

Dangerous Knowledge: Credit Value Adjustment with Credit Triggers

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First Version: Sept. 2009; This Version: December 15, 2010

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Abstract

We generalize the arbitrage-free valuation framework for counterparty credit risk (CCR) adjustments when credit triggers are allowed in the contract. The settlement of the deal for the investor could be either obliged or optional to execute when the counterparty hits the credit trigger before any default events from the two parties. General formulas for credit value adjustment (CVA) are given for all four cases: obliged unilateral, obliged bilateral, optional unilateral and optional bilateral. The unilateral CVA with an optional credit trigger is found to be the same as the unilateral CVA with an analogous obliged credit trigger. We show that adding credit triggers will decrease the unilateral CVA for both obliged and optional cases, which are in line with the motivation of investors to reduce CCR. However, adding credit triggers may not necessarily reduce bilateral CVA. Counter-intuitively, we show that the bilateral CVA may actually increase by adding credit triggers. Moreover, the increased amount of bilateral CVA due to credit triggers for one party is exactly the same amount of bilateral CVA reduced for the other party. The CVA calculation is subjected to large uncertainty of model risks, mostly due to the lack of data for calibrating jump-to-default probabilities. Some explicit models for obliged unilateral CVA are discussed with special caveats on the model assumptions. Numerical examples are also given to illustrate the model risk of CVA calculation due to the uncertainty of jump sizes, even though pure jump models are assumed.

Keywords: Counterparty Credit Risk; Unilateral (Bilateral) Credit Value Adjustment; Obligated (Optional) Credit Triggers; Mark-to-Credit;

1 Introduction

Although bankers have been talking about counterparty credit risk (CCR) for almost a decade, it is not until the most recent financial crisis that our understanding and practices of CCR are brought into question. The Basel Committee on Banking Supervision (2010) therefore proposes a series of changes to the Basel II framework on CCR capital requirements and related risk management practices in their recent consultative document.¹ In order to mitigate CCR, the international credit market has made some dramatic changes such as the introduction of new ISDA Credit Default Swaps (CDS) contracts and the introduction of central clearing counterparty for some credit derivatives.²

The traditional CCR management typically involves risk quantifications, such as *expected positive exposure* (EPE), based on *potential future exposure* (PFE) given counterparty default. The re-valuation of the portfolio at a future time bucket is based on Monte-Carlo simulation of the risk factors under *historical* measure.³ The risk management of CCR itself is evolving from passive risk measurements to one of active pricing and hedging practices. The price for CCR is called credit value adjustment (CVA) which has now become a very hot topic from front offices to back offices of every major financial institutions' capital markets. Different from the PFE quantification, risk factors are simulated under *risk-neutral* measure instead of historical measure for CVA pricing. Gregory & Kelly (2009) provide an excellent overview of this evolution of CCR from a practitioner's point of view. Zhu & Pykhtin (2007) give a very good discussion on both modeling exposure profiles and CVA pricing. The recent book by Gregory (2010) covers a wide range of different topics on CCR: from methodology, architecture and implementation to products, hedging and CVA pricing.

However, a rigorous mathematical formulation of CVA is a quite recent work. Brigo & Masetti (2006) give the first rigorous proof for the general unilateral counterparty risk pricing formula. Related literature on unilateral CVA calculation includes Brigo & Pallavicini (2007) on interest rates, Brigo & Bakkar (2009) on commodities and Brigo & Chourdakis (2008) on CDS. The general bilateral counterparty risk pricing formula is derived in Brigo & Capponi (2009) and Gregory (2009). Related literature on bilateral CVA includes Brigo, Pallavicini & Papatheodorou (2010) on interest rates, Brigo & Capponi (2009), Yi (2009) and Assefa, Bielecki, Crepey & Jeanblanc (2009) on CDS. CVA allocation at trade level is discussed by Pykhtin & Rosen (2009).

Initiated by the motivation of mitigating CCR, many institutions have engaged in credit trigger agreements to monitor their counterparty's creditworthiness. A credit trigger is usually a pre-specified credit level that is below the current rating of the counterparty, for example "BBB" when the counterparty is currently rated as "AA". When the counterparty's credit deteriorates to the credit trigger or below before default within maturity, the investor has the right to settle the deal with the counterparty at mark-to-market (MtM). Credit triggers can be set to both sides of the investor and the counterparty. The settlement of the deal for the investor could be either obliged or optional to execute when the counterparty hits the credit trigger before default. Previous literature on CVA does not consider credit triggers. In this article, we generalize the arbitrage-free valuation framework for CVA when credit triggers are allowed in the contract. General formulas are derived for all four cases: obliged unilateral, obliged bilateral, optional unilateral and optional bilateral. We point out that the increased amount of bilateral CVA due to credit triggers for one party is exactly the same amount of bilateral CVA reduced for the other party. This observation indicates that adding credit triggers may in fact increase bilateral CVA, which is against the motivation to add credit triggers in the first place. Caveat is also provided to warn the big model risk we are taking for CVA calculation with credit triggers.

The rest of this article is organized as follows. We start with a review of general CVA pricing without credit triggers in Section 2. Section 3 provides general formulas for both unilateral and bilateral CVA with either obliged or optional credit triggers. Section 4 discusses the large model risk of CVA with credit triggers from both theoretical and numerical point of views. Section 5 concludes this article. All proofs and some simple models for obliged unilateral CVA are given in appendices.

¹The treatment of CCR in the original Basel II is analysed by Fleck & Schmidt (2006) and Pykhtin & Zhu (2006)

²The readers are directed to Beumee, Brigo, Schiemert & Stoylo (2009) for a discussion on the new ISDA CDS and Duffie & Zhu (2009) on central clearing counterparty.

³The readers are directed to De Prisco & Rosen (2006) and Canabarro & Duffie (2004) for some CCR background and terminologies.

2 General CVA without Credit Triggers

We give a brief review of general CVA pricing without credit triggers in this section. Following the notations from Brigo & Capponi (2009), we refer to the names involved in the financial contract as follows: the investor is referred to as name “0”; the counterparty as name “2” and the reference entity as name “1”.⁴ Let τ_0 , τ_1 and τ_2 denote the default times of the investor, reference credit and the counterparty respectively. Default time of the reference entity τ_1 will affect the exposure at default of the counterparty. Let l_0 ($R_0 = 1 - l_0$), l_1 ($R_1 = 1 - l_1$) and l_2 ($R_2 = 1 - l_2$) denote the loss given default (recovery rate) of the investor, reference credit and the counterparty respectively. We will assume constant loss given default throughout this paper, although the general formulae can easily be extended to allow stochastic loss given default.

The probability space we are in is $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$ where the filtration $(\mathcal{G}_t)_t$ models the flow of total information of the whole market, including both market observable quantities and default events; and \mathbb{Q} is the risk-neutral measure. Let \mathcal{F}_t denote the right-continuous and complete sub-filtration which represents the market observable quantities and $\mathcal{H}_t := \sigma(\{\tau_0 \leq u \vee \tau_1 \leq u \vee \tau_2 \leq u : u \leq t\})$ denote the right-continuous filtration generated by the default events. Hence we have $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$. The expectation $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{G}_t]$ is under risk-neutral measure that leads to no-arbitrage prices.

Let t denote the valuation time and T denote the final maturity of the payoff. Let $\Pi_t^0(T_1, T_2)$ denote the discounted cash flows, from investor’s point of view, that are subjected to counterparty default. The cash flows in this notation are from time T_1 to T_2 and discounted to valuation time t . Let $\Pi_t(T_1, T_2)$ denote the analogous quantity when the counterparty is assumed to be default-free. Correspondingly, let $\Pi_t^2(T_1, T_2)$ denote the discounted cash flows, from the counterparty’s point of view, that are subjected to investor’s own default. It is clear that the following consistency equation holds

$$\Pi_t^2(T_1, T_2) = -\Pi_t^0(T_1, T_2). \quad (1)$$

The discounted MtM⁵ at some future time $g \in (t, T)$, denoted by A_g , is given by

$$A_g = \mathbb{E}_g[\Pi_g(g, T)]D(t, g);$$

where $D(t, g)$ denotes discounting factor that discounts cashflows at time g to time t . The payoffs depend on the underlying instruments or portfolio of instruments after taking into account either netting or no-netting agreements. From Brigo & Masetti (2006) and Brigo & Capponi (2009), the general counterparty risk pricing formulas are given by the following two propositions.

Proposition 2.1. [General Unilateral Counterparty Risk Pricing Formula]: *Assuming the investor is default-free, given $\tau_2 > t$, the price of the contract’s payoff under unilateral counterparty risk from the investor’s point of view is given by:*

$$\mathbb{E}_t[\Pi_t^0(t, T)] = \mathbb{E}_t[\Pi_t(t, T)] - CVA_0; \quad (2)$$

where CVA_0 is the investor’s unilateral credit value adjustment given by:

$$CVA_0 = l_2 \mathbb{E}_t[1_{\{\tau_2 \leq T\}} A_{\tau_2}^+]; \quad (3)$$

Assuming the counterparty is default-free, given $\tau_0 > t$, the price of the contract’s payoff under unilateral counterparty risk from the counterparty’s point of view is given by:

$$\mathbb{E}_t[\Pi_t^2(t, T)] = \mathbb{E}_t[-\Pi_t(t, T)] - CVA_2; \quad (4)$$

where CVA_2 is the counterparty’s unilateral credit value adjustment given by:

$$CVA_2 = l_0 \mathbb{E}_t[1_{\{\tau_0 \leq T\}} (-A_{\tau_0})^+]. \quad (5)$$

⁴In general, the underlying portfolio with the counterparty could have multiple reference entities. For illustrative purposes, we will assume one credit reference here. Distribution of time τ_1 will affect the exposure at default of the counterparty. For the rest of the paper, τ_1 does not appear explicitly in the events classification but implicitly sits in the exposure at default of the counterparty.

