot, \cdot, \cdot, \cdot)}(\cdot, \cdot, \cdot, \cdot, \cdot)$ functions, which involves confluent hypergeometric functions.

The yields to maturity for the default-free bond, $R(x_t, t, T)$, the yields to maturity for the defaultable bond, $R_j(x_t, x_t^j, t, T)$, and the credit spreads for the defaultable

bond $CS_j(x_t, x_t^j, t, T)$, for $j = 1, \dots, n$, are defined respectively as follows

$$\begin{aligned} R(x_t, t, T) &= -\frac{\log P(t, T)}{T-t}, \\ R_j(x_t, x_t^j, t, T) &= -\frac{\log P_j(t, T)}{T-t}, \\ CS_j(x_t, x_t^j, t, T) &= R_j(x_t, x_t^j, t, T) - R(x_t, t, T). \end{aligned}$$

Applying Propositions 4.1 and 4.2 and equation (26) we obtain

$$R(x_t, t, T) = -\frac{1}{T-t} \left[\log G^{CIR(a-\lambda_1, b+\lambda_2, c)}(t, T, x_t, 0, 1) \right], \quad (27)$$

$$\begin{aligned} R_j(x_t, x_t^j, t, T) &= -\frac{1}{T-t} \left[\log G^{CIR(a-\lambda_1, b+\lambda_2, c)}(t, T, x_t, \alpha_j, \beta_j + 1) \right. \\ &\quad \left. + \log A_j(t, T) - B_j(t, T)x_t^j \right], \end{aligned} \quad (28)$$

$$\begin{aligned} CS_j(x_t, x_t^j, t, T) &= -\frac{1}{T-t} \left[\log G^{CIR(a-\lambda_1, b+\lambda_2, c)}(t, T, x_t, \alpha_j, \beta_j + 1) \right. \\ &\quad \left. - \log G^{CIR(a-\lambda_1, b+\lambda_2, c)}(t, T, x_t, 0, 1) \right. \\ &\quad \left. + \log A_j(t, T) - B_j(t, T)x_t^j \right]. \end{aligned} \quad (29)$$

As we can see from the formula for function $G^{CIR(\cdot, \cdot, \cdot)}(\cdot, \cdot, \cdot, \cdot, \cdot)$, the yields to maturity, $R(x_t, t, T)$, for default-free bonds are non-affine in x_t . Similarly for defaultable bonds, the yields $R_j(x_t, x_t^j, t, T)$ are non-affine in x_t . However, both $R_j(x_t, x_t^j, t, T)$ and $CS_j(x_t, x_t^j, t, T)$ are affine in the idiosyncratic factors x_t^j , as indicated from the above equations. We say that this intensity-based model is *semi-affine*. This semi-affine property is very important when it comes to the calibration issue, which we will discuss in later sections.

5 Correlation Structure

In this section, we focus on the correlations between short spreads and interest rates, and the correlations among the short spreads of different firms. Both conditional (conditional on the initial state) and unconditional (stationary state) correlations are studied. Rich correlation structures are generated, while nonnegative interest rates and nonnegative short spreads are maintained.

5.1 Correlation of $r(t)$ and $h_j(t)$

Most empirical studies find that the default-free interest rate $r(t)$ and the credit spread $h_j(t)$ are negatively correlated, such as Longstaff & Schwartz (1995), Duffee (1999) and Collin-Dufresne, Goldstein & Martin (2001). Our model is flexible enough to allow both negative and positive correlations between $r(t)$ and $h_j(t)$. We first give a heuristic argument. Referring to equations (12-13), the correlations between $r(t)$ and $h_j(t)$ are solely determined by the coefficients α_j and β_j . Considering an extreme case when $\alpha_j = 0$ and $\beta_j > 0$, then the correlation between $r(t)$ and $h_j(t)$ is clearly positive. On the other hand, when $\beta_j = 0$ and $\alpha_j > 0$, $r(t)$ will be negatively correlated with $h_j(t)$, since $X(t)$ and $Y(t)$ are negatively correlated.