⁵Conditional on the realization of the paths of the market observable quantities upto time g , the discount factor $D(t, g)$ is constant.

Proposition 2.2. [*General Bilateral Counterparty Risk Pricing Formula*]: Let $\tau = \tau_0 \wedge \tau_2$. Given $\tau > t$, the price of the contract's payoff under bilateral counterparty risk from the investor's point of view is given by:

$$\mathbb{E}_t[\Pi_t^0(t, T)] = \mathbb{E}_t[\Pi_t(t, T)] - BCVA_0; \quad (6)$$

where $BCVA_0$ is the investor's bilateral credit value adjustment given by:

$$BCVA_0 = l_2 \mathbb{E}_t[1_{\{\tau=\tau_2 \leq T\}} A_{\tau_2}^+] - l_0 \mathbb{E}_t[1_{\{\tau=\tau_0 \leq T\}} (-A_{\tau_0})^+]; \quad (7)$$

Given $\tau > t$, the price of the contract's payoff under bilateral counterparty risk from the counterparty's point of view is given by:

$$\mathbb{E}_t[\Pi_t^2(t, T)] = \mathbb{E}_t[-\Pi_t(t, T)] - BCVA_2; \quad (8)$$

where $BCVA_2$ is the counterparty's bilateral credit value adjustment given by:

$$BCVA_2 = l_0 \mathbb{E}_t[1_{\{\tau=\tau_0 \leq T\}} (-A_{\tau_0})^+] - l_2 \mathbb{E}_t[1_{\{\tau=\tau_2 \leq T\}} A_{\tau_2}^+]. \quad (9)$$

According to the above propositions, the CVA is defined as the negated difference of the value of defaultable cash flows and the value of analogous default-free cash flows.

The CVA_0 effectively computes the expected loss at default of the counterparty within maturity from the investor's point of view (conditional on no previous counterparty default), assuming the investor is default-free. In general, the unilateral CVA of the investor is different from the unilateral CVA of the counterparty. Asymmetry of the unilateral CVA is due to the asymmetric assumptions underlying the unilateral CVA calculation. The investor's unilateral CVA is computed under the assumption that the investor will never default. On the other hand, the counterparty's unilateral CVA is calculated assuming the counterparty is default-free. This asymmetry breaks the consistency Equation (1).

The $BCVA_0$ is composed of the expected loss due to counterparty default before the investor (within maturity) minus the gain made by the investor in the event of self default before the counterparty (conditional on no previous default from both parties). It is clear that the bilateral CVA of the investor is exactly the negated bilateral CVA of the counterparty. This symmetry is due to the same consideration that both the investor and the counterparty are subjected to default. The bilateral CVA is in line with the consistency Equation (1). It is therefore preferable to calculate bilateral CVA in this sense.

3 General CVA with Credit Triggers

We extend the general CVA pricing formulas to allow credit triggers, where the credit trigger events could either be an obligation or an option. When the investor hits the credit trigger at the same time as the counterparty defaults, the default event overwrites the credit trigger agreement. Namely, in this scenario, the deal will be settled according to the default agreement. The default party will have to pay the other party a fractional portion of the MtM at default if the default party's MtM is negative or receive the full amount from the other party if the MtM is positive.

3.1 CVA with Obligated Credit Triggers

Let τ_0^b (τ_2^b) denote the first time the investor's (counterparty's) credit deteriorates to or below the credit trigger including default. The following equation clearly holds without any model assumptions:

$$1 = P(\tau_i^b \leq \tau_i) = P(\tau_i^b < \tau_i) + P(\tau_i^b = \tau_i); \quad i = 0, 2 \quad (10)$$

We will refer to events $1_{\{\tau_i^b < \tau_i\}}$ as *credit trigger events* and events $1_{\{\tau_i^b = \tau_i\}}$ as *jump-to-default events*.

By "obliged credit triggers", we mean that the investor is obliged to settle the deal when the counterparty hits the credit trigger for the first time before any default events, regardless of whether the investor is in the money or out of the money at the counterparty's credit trigger time. At this credit trigger time, the investor will realize a gain when in the money and will have to pay the total MtM to the counterparty when out of the money. Let

$$\tau^b = \tau_0^b \wedge \tau_2^b.$$

The following two propositions give the general formulas for calculating obliged CVA for both unilateral and bilateral cases.

Proposition 3.1. Given $\tau_2^b > t$ (given $\tau_0^b > t$), the unilateral CVA with obliged credit trigger CVA_0^b (CVA_2^b) for the investor (counterparty) is given by:

$$CVA_0^b = l_2 \mathbb{E}_t[1_{\{\tau_2^b = \tau_2 \leq T\}} A_{\tau_2}^+]; \quad (11)$$

$$CVA_2^b = l_0 \mathbb{E}_t[1_{\{\tau_0^b = \tau_0 \leq T\}} (-A_{\tau_0})^+]. \quad (12)$$

Particularly, when the credit trigger is the default state, this CVA becomes the same unilateral CVA without credit triggers.

Proposition 3.2. Given $\tau^b > t$, the bilateral CVA with obliged credit triggers of the investor $BCVA_0^b$ (counterparty $BCVA_2^b$) is given by:

$$BCVA_0^b = l_2 \mathbb{E}_t[1_{\{\tau_0 \geq \tau_2 = \tau^b \leq T\}} A_{\tau_2}^+] - l_0 \mathbb{E}_t[1_{\{\tau_2 \geq \tau_0 = \tau^b \leq T\}} (-A_{\tau_0})^+]; \quad (13)$$

$$BCVA_2^b = l_0 \mathbb{E}_t[1_{\{\tau_2 \geq \tau_0 = \tau^b \leq T\}} (-A_{\tau_0})^+] - l_2 \mathbb{E}_t[1_{\{\tau_0 \geq \tau_2 = \tau^b \leq T\}} A_{\tau_2}^+]. \quad (14)$$

Particularly, when the credit triggers for both parties are the default state, this CVA becomes the same bilateral CVA without credit triggers.

From Proposition 3.1, the obliged unilateral CVA_0^b computes the expected loss of the investor due to counterparty default at a time that is also the first time the counterparty's creditworthiness goes down to the credit trigger or below. The following scenarios are excluded from the obliged unilateral CVA calculation: the counterparty is first downgraded to the credit trigger or below but above the default level and then defaults at a later time within maturity. This is because that the deal will have to be settled for these scenarios and hence no loss will be realized due to counterparty's later default. Due to the exclusion of these scenarios, the unilateral CVA with obliged credit trigger will always be smaller or equal to the unilateral CVA without credit triggers. Mathematically, their difference is given by:

$$CVA_0 - CVA_0^b = l_2 \mathbb{E}_t[1_{\{\tau_2^b < \tau_2 \leq T\}} A_{\tau_2}^+].$$

From Proposition 3.2, the obliged bilateral $BCVA_0^b$ still consists of two components: the expected loss due to the counterparty's default at a time that is also the first time the counterparty's creditworthiness goes down to the credit trigger or below (within maturity and before the investor's default) minus the gain made by the investor in the event of self default at a time that is the first time the investor's creditworthiness goes down to the credit trigger or below (within maturity and before the counterparty's default; conditional on no previous credit trigger events for both parties). The following scenarios are excluded from the obliged bilateral CVA calculation: either the counterparty or the investor is first downgraded to their own credit triggers or below but above the default level and then a default from either party happens at a later time within maturity. When either party hits their own credit triggers before any default events, the deal will be settled from the obligation agreement. However, the obliged bilateral CVA is not necessarily smaller than the bilateral CVA without credit triggers, depending on the difference of the bilateral CVAs that is given by the following quantity:

$$q_1 := BCVA_0 - BCVA_0^b = l_2 \mathbb{E}_t[1_{\{\tau^b < \tau = \tau_2 \leq T\}} A_{\tau_2}^+] - l_0 \mathbb{E}_t[1_{\{\tau^b < \tau = \tau_0 \leq T\}} (-A_{\tau_0})^+].$$

The first term represents the expected loss at counterparty's default time (before the default of the investor; within maturity) after any credit trigger events from either parties. The second term represents the expected gain in the event of the investor's own default (before the default of the counterparty; within maturity) after any credit trigger events from either parties. Because the credit triggers are obliged, the deal will have to be settled once any credit trigger events happen from either side. Therefore, the expected losses due to these scenarios are not included in the obliged bilateral CVA calculation.

If this quantity q_1 is positive, the obliged bilateral CVA is smaller; if it is negative, the obliged bilateral CVA is larger. There are many factors that affect the sign of this quantity including initial creditworthiness of the two parties, level of credit triggers and correlations between default times and credit trigger times. When the credit trigger barriers of both parties are close to zero, then q_1 will be close to zero. Qualitatively, if the investor has a much better creditworthiness than the counterparty (for instance, less likely to default and smaller loss given default), then q_1 is more likely to be positive. As a result, the obliged bilateral CVA will be smaller than the bilateral CVA without credit triggers in this case.

We can also compute the difference of these two CVAs from the counterparty's point of view. It is clear that the decreased amount of CVA due to obliged credit triggers for one party is exactly the same amount of CVA increased for the other party.

3.2 CVA with Optional Credit Triggers

Most current contracts have optional credit triggers instead of obliged ones. By optionality, we mean that the investor has the right but not obligation to settle the deal when the counterparty hits the credit trigger before default. If the option is exercised, the investor will realize the discounted MtM of $A_{\tau_2^b}$ (denominated by today's dollar) from the counterparty and will not suffer any future loss due to counterparty's later default. If the investor does not settle at counterparty's credit trigger time, the realized value would be an expected discounted conditional fair value, denoted by $A_{\tau_2^b}^0$, of all future defaultable cash flows. Let us call this quantity $A_{\tau_2^b}^0$ discounted *mark-to-credit* (MtC) that is given by

$$A_{\tau_2^b}^0 = \mathbb{E}_{\tau_2^b}[\Pi_{\tau_2^b}^0(\tau_2^b, T)]D(t, \tau_2^b).$$

The optimal time to settle depends on the sign of the difference $A_{\tau_2^b} - A_{\tau_2^b}^0$, which stands for the discounted conditional CVA (without credit triggers) at the credit trigger time τ_2^b . The optimal time to settle is when this difference becomes positive. When only unilateral CVA is considered, this difference becomes

$$A_{\tau_2^b} - A_{\tau_2^b}^0 = \mathbb{E}_{\tau_2^b}[\text{CVA}_0(\tau_2^b, T)]D(t, \tau_2^b);$$

which is always non-negative, where $\text{CVA}_0(\tau_2^b, T)$ denotes the unilateral CVA at time τ_2^b . Therefore, in this case, the investor will settle for sure once the counterparty hits the credit trigger before default.

When bilateral CVA is considered, the difference of the discounted MtM and the discounted MtC becomes

$$A_{\tau_2^b} - A_{\tau_2^b}^0 = \mathbb{E}_{\tau_2^b}[\text{BCVA}_0(\tau_2^b, T)]D(t, \tau_2^b);$$

which could be either positive or negative, where $\text{BCVA}_0(\tau_2^b, T)$ denotes the bilateral CVA at time τ_2^b . When it is positive, the investor will settle to realize a gain from the counterparty. When it is negative, the investor should not settle in order to avoid immediate losses from negative CVA at trigger time. In the case when the investor does not settle, he may incur a future loss due to the counterparty's later default within maturity. Therefore, the bilateral CVA calculation should include expected losses due to these events as well.

From the counterparty's point of view, if the settlement occurs at the investor's credit trigger time τ_0^b , the expectation is to realize the discounted MtM of $-A_{\tau_0^b}$. Otherwise, the realized value of the investor would be the discounted MtC of $A_{\tau_0^b}^2$ which is given by

$$A_{\tau_0^b}^2 = \mathbb{E}_{\tau_0^b}[\Pi_{\tau_0^b}^2(\tau_0^b, T)]D(t, \tau_0^b).$$

We will refer to the scenario $A_{\tau_2^b} < A_{\tau_2^b}^0$ ($A_{\tau_2^b} > A_{\tau_2^b}^0$) as *out the money* (*in the money*) for the investor and the scenario $-A_{\tau_0^b} < A_{\tau_0^b}^2$ ($-A_{\tau_0^b} > A_{\tau_0^b}^2$) as *out the money* (*in the money*) for the counterparty. The following two propositions give general formulas for both unilateral and bilateral CVA pricing with optional credit triggers.

Proposition 3.3. *Given $\tau_2^b > t$ (given $\tau_0^b > t$), the unilateral CVA with optional credit triggers $\overline{\text{CVA}}_0^b$ ($\overline{\text{CVA}}_2^b$) for the investor (counterparty) equals corresponding unilateral CVA with obliged credit triggers:*

$$\overline{\text{CVA}}_0^b = \text{CVA}_0^b; \quad (15)$$

$$\overline{\text{CVA}}_2^b = \text{CVA}_2^b. \quad (16)$$

Particularly, when the credit trigger is the default state, this CVA becomes the same unilateral CVA without credit triggers.

Proposition 3.4. *Given $\tau^b > t$, the bilateral CVA with optional credit triggers $\overline{\text{BCVA}}_0^b$ ($\overline{\text{BCVA}}_2^b$) for the investor (counterparty) is given by:*

$$\overline{\text{BCVA}}_0^b = l_2 \mathbb{E}_t[1_{\{\tau_0 \geq \tau_2 = \tau \leq T\}} A_{\tau_2}^+] + l_2 \mathbb{E}_t[1_{\{\tau_0^b > \tau_2^b < \tau_2 = \tau \leq T, A_{\tau_2^b} < A_{\tau_2^b}^0\}} A_{\tau_2}^+] \quad (17)$$

$$+ l_2 \mathbb{E}_t[1_{\{\tau_2^b > \tau_0^b < \tau_2 = \tau \leq T, -A_{\tau_0^b} < A_{\tau_0^b}^2\}} A_{\tau_2}^+] - l_0 \mathbb{E}_t[1_{\{\tau_2 \geq \tau_0 = \tau^b \leq T\}} (-A_{\tau_0})^+]$$

$$- l_0 \mathbb{E}_t[1_{\{\tau_0^b > \tau_2^b < \tau_0 = \tau \leq T, A_{\tau_2^b} < A_{\tau_2^b}^0\}} (-A_{\tau_0})^+] - l_0 \mathbb{E}_t[1_{\{\tau_2^b > \tau_0^b < \tau_0 = \tau \leq T, -A_{\tau_0^b} < A_{\tau_0^b}^2\}} (-A_{\tau_0})^+]$$

$$\overline{\text{BCVA}}_2^b = -\overline{\text{BCVA}}_0^b. \quad (18)$$

Particularly, when the credit triggers for both parties are at the default state, this CVA becomes the same bilateral CVA without credit triggers.

From Proposition 3.3, the unilateral CVA with an optional credit trigger is the same as the obliged unilateral CVA with an analogous obliged credit trigger. This is similar to a vanilla American call option that should be valued as an analogous vanilla European call option.

From Proposition 3.4, the optional bilateral CVA has six terms. The first and the fourth term add up to exactly the obliged bilateral CVA. The second term denotes the expected loss at counterparty default time (before the investor's default and within maturity) before which there is a credit trigger event from the counterparty (before the trigger event from the investor). At the counterparty's credit trigger time, the investor has the option to settle. If out of the money, the investor will not settle and may therefore incur a loss due to counterparty's later default. The third term denotes the expected loss at counterparty default time (before the investor's default and within maturity) before which there is a credit trigger event from the investor (before the trigger event of the counterparty). At the investor's own credit trigger time, the counterparty (not the investor) has the option to settle. If the counterparty is out of the money (the investor is in the money), the counterparty will not settle and the investor may therefore incur a loss due to counterparty's later default. The fifth term denotes the expected gain from the investor's own default at a time before which the counterparty's credit trigger event has happened (before the credit trigger event of the investor). The sixth term represents the expected gain from the investor's own default at a time before which the investor's own credit trigger event has happened (before the credit trigger event of the counterparty).

When only unilateral CVA is considered, we have concluded that it is better for the investor to settle the deal at the credit trigger time regardless of moneyness. However, when bilateral CVA is considered, it may in fact be better off not to settle when the investor is out of the money at the counterparty credit trigger time (before any other default or credit trigger events) depending on the difference of these two bilateral CVAs which is given by:

$$\begin{aligned} \overline{\text{BCVA}}_0^b - \text{BCVA}_0^b &= l_2 \mathbb{E}_t [1_{\{\tau_0^b > \tau_2^b < \tau_2 = \tau \leq T, A_{\tau_2^b} < A_{\tau_2^b}^0\}} A_{\tau_2^b}^+] + l_2 \mathbb{E}_t [1_{\{\tau_2^b > \tau_0^b < \tau_2 = \tau \leq T, -A_{\tau_0^b} < A_{\tau_0^b}^2\}} A_{\tau_2^b}^+] \\ &\quad - l_0 \mathbb{E}_t [(1_{\{\tau_0^b > \tau_2^b < \tau_0 = \tau \leq T, A_{\tau_2^b} < A_{\tau_2^b}^0\}}) (-A_{\tau_0})^+] - l_0 \mathbb{E}_t [1_{\{\tau_2^b > \tau_0^b < \tau_0 = \tau \leq T, -A_{\tau_0^b} < A_{\tau_0^b}^2\}} (-A_{\tau_0})^+]. \end{aligned}$$

The financial meanings of each of the terms above have already been explained. When this difference is positive, it is better to settle regardless of moneyness; if it is negative, it is better not to settle when the investor is out of the money at counterparty's credit trigger time.

The optional bilateral CVA could be either larger or smaller than the bilateral CVA without credit triggers. Their difference is given by:

$$\begin{aligned} q_2 &:= \text{BCVA}_0 - \overline{\text{BCVA}}_0^b \\ &= l_2 \mathbb{E}_t [1_{\{\tau_0^b > \tau_2^b < \tau_2 = \tau \leq T, A_{\tau_2^b} \geq A_{\tau_2^b}^0\}} A_{\tau_2^b}^+] + l_2 \mathbb{E}_t [1_{\{\tau_2^b > \tau_0^b < \tau_2 = \tau \leq T, -A_{\tau_0^b} < A_{\tau_0^b}^2\}} A_{\tau_2^b}^+] \\ &\quad + l_2 \mathbb{E}_t [1_{\{\tau_0^b = \tau_2^b < \tau_2 = \tau \leq T\}} A_{\tau_2^b}^+] - l_0 \mathbb{E}_t [(1_{\{\tau_0^b > \tau_2^b < \tau_0 = \tau \leq T, A_{\tau_2^b} \geq A_{\tau_2^b}^0\}}) (-A_{\tau_0})^+] \\ &\quad - l_0 \mathbb{E}_t [1_{\{\tau_2^b > \tau_0^b < \tau_0 = \tau \leq T, -A_{\tau_0^b} < A_{\tau_0^b}^2\}} (-A_{\tau_0})^+] - l_0 \mathbb{E}_t [1_{\{\tau_2^b = \tau_0^b < \tau_0 = \tau \leq T\}} (-A_{\tau_0})^+]. \end{aligned}$$

The first term represents expected loss at counterparty default time (before the investor's default) before which the counterparty's credit trigger event has happened (before the investor's credit trigger time) at which time the investor is in or at the money. The investor will settle the deal in this scenario and hence the value is excluded from the optional bilateral CVA calculation. This is similar for the second, fourth and fifth terms. The third term represents the expected loss at counterparty default time (before the investor's default) that is before the simultaneous credit trigger events from both parties. When simultaneous credit trigger events happen, the deal will be settled for sure. This is because the optionality is bilateral and one of the two parties will be in the money. Similarly for the last term.