Next, let us give a quantitative analysis on the correlation structure. The covariance of $r(t)$ and $h_j(t)$ can be directly calculated through (conditional on time zero)

$$Cov[r(t), h_j(t)] = \alpha_j(1 - \mathbb{E}[X(t)]\mathbb{E}[Y(t)]) + \beta_j Var[Y(t)], \quad (30)$$

where $\mathbb{E}[X(t)]$, $\mathbb{E}[Y(t)]$ and $Var[Y(t)]$ are given in equations (2-5) and equations (8-11). As time $t \rightarrow +\infty$, the covariance $Cov[r(t), h_j(t)]$ will converge asymptotically to the unconditional covariance

$$Cov[r, h_j] = \alpha_j(1 - \mathbb{E}[X]\mathbb{E}[Y]) + \beta_j Var[Y], \quad (31)$$

where $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $Var[Y]$ are given in equations (2-5) and equations (8-11). This can be simplified into

$$Cov[r, h_j] = \frac{c^2}{2a - c^2} (D_2\beta_j - \alpha_j). \quad (32)$$

where

$$D_2 = \frac{2b^2}{(a - c^2)(2a - c^2)}. \quad (33)$$

Recall that $a > c^2$, therefore, both $2a - c^2$ and $a - c^2$ are positive, which implies that $D_2 > 0$. The sign of the asymptotic correlation of r and h_j is hence determined by the sign of $D_2\beta_j - \alpha_j$. If $\alpha_j = \beta_j D_2$, then $Cov[r, h_j] = 0$; if $\alpha_j > \beta_j D_2$, then $Cov[r, h_j] < 0$; if $\alpha_j < \beta_j D_2$, then $Cov[r, h_j] > 0$. Since empirical findings suggest

that the interest rate and short spreads are mostly negatively correlated, we expect to have most of the cases that $\alpha_j > \beta_j D_2$.

The correlation coefficient of $r(t)$ and $h_j(t)$, $\rho_j(t)$, and its asymptotic limit ρ_j , can thus be calculated by

$$\rho_j(t) := \frac{\text{Cov}[r(t), h_j(t)]}{\sqrt{\text{Var}[Y(t)]\text{Var}[h_j(t)]}}, \quad (34)$$

$$\rho_j := \frac{\text{Cov}[r, h_j]}{\sqrt{\text{Var}[Y]\text{Var}[h_j]}}, \quad (35)$$

where $\text{Cov}[r(t), h_j(t)]$ and $\text{Cov}[r, h_j]$ are given in equations (30) and (31) respectively.

5.2 Correlation of $h_i(t)$ and $h_j(t)$

The short spreads $h_i(t)$ and $h_j(t)$, for different firms i and j , can be positively or negatively correlated as well. As in the previous section, a similar heuristic arguments can be given for the correlation between $h_i(t)$ and $h_j(t)$. Recall the following equations, for $i \neq j$,

$$h_i(t) = \alpha_i X(t) + \beta_i Y(t) + X_i(t),$$

$$h_j(t) = \alpha_j X(t) + \beta_j Y(t) + X_j(t).$$

The correlation between $h_i(t)$ and $h_j(t)$ is solely determined by α_i , α_j , β_i and β_j . Consider an extreme case when $\alpha_i = \alpha_j = 0$ and $\beta_i > 0$, $\beta_j > 0$ (or $\alpha_i > 0$, $\alpha_j > 0$ and $\beta_i = \beta_j = 0$), the correlation between $h_i(t)$ and $h_j(t)$ will be positive. On the other hand, when $\alpha_i = \beta_j = 0$ and $\beta_i > 0$, $\alpha_j > 0$ (or $\alpha_j = \beta_i = 0$ and $\beta_j > 0$, $\alpha_i > 0$), $h_i(t)$ will be negatively correlated with $h_j(t)$, since $X(t)$ and $Y(t)$ are negatively correlated.

The covariance of $h_i(t)$ and $h_j(t)$, $\text{Cov}[h_i(t), h_j(t)]$ can be expressed as (conditional on time zero)

$$\begin{aligned} \text{Cov}[h_i(t), h_j(t)] &= \alpha_i \alpha_j \text{Var}[X(t)] + \beta_i \beta_j \text{Var}[Y(t)] \\ &\quad + (\alpha_i \beta_j + \alpha_j \beta_i)(1 - \mathbb{E}[X(t)]\mathbb{E}[Y(t)]), \end{aligned} \quad (36)$$

where $Var[X(t)]$, $Var[Y(t)]$, $\mathbb{E}[X(t)]$ and $\mathbb{E}[Y(t)]$ are given in previous sections. As $t \rightarrow +\infty$, the above equation tends to the unconditional covariance

$$Cov[h_i, h_j] = \alpha_i \alpha_j Var[X] + \beta_i \beta_j Var[Y] + (\alpha_i \beta_j + \alpha_j \beta_i)(1 - \mathbb{E}[X]\mathbb{E}[Y]). \quad (37)$$

which can also be written as

$$\begin{aligned} Cov[h_i, h_j] &= \alpha_i Cov\left[\frac{1}{r}, h_j\right] + \beta_i Cov[r, h_j], \\ &= \alpha_j Cov\left[\frac{1}{r}, h_i\right] + \beta_j Cov[r, h_i]. \end{aligned}$$

This expression will be very useful for calibration of α_j and β_j , if we believe that the observed time series of $r(t)$, $h_i(t)$ and $h_j(t)$ are stationary. Some simple algebra will lead to an explicit expression of $Cov[h_i, h_j]$ as follows

$$Cov[h_i, h_j] = \frac{c^2}{2a - c^2} \left[\alpha_i \left(\frac{\alpha_j}{D_1} - \beta_j \right) + \beta_i (D_2 \beta_j - \alpha_j) \right], \quad (38)$$

where D_2 is given in equation (33) and D_1 is defined as

$$D_1 = \frac{2b^2}{a(2a - c^2)}. \quad (39)$$

It is easy to see that $D_2 > D_1 > 0$, given that $a > c^2$.

The correlation coefficient of $h_i(t)$ and $h_j(t)$, $\rho_{ij}(t)$, and its asymptotic limit, ρ_{ij} can thus be calculated by

$$\rho_{ij}(t) := \frac{Cov[h_i(t), h_j(t)]}{\sqrt{Var[h_i(t)]Var[h_j(t)]}}, \quad (40)$$

$$\rho_{ij} := \frac{Cov[h_i, h_j]}{\sqrt{Var[h_i]Var[h_j]}}, \quad (41)$$

where $Cov[h_i(t), h_j(t)]$ and $Cov[h_i, h_j]$ are given in equations (36) and (38) respectively, and $Var[h_j(t)]$ and $Var[h_j]$, for $j = 1, \dots, n$, are given by

$$Var[h_j(t)] = \alpha_j^2 Var[X(t)] + \beta_j^2 Var[Y(t)] + 2\alpha_j \beta_j (1 - \mathbb{E}[X(t)]\mathbb{E}[Y(t)]) + Var[X_j(t)],$$

$$Var[h_j] = \alpha_j^2 Var[X] + \beta_j^2 Var[Y] + 2\alpha_j \beta_j (1 - \mathbb{E}[X]\mathbb{E}[Y]) + Var[X_j].$$

All of these formulae can be written in explicit forms.

The correlations we have discussed so far are under the physical measure. Similar arguments will follow if we consider the correlations under the risk-neutral measure.

The formulae for $Cov[r(t), h_j(t)]$ and $Cov[h_i(t), h_j(t)]$, under the risk-neutral measure, will be the same as in equations (30) and (36) respectively, with a replaced by $a - \lambda_1$ and b replaced by $b + \lambda_2$. The formulae for $\rho_j(t)$ and $\rho_{ij}(t)$, under the risk-neutral measure will be the same as in equations (34) and (40), with a, b, a_j and b_j replaced by $a - \lambda_1, b + \lambda_2, a_j - \lambda_1^j$ and $b_j + \lambda_2^j$ respectively.

6 Numerical Illustration

Figure 1 plots the time series of U.S. 5-year treasury yields versus the 5-year Financial Sector CDS spreads of AA-rated, which are daily market data taken from Bloomberg from the same period: March 4, 2002 to August 30, 2006. It can be seen from the graph that the 5-year treasury yield went down first from March 2002 and then went up from October 2003 and continued the upward trend to August 2006. The 5-year CDS spread behaved in an inverse pattern: it went up first and then went down. The statistical correlation coefficient in Figure 1 is calculated as if the two time series have reached stationary state. The calculated number -0.5741 only serves as an estimate from the two time series. The accuracy of the estimate may be questioned, but what is more important is a strong negative correlation is presented. ²

a	b	c	x_0	a_1	b_1	c_1	x_0^1	α_1	β_1	λ_1	λ_2	λ_1^1	λ_2^1
3.09	0.13	1.3	18	0.006	3	0.002	0.001	0.0001	0.002	0	0	0	0

Table 1: Base Case Parameters for Simulation.