If the quantity $q_2 > 0$, the bilateral CVA without credit triggers is larger; if $q_2 < 0$, the bilateral CVA with optional credit triggers is larger. Similar to q_1 , there are many factors that determine the sign of quantity q_2 . When both credit triggers are at the default state, q_2 goes to zero. Again, qualitatively, if the investor has a much better creditworthiness than the counterparty (for instance, it is less likely to default with a smaller loss given default), then q_2 is more likely to be positive. As a result, the optional bilateral CVA will be smaller than the bilateral CVA without credit triggers in this case.

Similar to the case of obliged bilateral CVA, the reduced amount of bilateral CVA due to optional credit triggers for one party is exactly the same amount of bilateral CVA increased for the other party. Based on bilateral CVA calculation for both obliged and optional credit triggers, we observe that adding credit triggers may sometimes produce larger CVA which contradicts the original motivation of adding credit triggers. Therefore, it is only reasonable to add credit triggers when the investor has much better creditworthiness than the counterparty. However, the counterparty with a much worse creditworthiness would not agree to add credit triggers. Alternatively, the counterparty may require to set a very high credit trigger level for the investor. On the other hand, the investor would not agree the deal if the credit trigger level is too high.

4 The Model Risk of CVA with Credit Triggers

In this section, we provide a general modeling formulation and then discuss the impact of different model assumptions on CVA numbers. We find that the CVA with credit triggers is highly model dependent, especially when considering the lack of data for calibration of jump-to-default probabilities. A numerical example is also given to illustrate this model risk.

4.1 General Formulation

Let real valued process $X_{t,i}$ denote the creditworthiness of name i at time t with initial $x_{0,i} > 0$. Without loss of generality, let 0 be the absorbing default state. Let constant b_i denote the credit trigger barrier of name i with $0 \leq b_i < x_{0,i}$. Default time τ_i and credit trigger time τ_i^b are thus defined respectively as follows:

$$\tau_i = \inf\{t \geq 0 : X_{t,i} \leq 0\}; \quad \tau_i^b = \inf\{t \geq 0 : X_{t,i} \leq b_i\}.$$

From the above definition, it is obvious that Equation (10) is satisfied. This is a general formulation of the problem. In order to compute CVA explicitly, the joint distribution of $X_{t,i}$ needs to be specified.

4.2 Diffusion, Pure Jumps or Jump Diffusion?

The choices of $X_{t,i}$ will essentially affect the CVA numbers. In general, one could specify the creditworthiness $X_{t,i}$ to be either a diffusion, or a pure jump or a jump diffusion.

If $X_{t,i}$ is assumed to be a diffusion for both $i = 0, 2$, both unilateral and bilateral CVA with obliged credit triggers are zero based on Propositions 3.1 and 3.2. However, both unilateral and bilateral CVA without credit triggers are non-zero in general even for diffusion models. For optional credit triggers under diffusion models, from Propositions 3.3 and 3.4, the unilateral CVA is zero and the bilateral CVA is simplified to the following equation:

$$\begin{aligned} \overline{\text{BCVA}}_0^b &= l_2 \mathbb{E}_t[1_{\{\tau_0^b > \tau_2^b < \tau_2 = \tau \leq T, A_{\tau_2} < A_{\tau_2}^0\}} A_{\tau_2}^+] + l_2 \mathbb{E}_t[1_{\{\tau_2^b > \tau_0^b < \tau_2 = \tau \leq T, -A_{\tau_0} < A_{\tau_0}^2\}} A_{\tau_2}^+] \\ &\quad - l_0 \mathbb{E}_t[(1_{\{\tau_0^b > \tau_2^b < \tau_0 = \tau \leq T, A_{\tau_2} < A_{\tau_2}^0\}})(-A_{\tau_0})^+] - l_0 \mathbb{E}_t[1_{\{\tau_2^b > \tau_0^b < \tau_0 = \tau \leq T, -A_{\tau_0} < A_{\tau_0}^2\}}(-A_{\tau_0})^+]. \end{aligned}$$

If only $X_{t,2}$ is assumed to be a diffusion while $X_{t,0}$ can jump-to-default, then $\overline{\text{BCVA}}_0^b$ will be negative. On the other hand, if only $X_{t,0}$ is assumed to be a diffusion while $X_{t,2}$ can jump-to-default, then $\overline{\text{BCVA}}_0^b$ will be positive. The unilateral CVA numbers could vary from zero to non-zero while the bilateral CVA numbers could switch signs depending on different model settings.

If $X_{t,i}$ is assumed to be a pure jump for both $i = 0, 2$, the general CVA formula can not be simplified further. It is therefore essential to compute jump-to-default probabilities in all cases. This jump-to-default probability also depends on the distribution assumption of the jump sizes. Consider a pure jump model, if the jumps follow a distribution that takes positive mass only at $(-\infty, -x_0)$ and zero at anywhere else, then the probability $P(\tau^b = \tau \leq T)$ will be exactly the same as $P(\tau \leq T)$. On the other hand, if the jumps follow a distribution that takes positive mass only at $(-b, +\infty)$ and zero at anywhere else, then $P(\tau^b = \tau \leq T) = 0$.

Based on this analysis, we conclude that the choices of different models for $X_{t,i}$ have very large impact on CVA calculations.

4.3 Probabilities Needed for CVA Calculation

It is essential to compute the probabilities of the following kinds for CVA calculation:

$$\begin{aligned} &\mathbb{E}_t[1_{\{\tau_i^b=\tau_i\leq T\}}]; & \mathbb{E}_t[1_{\{\tau=\tau_i=\tau_i^b=\tau^b\leq T\}}]; \\ &\mathbb{E}_t[1_{\{\tau_2^b<\tau_2\leq T, A_{\tau_2^b}<A_{\tau_2^b}^0\}}]; & \mathbb{E}_t[1_{\{\tau_2^b>\tau_0^b<\tau_2=\tau\leq T, A_{\tau_2^b}<A_{\tau_2^b}^0\}}]. \end{aligned}$$

We will give some explicit models to discuss the computation of the first probability $\mathbb{E}_t[1_{\{\tau_i^b=\tau_i\leq T\}}]$ in appendices. The second probability needs the joint distribution of τ_0, τ_2, τ_0^b and τ_2^b . The third probability involves the joint distribution of $\tau_2, \tau_2^b, A_{\tau_2^b}$ and $A_{\tau_2^b}^0$. The most complicated probability is the last one, which involves the joint distribution of $\tau_0, \tau_2, \tau_0^b, \tau_2^b, A_{\tau_2^b}$ and $A_{\tau_2^b}^0$. In general these probabilities are very complicated to compute and one may have to rely on Monte-Carlo simulation.

4.4 Calibration Issues

From the previous subsection, we see the difficulties of computing those probabilities. Even though one can compute these probabilities, what is more difficult is the lack of data for calibrating jump-to-default probabilities.

In practice, we may have term structures of default probabilities and transition probabilities. These are all cumulative default (transition) probabilities. For the default probability, it contains the scenarios of both diffuse-to-default and jump-to-default events. However, no information is available in the market to disentangle these two different types of default probabilities, unless we assume that default can only occur through jumps (or only occur through diffusion). Consequently, we may find two very different models that match exactly the default probability term structure and the transition matrix, however producing very different CVA numbers. Therefore the CVA with credit triggers exhibits a large model risk. To avoid these difficulties, some institutions are using the Poor Man’s estimate which is an approximation based on the transition matrix approach. However, this approximation is under an invalid assumption, see appendix for details.

4.5 Numerical Example

In this example, we assume that the only data available for calibration is the PD term structure of the counterparty which is given by the circles in Figure 1⁶. Both the Normal Jump (NJ) model and the Negative Exponential Jump (NEJ) model are calibrated to this term structure. The description of these two models are given in the appendix. Both models can match the term structure reasonably well. However, we will show that they produce different obliged unilateral CVA numbers. For illustrative purpose, here we simply assume exposure at default is constant one.

Figure 1 shows the given default probability term structure with tenors 1-year, 2-year, ..., 15-year and the fitted term structures from both NJ and NEJ models. The calibration is done by minimizing the root mean square error (RMSE), where the error is computed as percentage relative error. Parameters μ, x_0 (normalized) and λ are calibrated for the NJ model and parameters β, x_0 and λ are backed out for the NEJ model. Table 1 shows the calibrated parameters for these two models and they share the same RMSE which is around 2.68%. As expected, the NEJ model has a larger calibrated x_0 and a smaller calibrated λ than the NJ model. This is because the NEJ model has only negative jumps. In order to match the same default probability with the NJ model with two sided jumps, x_0 is expected to be larger and λ is expected to be smaller. Based on the calibrated parameters, the credit trigger level

	μ	x_0	λ	β	RMSE
NJ	0.0000	2.0000	0.4000	N.A.	2.68%
NEJ	N.A.	5.7785	0.1276	0.4028	2.68%

Table 1: Calibrated Parameters for the Normal Jump (NJ) Model and the Negative Exponential Jump (NEJ) Model.

⁶In practice, PD term structures are usually obtained by bootstrapping from CDS curves. In this example, we assume a hypothetical PD curve.

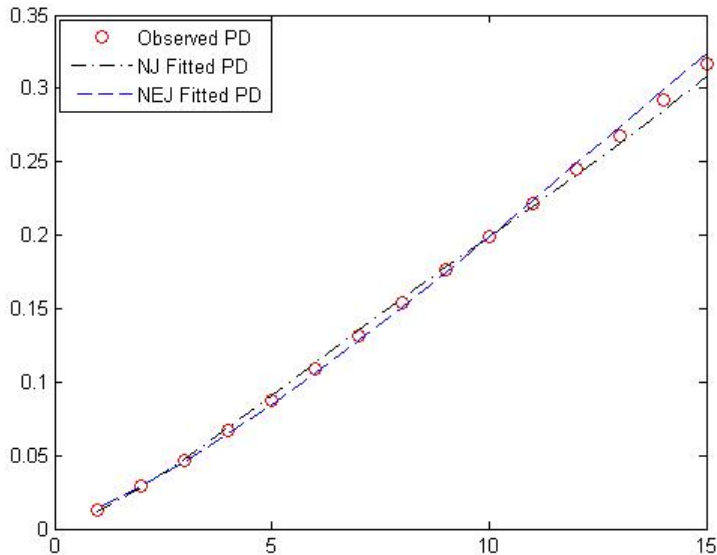


Figure 1: Observed Default Probability Term Structure vs. Fitted Ones.

for the NJ model is in the range of $(0, 2)$ while $(0, 5.7785)$ for the NEJ model. A credit trigger level of the same amount, say 1, for these two models have a different creditworthiness. In order to be fair for a comparison, the p^b is matched to compute comparable credit trigger levels. This also depends on the maturity one is looking at. For a given maturity, say $T = 1$, and a given credit trigger level of the NJ model, say $b = 0.5$, a comparable credit trigger for the NEJ model (2.0 in this case) is found by equating the p^b of these two models. Based on the calibrated parameters together with the credit triggers, we can compute the following trigger ratios:

$$Trigger\ Ratio := P(\tau^b = \tau \leq T) / P(\tau \leq T).$$

Table 2 shows a comparison of trigger ratios for these two models for varying maturities T and varying credit triggers. There are several observations in the table. First, the trigger ratios are different for these two models although both can match the observed PD term structure very well. Second, for a given fixed T , the discrepancy becomes smaller as the credit trigger becomes smaller. This is intuitive, since both will go to one as the barrier approaches zero. Third, the trigger ratios of NEJ are consistently higher than the ones of NJ for all the cases. This example quantitatively illustrates the model risk that exhibits in the CVA calculation with credit triggers.

5 Conclusions and Recommendations

In this article, we have generalized the arbitrage-free valuation framework for CCR adjustments when credit triggers are allowed in the contract. General formulas are derived for all four cases: obliged unilateral, obliged bilateral, optional unilateral and optional bilateral. The unilateral CVA with an optional credit trigger is found to be equal to the unilateral CVA with an analogous obliged credit trigger. Based on the general formulas, we show that the unilateral CVA is indeed reduced by adding credit triggers, either obliged or optional. This is in line with the original motivation of adding credit triggers in the contract. It is therefore always beneficial to enter into a unilateral credit trigger agreement. However, adding credit triggers may not necessarily decrease bilateral CVA. In fact, the increased amount of bilateral CVA due to credit triggers for one party is exactly the same amount of bilateral CVA reduced for the other party. It is therefore not always beneficial to enter into a bilateral credit trigger agreement.

We also point out the model risk of CVA calculation. This risk largely comes from the lack of data for calibrating jump-to-default probabilities. Finally we numerically calibrated two models and show that they produce different CVA numbers even though both match the same probability term structure.

$T = 1$	p^b	2.89%	6.12%	11.12%
Credit Barrier	NJ	0.5000	1.0000	1.5000
	NEJ	2.0171	4.0000	5.5816
Trigger Ratios	NJ	89.26%	77.32%	68.39%
	NEJ	95.23%	90.70%	87.19%
$T = 5$	p^b	15.67%	25.61%	38.76%
Credit Barrier	NJ	0.5000	1.0000	1.5000
	NEJ	2.0354	3.6850	5.098
Trigger Ratios	NJ	72.90%	48.01%	31.63%
	NEJ	81.58%	68.61%	58.73%
$T = 10$	p^b	29.06%	40.96%	54.94%
Credit Barrier	NJ	0.5000	1.0000	1.5000
	NEJ	1.5971	3.1055	4.4574
Trigger Ratios	NJ	64.05%	36.55%	20.26%
	NEJ	76.73%	58.91%	45.84%
$T = 15$	p^b	40.30%	51.68%	64.21%
Credit Barrier	NJ	0.5000	1.0000	1.5000
	NEJ	1.1492	2.5400	3.8481
Trigger Ratios	NJ	52.62%	27.96%	14.26%
	NEJ	78.23%	57.29%	42.03%

Table 2: Trigger Ratio Comparison of the Normal Jump (NJ) Model and the Negative Exponential Jump (NEJ) Model.

This example quantitatively illustrates the model risk embedded in the credit trigger CVA calculation. The model risk may be mitigated by consulting business experts who may have a prior projection of the future paths of the creditworthiness of the counterparty based on their experience and expert judgments. It is also instructive to look at the risk numbers generated from several different models. This comparison study may then be combined with expert judgment to come to a final selection of the model that fits the situation we are in. However, we recommend considering other CCR mitigation methods, such as using collaterals, before using credit triggers, due to large model risks involved.

Appendices

A Simple Models for Obligated Unilateral CVA

Minding the difficulties mentioned in previous sections, one can still do something for the unilateral CVA calculation with credit triggers if we assume a pure jump model for $X_{t,i}$. The motivation of adding a credit trigger to the agreement is to reduce CVA and hence increase the value of the discounted defaultable cash flow. However, as we have shown, adding a credit trigger may actually increase the CVA for the bilateral case. The unilateral CVA calculation becomes important because it indeed lowers the unilateral CVA without credit triggers, which is in line with the motivation of adding credit triggers in the agreement. When the investor confidently does not expect to default within maturity of the contract, it is then reasonable for the investor to compute the CVA based on unilateral calculation.

In this section, we provide several simple models to calculate obligated unilateral CVA. The examples we are going to discuss are well-known in the credit risk literature such as Lando (2004). We augmented existing results by providing the formulas for jump-to-default probabilities. Since we are dealing with a unilateral case, we will suppress the name index in this section. All calculations are from the investor's point of view. The default time and credit trigger time in this section are therefore referred to the counterparty.

A.1 Poor Man's Estimate

For given time T , the default event $1_{\{\tau \leq T\}}$ and the credit trigger event $1_{\{\tau^b \leq T\}}$ are positively correlated with correlation given by:

$$\text{Corr}[1_{\{\tau \leq T\}}, 1_{\{\tau^b \leq T\}}] = \sqrt{\frac{p^b(1-p)}{p(1-p^b)}};$$

where $p := P(\tau \leq T)$, $p^b := P(\tau^b \leq T)$. This correlation will approach 1 when b approaches zero and it will be close to zero only when p^b is far more less than p . Let us define another stopping $\hat{\tau}^b$ as

$$\hat{\tau}^b = \inf\{t > 0 : 0 < X_t \leq b\}.$$

It is clear that we have $\tau^b = \tau \wedge \hat{\tau}^b$. If X_t is a diffusion, then $\tau^b = \hat{\tau}^b$ with probability one. The correlation between the default event $1_{\{\tau \leq T\}}$ and the event $1_{\{\hat{\tau}^b \leq T\}}$ is given by:

$$\begin{aligned} \text{Corr}[1_{\{\tau \leq T\}}, 1_{\{\hat{\tau}^b \leq T\}}] &= \text{Corr}[1_{\{\tau > T\}}, 1_{\{\hat{\tau}^b > T\}}] = \frac{1 - p^b - (1-p)(1 - \hat{p}^b)}{\sqrt{p\hat{p}^b(1-p)(1 - \hat{p}^b)}} \\ &\leq \sqrt{\frac{\hat{p}^b(1-p)}{p(1 - \hat{p}^b)}} \leq \text{Corr}[1_{\{\tau \leq T\}}, 1_{\{\tau^b \leq T\}}]; \end{aligned}$$

where $\hat{p}^b := P(\hat{\tau}^b \leq T)$. This shows that τ and $\hat{\tau}^b$ are not independent. Nevertheless, if one assumes the following approximations⁷:

$$P(\hat{\tau}^b \geq \tau, T_{i-1} < \tau \leq T_i) \approx P(\hat{\tau}^b \geq T_i, T_{i-1} < \tau \leq T_i) \approx P(\hat{\tau}^b \geq T_i)P(T_{i-1} < \tau \leq T_i);$$

we have:

$$\begin{aligned} P(\tau^b = \tau \leq T) &= P(\hat{\tau}^b \geq \tau \leq T) = \sum_{T_i}^T P(\hat{\tau}^b \geq \tau, T_{i-1} < \tau \leq T_i) \\ &\approx \sum_{T_i}^T P(\hat{\tau}^b > T_i)(P(\tau \leq T_i) - P(\tau \leq T_{i-1})). \end{aligned}$$

This provides a very rough estimate of $P(\tau^b = \tau \leq T)$. A nice feature about this approximation is the form of its analytical formula which is a summation of scaled PD differentials. The scale factor $P(\hat{\tau}^b > T_i)$ goes to one when b approaches zero, making $P(\tau^b = \tau \leq T)$ consistently converge to $P(\tau \leq T)$. Probabilities $P(\hat{\tau}^b > T_i)$ and $P(\tau \leq T_i)$ need to be calibrated from the market data. One possibility is to use the transition matrix. Certain assumptions, such as homogeneous assumption, have to be made for the transition matrix, because usually only a 1-year transition matrix is available. The probability $P(\hat{\tau}^b > T_i)$ can be computed by aggregating the T_i transition probabilities from the trigger rating to all the ratings that are above credit trigger rating plus the default probability. Certainly, a caveat should be given on the neutrality of the default probability. The CVA calculation is based on risk-neutral valuation and hence risk-neutral default probabilities should be used. However, the transition matrix provided from rating agencies are obtained from historical data. The historical transition matrix should be modified to risk-neutral transition probability for the CVA calculation.