Figure 2 shows simulated sample paths of the interest rate and the short spread processes using parameters given in table 1.³ It is clear from the graph that the

²Treasury yield time series are available for maturities 1/12, 3/12, 6/12 1, 2, 3, 5, 7, 10, and 20-year. Financial AA CDS time series are available for maturities 1, 3, 5 and 10-year. These time series show a similar fashion as in Figure 1. The reason we choose 5-year data for illustration is because the the 5-year CDS has best liquidity.

³As recommended by Alfonsi (2005), the explicit discretization scheme was used to simulate CIR processes.

simulated interest rate process is negatively correlated with the short spread process, hence mimicking the real observed time series shown in Figure 1. This is not surprising, because for the base case parameters, we can easily deduce that $D_2\beta_1 = 1.0754 \times 10^{-5} < \alpha_1 = 0.0001$. To quantify this negative correlation, Figure 3 plots the correlation coefficient of $r(t)$ and $h_1(t)$ from equations (34) and (35). We see that the strength of the negative correlation diminishes gradually to the asymptotic correlation as time progresses.

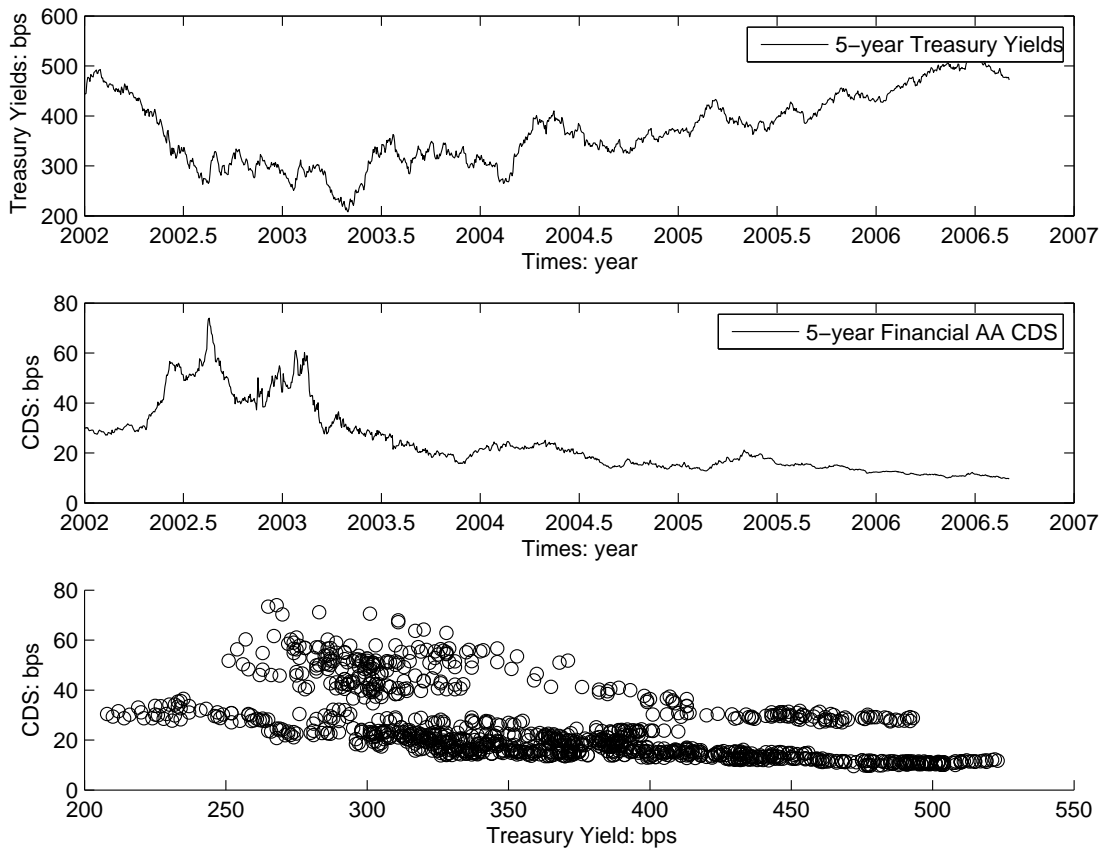


Figure 1: Time series of 5-year Treasury yield vs. time series of 5-year CDS spreads (daily) of Financial Sector, AA-rated. Dates: March 4, 2002 to August 30, 2006. Resources: Bloomberg. The statistical correlation coefficient between the two time series is -0.5741.