A.2 Compound Poisson Model

In this subsection we will give examples of continuous-state-continuous-time model. Let the creditworthiness of the counterparty X_t be a compound Poisson process starting at $x_0 > b \geq 0$. More specifically, it is given by:

$$X_t = \sum_{i=1}^{N_t} Y_i + x_0; \tag{19}$$

where $N_t \sim \text{Poisson}(\lambda)$ and Y_1, Y_2, \dots are i.i.d. which are independent of each other and are also independent of N_t . We will discuss some examples of Y_i after we arrive at some general formulas.

⁷The first approximation is termed the *Minimal Approximation* as discussed in Yi (2009).

Proposition A.1. For the compound Poisson model, we have the following equalities:

$$P(\tau > T) = \sum_{n=0}^{\infty} F_Y^n(x_0) \frac{(\lambda T)^n e^{-\lambda T}}{n!}; \quad (20)$$

$$P(\tau^b = \tau \leq T) = \sum_{n=1}^{\infty} G_Y^n(x_0, b) \frac{(\lambda T)^n e^{-\lambda T}}{n!}. \quad (21)$$

where $F_Y^n(x_0)$ and $G_Y^n(x_0, b)$ are given by:

$$\begin{aligned} F_Y^n(x_0) &= P(Y_1 + x_0 > 0, Y_1 + Y_2 + x_0 > 0, \dots, Y_1 + \dots + Y_n + x_0 > 0); \quad n = 1, 2, 3, \dots \\ G_Y^n(x_0, b) &= G_Y^{n-1}(x_0, b) + P(Y_1 + x_0 > b, \dots, Y_1 + \dots + Y_{n-1} + x_0 > b, Y_1 + \dots + Y_n + x_0 \leq 0) \end{aligned}$$

with $F_Y^0(x_0) = 1$ and $G_Y^1(x_0, b) = P(Y_1 + x_0 \leq 0)$.

For a given distribution of Y_i , the above proposition provides the general formula for computing two important probabilities that are needed for unilateral CVA calculation. Multi-dimensional integrations are involved in these formulas. Functions $F_Y^n(x_0)$ and $G_Y^n(x_0, b)$ are probabilities themselves and thus are less than one. The series representation in the above formulas actually converge fast considering the growth of $n!$. Let us then consider two explicit examples in which the distribution of Y_i is assumed to be i.i.d. normal and i.i.d. gamma respectively.

Example A.1. Let $Y_i \sim N(\mu, \sigma^2)$ in the compound Poisson model. Let Φ denote one dimensional standard normal cumulative distribution function (cdf) and $\Phi_n(\mathbf{x}, \mathbf{u}, \Sigma)$ denote n -dimensional normal cdf with mean vector \mathbf{u} covariance matrix Σ . Then functions $F_Y^n(x_0)$ and $G_Y^n(x_0, b)$ can be computed explicitly from:

$$\begin{aligned} F_Y^n(x_0) &= \Phi_n(\mathbf{x}_F, \mathbf{u}_F, \Sigma_F); \quad n = 1, 2, 3, \dots \\ G_Y^n(x_0, b) &= G_Y^{n-1}(x_0, b) + \Phi_n(\mathbf{x}_G, \mathbf{u}_G, \Sigma_G); \quad n = 2, 3, \dots \end{aligned}$$

with $G_Y^1(x_0, b) = \Phi(-\frac{x_0 + \mu}{\sigma})$ and

$$\begin{aligned} \mathbf{x}_F &= [x_0, x_0, \dots, x_0, x_0]_{(1 \times n)}; \\ \mathbf{u}_F &= -[\mu, 2\mu, \dots, (n-1)\mu, n\mu]_{(1 \times n)}; \\ \mathbf{x}_G &= [x_0 - b, x_0 - b, \dots, x_0 - b, -x_0]_{(1 \times n)}; \\ \mathbf{u}_G &= -[\mu, 2\mu, \dots, (n-1)\mu, -n\mu]_{(1 \times n)}; \\ \Sigma_F &= \sigma^2 \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & n \end{bmatrix}_{(n \times n)}; \\ \Sigma_G &= \sigma^2 \begin{bmatrix} 1 & 1 & \dots & 1 & -1 \\ 1 & 2 & \dots & 2 & -2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & \dots & n-1 & -(n-1) \\ -1 & -2 & -3 & -(n-1) & n \end{bmatrix}_{(n \times n)}. \end{aligned}$$

In this simple example, there are actually only three parameters: μ/σ , x_0/σ and λ .

Example A.2. Let $-Y_i \sim \text{Exp}(1/\beta)$ in the compound Poisson model. Then functions $F_Y^n(x_0)$ and $G_Y^n(x_0, b)$ can be computed explicitly from:

$$\begin{aligned} F_Y^n(x_0) &= \frac{\gamma(n, \beta x_0)}{\Gamma(n)}; \quad n = 1, 2, 3, \dots \\ G_Y^n(x_0, b) &= G_Y^{n-1}(x_0, b) + \frac{\beta^{n-1} e^{-\beta x_0} (x_0 - b)^{n-1}}{\Gamma(n)}; \quad n = 2, 3, \dots \end{aligned}$$

with $G_Y^1(x_0, b) = e^{-\beta x_0}$ and $\gamma(s, x)$ denotes the incomplete gamma function given by

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt.$$

In this simple example, we again only have three parameters: x_0 , β and λ .

Example A.1 is trivial and we will skip the proof. The proof for Example A.2 is given in the appendix. For this one sided negative exponential jump example, we have used the fact that summation of n i.i.d $Exp(1/\beta)$ is equal to $Gamma(n, 1/\beta)$ in distribution.

B Proof of Propositions

B.1 Proof of Proposition 3.1

It is sufficient to prove Equation (11). There are three scenarios for the defaultable cash flow $\Pi_t^0(T_1, T_2)$. If $\tau_2^b > T$, there is neither default nor credit trigger event of the counterparty during the life of the product. The counterparty has no problem in repaying the investor in this case. If $\tau_2 = \tau_2^b \leq T$, the counterparty jumps to default before the maturity. The investor has realized the cash flow from time t to default time τ_2 . At time τ_2 , the MtM of the remaining cash flow will be calculated. If the MtM is positive, the investor can only recover a fraction of it. If it is negative, the investor will have to pay the full amount to the counterparty. If $\tau_2 > \tau_2^b \leq T$, the counterparty hits the credit trigger before default. The investor has realized the cash flow from time t to trigger time τ_2^b . By the obligation agreement of the trigger event, the MtM of the contract will be exchanged at the trigger time. Then, we have the following equation:

$$\begin{aligned} \Pi_t^0(t, T) &= 1_{\{\tau_2^b > T\}} \Pi_t(t, T) + 1_{\{\tau_2 = \tau_2^b \leq T\}} [\Pi_t(t, \tau_2) + R_2 A_{\tau_2}^+ - (-A_{\tau_2})^+] \\ &\quad + 1_{\{\tau_2 > \tau_2^b \leq T\}} [\Pi_t(t, \tau_2^b) + A_{\tau_2^b}]. \end{aligned} \quad (22)$$

It follows that we only need to prove the following equality:

$$\mathbb{E}_t[\Pi_t^0(t, T)] = \mathbb{E}_t[\Pi_t(t, T)] - l_2 \mathbb{E}_t[1_{\{\tau_2 = \tau_2^b \leq T\}} A_{\tau_2}^+]. \quad (23)$$

Let us rewrite the terms inside the expectation in the right hand side of above Equation as:

$$1_{\{\tau_2^b > T\}} \Pi_t(t, T) + 1_{\{\tau_2^b \leq T\}} \Pi_t(t, T) + R_2 1_{\{\tau_2 = \tau_2^b \leq T\}} A_{\tau_2}^+ - 1_{\{\tau_2 = \tau_2^b \leq T\}} A_{\tau_2}^+$$

Comparing the above formula to Equation (22), it is then equivalent to prove that:

$$\begin{aligned} &\mathbb{E}_t[1_{\{\tau_2 = \tau_2^b \leq T\}} [\Pi_t(t, \tau_2) - (-A_{\tau_2})^+] + 1_{\{\tau_2 > \tau_2^b \leq T\}} [\Pi_t(t, \tau_2^b) + A_{\tau_2^b}]] \\ &= \mathbb{E}_t[1_{\{\tau_2^b \leq T\}} \Pi_t(t, T) - 1_{\{\tau_2 = \tau_2^b \leq T\}} A_{\tau_2}^+] \end{aligned}$$

This is equivalent to:

$$\begin{aligned} &\mathbb{E}_t[1_{\{\tau_2^b \leq T\}} \Pi_t(t, \tau_2) + 1_{\{\tau_2 > \tau_2^b \leq T\}} A_{\tau_2^b} - 1_{\{\tau_2 = \tau_2^b \leq T\}} (-A_{\tau_2})^+] \\ &= \mathbb{E}_t[1_{\{\tau_2^b \leq T\}} \Pi_t(t, T) - 1_{\{\tau_2 = \tau_2^b \leq T\}} A_{\tau_2}^+] \end{aligned}$$

Using the trivial equality $f = f^+ - (-f)^+$, the above equation is equivalent to:

$$\mathbb{E}_t[1_{\{\tau_2^b \leq T\}} A_{\tau_2^b}] = \mathbb{E}_t[1_{\{\tau_2^b \leq T\}} \Pi_t(\tau_2^b, T)]$$

We can see that the above equality is true by conditioning on the information at time τ_2^b .