Figure 4 shows the term structure of treasury yields, corporate yields and credit spreads with base case parameters. The term structure of treasury yields and corpo-

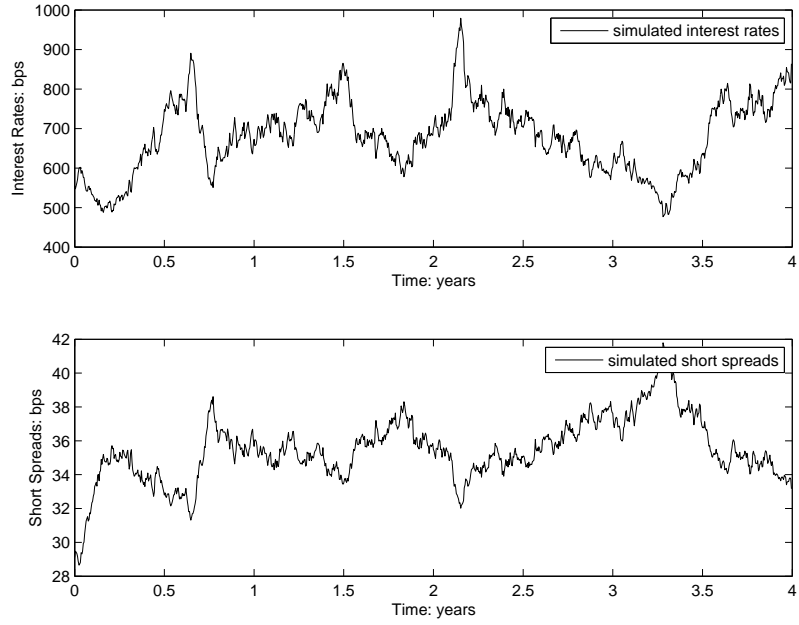


Figure 2: Simulated Sample Paths of the Interest Rate and the Short Spread Processes with Base Case Parameters.

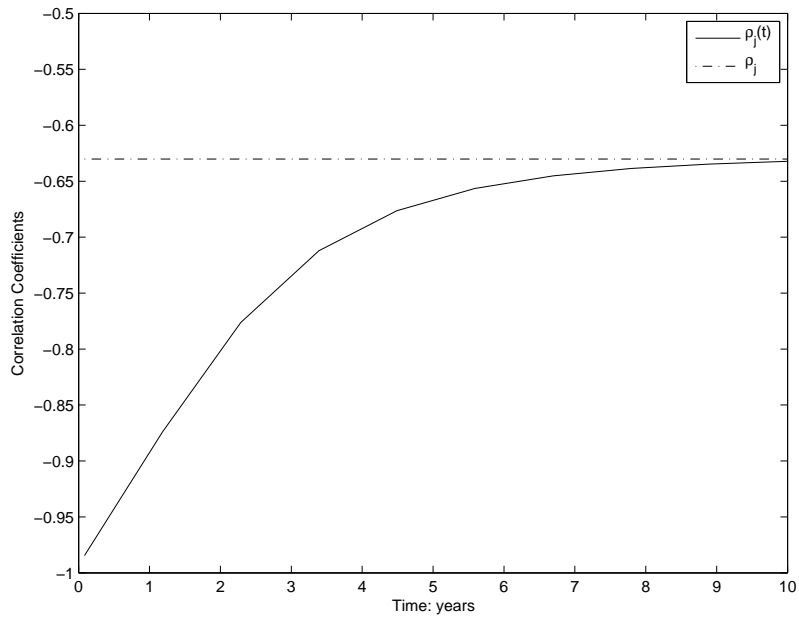


Figure 3: Correlation Coefficients of Interest Rates and Short Spreads with Base Case Parameters.

rate yields are hump-shaped for the base case parameters and the term structure of credit spreads is upwarding shaped. However, various shapes of these term structures could be generated using different parameters, including flat, humped, upwarding and downwarding shaped.

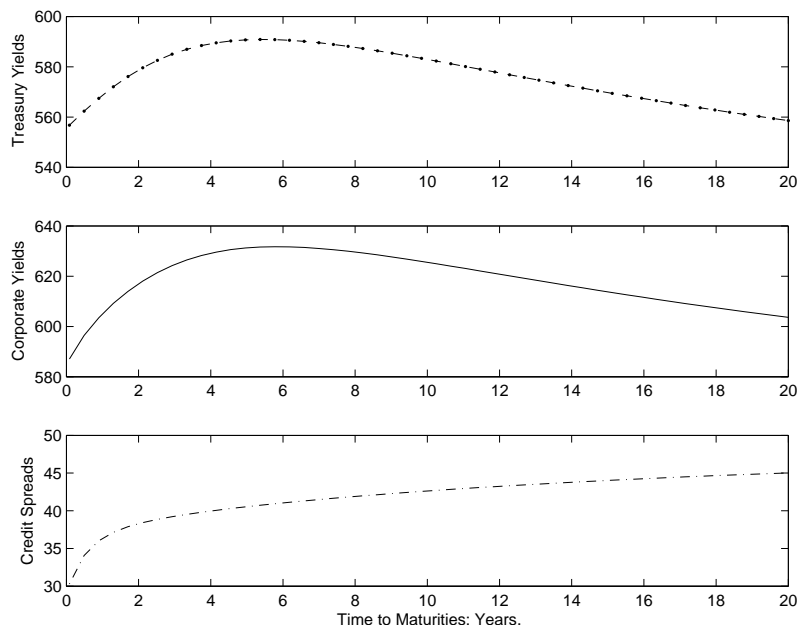


Figure 4: Simulated Treasury Yields Curve, Corporate Yields Curve and Credit Spreads Curve (at time 0) with Base Case Parameters.

7 Calibration Issues

Despite the elegant mathematics of our model, it is unfortunately the case that the calibration has proved to be difficult. In this section, we outline our proposed method and give possible reasons why we failed to complete.

We propose a two-step calibration of our model by using the Extended Kalman Filter (EKF) in conjunction with the Quasi-Maximum Likelihood Estimation (QMLE).⁴

The first step is to calibrate the iCIR interest rate process to observed Treasury yield

⁴The readers are directed to Duffee (1999), Chen & Scott (2003), Duan & Simonato (1999) and Yi (2005) for more information on EKF and QMLE.

time series with different maturities. The *measurement equation* of the EKF is given by Equation 27. Since the Treasury yield is non-affine in the iCIR-factor, linearization of the measurement equation is needed for the EKF scheme. The *transition equation* for the EKF is obtained by discretizing the iCIR stochastic differential equation (SDE) given by Equation 7. Assume that we have backed out the parameters in the iCIR interest rate model and have filtered out the time series of the interest rate in the first step. Therefore, parameters a , b , c , λ_1 and λ_2 , and the time series of the common factor $X(t)$ are known in step two.

In the second step, for each company j , the short spread process can be calibrated by using Kalman Filter (KF) combined with QMLE. The measurement equation of the KF is given by Equation (28) or (29). The transition equation is obtained by discretizing the CIR SDE given by Equation 14. As we have discussed, both the corporate yields $R_j(x_t, x_t^j, t, T)$ and the credit spreads $CS_j(x_t, x_t^j, t, T)$ are affine in the idiosyncratic factor $X_j(t)$. Hence, the standard KF can be used. The parameters a_j , b_j , c_j , λ_1^j , λ_2^j , α_j and β_j are obtained from step two. The time series of the idiosyncratic factor $X_j(t)$ are filtered out from this step as well.

A number of difficulties were encountered when we empirically implemented the EKF and QMLE scheme for the first step. These difficulties possibly come from the following reasons. First, linearization of the measurement equation. Since the linearization is a first order Taylor approximation of a very complicated function, the accuracy of the approximation should be questioned. Second, non-Gaussian innovations in transition equations. Standard KF requires Gaussian innovations in the transition equation. This is violated for both CIR and iCIR models. Duan & Simonato (1999) claimed that QMLE is still a consistent way to estimate parameters for the CIR interest rate model. However, we are not clear if this is the case for the iCIR interest rate model.