B.2 Proof of Proposition 3.2

It is sufficient to prove Equation (13). When considering the bilateral agreement, there are actually five scenarios. If $\tau^b > T$, there is neither defaults or credit trigger events for both parties and hence all the cash flows are exchanged. If $\tau_0 > \tau_2 = \tau^b \leq T$, the counterparty first jumps to default. The investor has

realized the cash flows from time t to default time τ . Similarly, the MtM is computed and the investor will receive a fraction of it if it is positive and will have to pay the full amount to the counterparty if it is negative. If $\tau_2 > \tau_0 = \tau^b \leq T$, the investor's first jumps to default. The investor has realized the cash flows from time t to default time τ . If the MtM is positive, the investor will receive the full amount since the counterparty is not in default. If the MtM is negative, the investor only have to pay a fraction of the MtM to the counterparty. If $\tau_2 = \tau_0 = \tau^b \leq T$, the investor and the counterparty simultaneous defaults. The investor has realized the cash flows up to the default time. The investor will also receive a recovery value of the MtM provided it is positive and will only pay a fraction of it to the counterparty if it is negative. Finally, if $\tau > \tau^b \leq T$, the MtM will be exchanged by the obliged agreement. Therefore, we have:

$$\begin{aligned}\Pi_t^0(t, T) &= 1_{\{\tau^b > T\}} \Pi_t(t, T) + 1_{\{\tau_0 > \tau_2 = \tau^b \leq T\}} [\Pi_t(t, \tau_2) + R_2 A_{\tau_2}^+ - (-A_{\tau_2})^+] \\ &\quad + 1_{\{\tau_2 > \tau_0 = \tau^b \leq T\}} [\Pi_t(t, \tau_0) + A_{\tau_0}^+ - R_0 (-A_{\tau_0})^+] \\ &\quad + 1_{\{\tau_2 = \tau_0 = \tau^b \leq T\}} [\Pi_t(t, \tau_0) + R_2 A_{\tau_0}^+ - R_0 (-A_{\tau_0})^+] \\ &\quad + 1_{\{\tau > \tau^b \leq T\}} [\Pi_t(t, \tau^b) + A_{\tau^b}].\end{aligned}\tag{24}$$

It is then equivalent to prove that:

$$\mathbb{E}_t[\Pi_t^0(t, T)] = \mathbb{E}_t[\Pi_t(t, T)] - l_2 \mathbb{E}_t[1_{\{\tau_0 \geq \tau_2 = \tau^b \leq T\}} A_{\tau_2}^+] + l_0 \mathbb{E}_t[1_{\{\tau_2 \geq \tau_0 = \tau^b \leq T\}} (-A_{\tau_0})^+];\tag{25}$$

Let us rewrite the terms in the expectation of the right hand side of the above equation as follows:

$$\begin{aligned}1_{\{\tau^b > T\}} \Pi_t(t, T) + 1_{\{\tau^b \leq T\}} \Pi_t(t, T) + (R_2 - 1) 1_{\{\tau_0 > \tau_2 = \tau^b \leq T\}} A_{\tau_2}^+ - (R_0 - 1) 1_{\{\tau_2 > \tau_0 = \tau^b \leq T\}} (-A_{\tau_0})^+ \\ + (R_2 - 1) 1_{\{\tau_0 = \tau_2 = \tau^b \leq T\}} A_{\tau_2}^+ - (R_0 - 1) 1_{\{\tau_2 = \tau_0 = \tau^b \leq T\}} (-A_{\tau_0})^+\end{aligned}$$

Comparing the above formula to Equation (24), it is then equivalent to prove:

$$\begin{aligned}\mathbb{E}_t [1_{\{\tau_0 > \tau_2 = \tau^b \leq T\}} [\Pi_t(t, \tau_2) - (-A_{\tau_2})^+] + 1_{\{\tau_2 > \tau_0 = \tau^b \leq T\}} [\Pi_t(t, \tau_0) + A_{\tau_0}^+] \\ + 1_{\{\tau_0 = \tau_2 = \tau^b \leq T\}} \Pi_t(t, \tau^b) + 1_{\{\tau > \tau^b \leq T\}} [\Pi_t(t, \tau^b) + A_{\tau^b}]] \\ = \mathbb{E}_t [1_{\{\tau^b \leq T\}} \Pi_t(t, T) - 1_{\{\tau_0 > \tau_2 = \tau^b \leq T\}} A_{\tau_2}^+ + 1_{\{\tau_2 > \tau_0 = \tau^b \leq T\}} (-A_{\tau_0})^+ - 1_{\{\tau_0 = \tau_2 = \tau^b \leq T\}} A_{\tau^b}].\end{aligned}$$

This is equivalent to:

$$\begin{aligned}\mathbb{E}_t [1_{\{\tau^b \leq T\}} \Pi_t(t, \tau^b) - 1_{\{\tau_0 > \tau_2 = \tau^b \leq T\}} (-A_{\tau_2})^+ + 1_{\{\tau_2 > \tau_0 = \tau^b \leq T\}} A_{\tau_0}^+ + 1_{\{\tau > \tau^b \leq T\}} A_{\tau^b}] \\ = \mathbb{E}_t [1_{\{\tau^b \leq T\}} \Pi_t(t, T) - 1_{\{\tau_0 > \tau_2 = \tau^b \leq T\}} A_{\tau_2}^+ + 1_{\{\tau_2 > \tau_0 = \tau^b \leq T\}} (-A_{\tau_0})^+ - 1_{\{\tau_0 = \tau_2 = \tau^b \leq T\}} A_{\tau^b}]\end{aligned}$$

This is equivalent to:

$$\mathbb{E}_t [1_{\{\tau^b \leq T\}} A_{\tau^b}] = \mathbb{E}_t [1_{\{\tau^b \leq T\}} \Pi_t(\tau^b, T)]$$

The above formula is true by applying the tower property of conditional expectation.

B.3 Proof of Proposition 3.3

It is sufficient to prove Equation (15). For unilateral case, since the event indicator $1_{\{A_{\tau_2^b} < A_{\tau_2^b}^0\}}$ equals zero with probability one, we can rewrite Equation (15) as follows:

$$\overline{\text{CVA}}_0^b = l_2 \mathbb{E}_t [(1_{\{\tau_2^b = \tau_2 \leq T\}} + 1_{\{\tau_2^b < \tau_2 \leq T, A_{\tau_2^b} < A_{\tau_2^b}^0\}}) A_{\tau_2}^+].$$

When the investor chooses to settle the deal only when it is in or at the money at $\tau_2^b < \tau_2 \leq T$, there are four scenarios. If $\tau_2^b > T$, the investor will receive full cash flows. If $\tau_2 = \tau_2^b \leq T$, the counterparty first jumps to default. Besides realized cash flows from time t to τ_2 , the investor will only recover a fraction of the MtM if the MtM is positive and will have to pay it full if it is negative. If $\tau_2^b \leq T \leq \tau_2$, the investor will settle when it is in or at the money, receiving the full positive MtM; the investor will not settle when it is out of the money and will wait to receive the full rest remaining cash flows. If $\tau_2^b < \tau_2 \leq T$, the deal will be settled when the investor is in or at the money at τ_2^b ; when it is out of the money at τ_2^b , the investor will first realize the cash flows from time τ_2^b to time τ_2 and then realize the

fractional recovery if any or pay the full amount MtM to the counterparty if necessary. Therefore, we have:

$$\begin{aligned}\Pi_t^0(t, T) &= 1_{\{\tau_2^b > T\}} \Pi_t(t, T) + 1_{\{\tau_2 = \tau_2^b \leq T\}} [\Pi_t(t, \tau_2) + R_2 A_{\tau_2}^+ - (-A_{\tau_2})^+] \\ &\quad + 1_{\{\tau_2^b \leq T < \tau_2\}} [\Pi_t(t, \tau_2^b) + 1_{\{A_{\tau_2^b} \geq A_{\tau_2^b}^0\}} A_{\tau_2^b} + 1_{\{A_{\tau_2^b} < A_{\tau_2^b}^0\}} \Pi_t(\tau_2^b, T)] \\ &\quad + 1_{\{\tau_2^b < \tau_2 \leq T\}} [\Pi_t(t, \tau_2^b) + 1_{\{A_{\tau_2^b} \geq A_{\tau_2^b}^0\}} A_{\tau_2^b} + 1_{\{A_{\tau_2^b} < A_{\tau_2^b}^0\}} (\Pi_t(\tau_2^b, \tau_2) + R_2 A_{\tau_2}^+ - (-A_{\tau_2})^+)]\end{aligned}\quad (26)$$

It is then equivalent to prove that:

$$\mathbb{E}_t[\Pi_t^0(t, T)] = \mathbb{E}_t[\Pi_t(t, T)] - l_2 \mathbb{E}_t[(1_{\{\tau_2^b = \tau_2 \leq T\}} + 1_{\{\tau_2^b < \tau_2 \leq T, A_{\tau_2^b} < A_{\tau_2^b}^0\}}) A_{\tau_2}^+]. \quad (27)$$

Let us rewrite Equation (26) as:

$$\begin{aligned}\Pi_t^0(t, T) &= 1_{\{\tau_2^b > T\}} \Pi_t(t, T) + 1_{\{\tau_2 = \tau_2^b \leq T\}} [\Pi_t(t, \tau_2) + R_2 A_{\tau_2}^+ - (-A_{\tau_2})^+] \\ &\quad + 1_{\{\tau_2 > \tau_2^b \leq T\}} [\Pi_t(t, \tau_2^b) + A_{\tau_2^b}] \\ &\quad - 1_{\{\tau_2 > \tau_2^b \leq T\}} A_{\tau_2^b} + 1_{\{\tau_2^b \leq T < \tau_2\}} [1_{\{A_{\tau_2^b} \geq A_{\tau_2^b}^0\}} A_{\tau_2^b} + 1_{\{A_{\tau_2^b} < A_{\tau_2^b}^0\}} \Pi_t(\tau_2^b, T)] \\ &\quad + 1_{\{\tau_2^b < \tau_2 \leq T\}} [1_{\{A_{\tau_2^b} \geq A_{\tau_2^b}^0\}} A_{\tau_2^b} + 1_{\{A_{\tau_2^b} < A_{\tau_2^b}^0\}} (\Pi_t(\tau_2^b, \tau_2) + R_2 A_{\tau_2}^+ - (-A_{\tau_2})^+)].\end{aligned}$$