Another major obstacle is computing the hypergeometric function for different parameters for hundreds of times (in both steps). Although this function is a Matlab built-in function, it is impossible to calculate for certain input parameters. A better numerical approximation of the hypergeometric function may be needed, which is

beyond the scope of this paper.

8 Extension of The Model

We have considered a model where $r(t)$ is assumed to be a one factor iCIR process and the short spread $h_j(t)$ is assumed to be a two factor model. However, this model is not restrictive and it can be easily generalized to a model which has three or more factors.

The interest rate and the short spreads could be modeled as multi-CIR-factor, or multi-iCIR-factor, or even linear combinations of CIR and iCIR factors. Negative correlations between the interest rate and the short spread are generated through the common CIR factor (factors) and its (their) inverse(s).

It can be proved that, if $Cov[r, h_i] < 0$ and $Cov[r, h_j] < 0$, then $Cov[h_i, h_j] > 0$ for the semi-affine intensity-based model we have discussed. However, in reality, the short spreads of two firms are not necessarily positively correlated given that they are both negatively correlated with the interest rate. To the authors' best knowledge, the extant factor models in the literature are not able to incorporate this feature without violating the nonnegativity property. However, our model can easily be adjusted to do so by adding one additional CIR factor to the short spread as follows:

$$\begin{aligned} r(t) &= \frac{1}{X(t)}, \\ h_i(t) &= \alpha_i X(t) + \beta_i \frac{1}{X(t)} + p_i H(t) + q_i \frac{1}{H(t)} + X_i(t), \end{aligned}$$

where p_i, q_i are nonnegative constants and $H(t)$ is an independent $CIR(a_h, b_h, 1)$ with $a_h > 1$ and $b_h > 0$. The factor $H(t)$ is thus shared by all the firms, but not by the interest rate. The correlations between $h_i(t)$ and $h_j(t)$ are thus determined not only by $\alpha_i, \alpha_j, \beta_i$ and β_j , but also p_i, p_j, q_i and q_j . To see that $Cov[r, h_i] < 0$, $Cov[r, h_j] < 0$ and $Cov[h_i, h_j] < 0$ can coexist for the model specified above, let us consider the following example. Consider two firms i and j , assume that $\beta_i = \beta_j = p_i = q_j = 0$

and $p_j > 0$, $q_i > 0$. Therefore

$$\begin{aligned} h_i(t) &= \alpha_i X(t) + q_i \frac{1}{H(t)} + X_i(t), \\ h_j(t) &= \alpha_j X(t) + p_j H(t) + X_j(t). \end{aligned}$$

It is obvious that both $h_i(t)$ and $h_j(t)$ are negatively correlated with the interest rate $r(t)$. In this case, the covariance of $h_i(t)$ and $h_j(t)$, $Cov[h_i(t), h_j(t)]$, can be written as

$$Cov[h_i(t), h_j(t)] = \alpha_i \alpha_j Var[X(t)] + q_i p_j Cov[\frac{1}{H(t)}, H(t)].$$

Since $Cov[\frac{1}{H(t)}, H(t)] < 0$, negative $Cov[h_i(t), h_j(t)]$ can be obtained by choosing sufficiently large product of q_i and p_j .

The present work of this paper should thus be regarded more as a framework of multi-factor models which can generalize rich correlation structure, while maintaining nonnegativity property, than as a specific model we have discussed in previous sections.

9 Summary

In this chapter, we have introduced the iCIR process to the literature of intensity-based factor models on credit risk. In this framework, the short spread for each firm not only depends on the common factor, but also depends on the inverse of the common factor. By doing this, we added much flexibility of correlation structures among the interest rates and the short spreads, and among the short spreads of different firms, while both the interest rates and the short spreads remain nonnegative.

The pricing formulae of both the default-free and defaultable bonds were derived in explicit forms by utilizing the recent findings of stochastic integrals by Hurd & Kuznetsov (2006). The term structure of the credit spreads are non-affine in the common factor, but are affine in the idiosyncratic factor. This allows us to use the KF and QMLE to conduct the calibration of the short spread process, once the interest rate process is fitted. The affine intensity-based modeling is thus extended to a more general semi-affine family.

Numerical illustrations showed that our model is able to generate various shapes of the term structure of credit spreads which are consistent with the empirical findings. In addition, we were able to generate negatively correlated time series of interest rates and short spreads, without violating the nonnegativity property, mimicking the real observations as seen in Figure 1.

10 Appendix II

Lemma 10.1 (*Hurd & Kuznetsov (2006)*) *Let $X(t)$ be a CIR process given by equation (1). Let $CIR(a,b,c)$ denote this process with parameters a, b, c . Assume that $X(t) = x_t$ and it has unattainable boundaries. Define function $G^{CIR(a,b,c)}(t, T, x_t, l_1, l_2)$ associated with this $CIR(a,b,c)$ as follows*

$$G^{CIR(a,b,c)}(t, T, x_t, l_1, l_2) = E_t \left[\exp \left(- \int_t^T l_1 X(s) + \frac{l_2}{X(s)} ds \right) \right],$$

where l_1, l_2 are constants. Then, the function $G^{CIR(a,b,c)}(t, T, x_t, l_1, l_2)$ admits explicit expression given by

$$\begin{aligned} G^{CIR(a,b,c)}(t, T, x_t, l_1, l_2) &= \left(1 + \frac{v_1}{\gamma(t, T)} \right)^{-\frac{2a}{c^2}} \exp \left\{ v_1 \left[x_t + a(T-t) - \frac{y(x_t, t, T)}{\gamma(t, T)} \right] \right\} \\ &\quad \times \frac{\Gamma(v_3 - v_2)}{\Gamma(v_3)} M(v_2, v_3, -y(x_t, t, T)) y(x_t, t, T)^{v_2} \end{aligned}$$

where

$$\begin{aligned} v_1 &= \frac{b - \sqrt{b^2 + 2l_1 c^2}}{c^2}, \\ v_2 &= \frac{\sqrt{(2a - c^2)^2 + 8l_2 c^2} - (2a - c^2)}{2c^2}, \\ v_3 &= \frac{\sqrt{(2a - c^2)^2 + 8l_2 c^2} + c^2}{c^2}, \\ \gamma(t, T) &= \frac{2\sqrt{b^2 + 2l_1 c^2}}{c^2(1 - \exp\{-(T-t)\sqrt{b^2 + 2l_1 c^2}\})}, \\ y(x_t, t, T) &= \frac{x_t \gamma(t, T)^2 \exp\{-(T-t)\sqrt{b^2 + 2l_1 c^2}\}}{v_1 + \gamma(t, T)}. \end{aligned}$$

Please refer to Hurd & Kuznetsov (2006) for the proof.

- **Proof of Proposition 4.1:** From equation (23), the default-free zero coupon bond price, $P(t, T)$, can also be written as

$$\begin{aligned} P(t, T) &= \mathbb{E}_t^Q \left[\exp \left(- \int_t^T \frac{1}{X(s)} ds \right) \right] \\ &= G^{CIR(a-\lambda_1, b+\lambda_2, c)}(t, T, x_t, 0, 1) \end{aligned}$$

where the Q -dynamics of $X(t)$ is given by equation (20). The second step comes from invoking the above lemma.

- **Proof of Proposition 4.2:** According to equation (23), applying equations (12) and (13), the defaultable zero coupon bond price, $P_j(t, T)$, can be calculated as

$$\begin{aligned} P_j(t, T) &= \mathbb{E}_t^Q \left[\exp \left(- \int_t^T \frac{1}{X(s)} + \alpha_j X(s) + \beta_j \frac{1}{X(s)} + X_j(s) ds \right) \right] \\ &= \mathbb{E}_t^Q \left[\exp \left(- \int_t^T \alpha_j X(s) + \frac{1 + \beta_j}{X(s)} ds \right) \right] \mathbb{E}_t^Q \left[\exp \left(- \int_t^T X_j(s) ds \right) \right] \\ &= G^{CIR(a-\lambda_1, b+\lambda_2, c)}(t, T, x_t, \alpha_j, 1 + \beta_j) \\ &\quad \times G^{CIR(a_j-\lambda_1^j, b_j+\lambda_2^j, c_j)}(t, T, x_t^j, 1, 0). \end{aligned}$$

For the second step, we utilized the fact that $X_j(t)$ are independent of $X(t)$, for $j = 1, \dots, n$. The last step follows by invoking the above lemma.

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