Taking expectation of both sides, from the arguments in Proposition 3.1, the first three terms add up to $\mathbb{E}_t[\Pi_t(t, T) - l_2 1_{\{\tau_2 = \tau_2^b \leq T\}} A_{\tau_2}^+]$. Then it is equivalent to show that:

$$\begin{aligned}&\mathbb{E}_t \left[-1_{\{\tau_2 > \tau_2^b \leq T\}} A_{\tau_2^b} + 1_{\{\tau_2^b \leq T < \tau_2\}} [1_{\{A_{\tau_2^b} \geq A_{\tau_2^b}^0\}} A_{\tau_2^b} + 1_{\{A_{\tau_2^b} < A_{\tau_2^b}^0\}} \Pi_t(\tau_2^b, T)] \right. \\ &\quad \left. + 1_{\{\tau_2^b < \tau_2 \leq T\}} [1_{\{A_{\tau_2^b} \geq A_{\tau_2^b}^0\}} A_{\tau_2^b} + 1_{\{A_{\tau_2^b} < A_{\tau_2^b}^0\}} (\Pi_t(\tau_2^b, \tau_2) + R_2 A_{\tau_2}^+ - (-A_{\tau_2})^+)] \right] \\ &= \mathbb{E}_t[(R_2 - 1) 1_{\{\tau_2^b < \tau_2 \leq T, A_{\tau_2^b} < A_{\tau_2^b}^0\}} A_{\tau_2}^+]\end{aligned}$$

This is equivalent to show that:

$$\mathbb{E}_t[1_{\{\tau_2^b \leq T < \tau_2\}} \Pi_t(\tau_2^b, T) + 1_{\{\tau_2^b < \tau_2 \leq T\}} (\Pi_t(\tau_2^b, \tau_2) + A_{\tau_2})] = \mathbb{E}_t[1_{\{\tau_2 > \tau_2^b \leq T\}} A_{\tau_2^b}]. \quad (28)$$

The RHS of the above equality can be rewritten as

$$\mathbb{E}_t[1_{\{\tau_2 > \tau_2^b \leq T\}} A_{\tau_2^b}] = \mathbb{E}_t[(1_{\{\tau_2^b \leq T < \tau_2\}} + 1_{\{\tau_2^b < \tau_2 \leq T\}}) A_{\tau_2^b}].$$

This indicates that it is sufficient to show the following equalities

$$\begin{aligned}\mathbb{E}_t[1_{\{\tau_2^b \leq T < \tau_2\}} \Pi_t(\tau_2^b, T)] &= \mathbb{E}_t[1_{\{\tau_2^b \leq T < \tau_2\}} A_{\tau_2^b}]; \\ \mathbb{E}_t[1_{\{\tau_2^b < \tau_2 \leq T\}} (\Pi_t(\tau_2^b, \tau_2) + A_{\tau_2})] &= \mathbb{E}_t[1_{\{\tau_2^b < \tau_2 \leq T\}} A_{\tau_2^b}].\end{aligned}$$

These two equalities can be proved by using tower property again.

B.4 Proof of Proposition 3.4

Similar arguments can imply that $\Pi_t^0(t, T)$ is given by:

$$\begin{aligned}
\Pi_t^0(t, T) &= 1_{\{\tau^b > T\}} \Pi_t(t, T) + 1_{\{\tau_0 > \tau_2 = \tau^b \leq T\}} [\Pi_t(t, \tau_2) + R_2 A_{\tau_2}^+ - (-A_{\tau_2})^+] \\
&\quad + 1_{\{\tau_2 > \tau_0 = \tau^b \leq T\}} [\Pi_t(t, \tau_0) + A_{\tau_0}^+ - R_0 (-A_{\tau_0})^+] \\
&\quad + 1_{\{\tau_2 = \tau_0 = \tau^b \leq T\}} [\Pi_t(t, \tau_0) + R_2 A_{\tau_0}^+ - R_0 (-A_{\tau_0})^+] \\
&\quad + 1_{\{\tau^b \leq T < \tau\}} \left[\Pi_t(t, \tau^b) + 1_{\{\tau_2^b < \tau_0^b\}} [1_{\{A_{\tau_2^b} \geq A_{\tau_0^b}^0\}} A_{\tau_2^b} + 1_{\{A_{\tau_2^b} < A_{\tau_0^b}^0\}} \Pi_t(\tau^b, T)] \right. \\
&\quad \quad \quad + 1_{\{\tau_0^b < \tau_2^b\}} [1_{\{-A_{\tau_0^b} \geq A_{\tau_2^b}^2\}} A_{\tau_0^b} + 1_{\{-A_{\tau_0^b} < A_{\tau_2^b}^2\}} \Pi_t(\tau^b, T)] \\
&\quad \quad \quad \left. + 1_{\{\tau_2^b = \tau_0^b\}} [A_{\tau^b} + \Pi_t(\tau^b, T)] \right] \\
&\quad + 1_{\{\tau^b < \tau_2 < \tau_0, \tau_2 \leq T\}} \left[\Pi_t(t, \tau^b) + 1_{\{\tau_2^b < \tau_0^b\}} [1_{\{A_{\tau_2^b} \geq A_{\tau_0^b}^0\}} A_{\tau_2^b} + 1_{\{A_{\tau_2^b} < A_{\tau_0^b}^0\}} (\Pi_t(\tau^b, \tau_2) + R_2 A_{\tau_2}^+ - (-A_{\tau_2})^+)] \right. \\
&\quad \quad \quad + 1_{\{\tau_0^b < \tau_2^b\}} [1_{\{-A_{\tau_0^b} \geq A_{\tau_2^b}^2\}} A_{\tau_0^b} + 1_{\{-A_{\tau_0^b} < A_{\tau_2^b}^2\}} (\Pi_t(\tau^b, \tau_2) + R_2 A_{\tau_2}^+ - (-A_{\tau_2})^+)] \\
&\quad \quad \quad \left. + 1_{\{\tau_0^b = \tau_2^b\}} [A_{\tau^b} + \Pi_t(\tau^b, T)] \right] \\
&\quad + 1_{\{\tau^b < \tau_0 < \tau_2, \tau_0 \leq T\}} \left[\Pi_t(t, \tau^b) + 1_{\{\tau_2^b < \tau_0^b\}} [1_{\{A_{\tau_2^b} \geq A_{\tau_0^b}^0\}} A_{\tau_2^b} + 1_{\{A_{\tau_2^b} < A_{\tau_0^b}^0\}} (\Pi_t(\tau^b, \tau_0) + A_{\tau_0}^+ - R_0 (-A_{\tau_0})^+)] \right. \\
&\quad \quad \quad + 1_{\{\tau_0^b < \tau_2^b\}} [1_{\{-A_{\tau_0^b} \geq A_{\tau_2^b}^2\}} A_{\tau_0^b} + 1_{\{-A_{\tau_0^b} < A_{\tau_2^b}^2\}} (\Pi_t(\tau^b, \tau_0) + A_{\tau_0}^+ - R_0 (-A_{\tau_0})^+)] \\
&\quad \quad \quad \left. + 1_{\{\tau_0^b = \tau_2^b\}} [A_{\tau^b} + \Pi_t(\tau^b, T)] \right] \\
&\quad + 1_{\{\tau^b < \tau_0 = \tau_2 \leq T\}} \left[\Pi_t(t, \tau^b) + 1_{\{\tau_2^b < \tau_0^b\}} [1_{\{A_{\tau_2^b} \geq A_{\tau_0^b}^0\}} A_{\tau_2^b} + 1_{\{A_{\tau_2^b} < A_{\tau_0^b}^0\}} (\Pi_t(t, \tau_0) + R_2 A_{\tau_0}^+ - R_0 (-A_{\tau_0})^+)] \right. \\
&\quad \quad \quad + 1_{\{\tau_0^b < \tau_2^b\}} [1_{\{-A_{\tau_0^b} \geq A_{\tau_2^b}^2\}} A_{\tau_0^b} + 1_{\{-A_{\tau_0^b} < A_{\tau_2^b}^2\}} (\Pi_t(t, \tau_0) + R_2 A_{\tau_0}^+ - R_0 (-A_{\tau_0})^+)] \\
&\quad \quad \quad \left. + 1_{\{\tau_0^b = \tau_2^b\}} [A_{\tau^b} + \Pi_t(\tau^b, T)] \right]
\end{aligned}$$

We have used the fact that the deal will be settled for sure when simultaneous credit trigger events happen, namely when $\tau_0^b = \tau_2^b$ at which both parties have the right to settle. Since there will be one party who is in the money or at the money, the deal will be settled for sure. The rest of the proof is straightforward.

B.5 Proof of Proposition A.1

From the definition of τ we have:

$$\begin{aligned}
P(\tau > T) &= P(\min_{0 < u \leq T} X_u > 0) \\
&= P(\min \{Y_1 + x_0, Y_1 + Y_2 + x_0, \dots, Y_1 + \dots + Y_{N_T} + x_0\} > 0) \\
&= P(Y_1 + x_0 > 0, Y_1 + Y_2 + x_0 > 0, \dots, Y_1 + \dots + Y_{N_T} + x_0 > 0) \\
&= \sum_{n=0}^{\infty} F_Y^n(x_0) \frac{(\lambda T)^n e^{-\lambda T}}{n!}
\end{aligned}$$

For the second equality, we prove by conditioning on the number of jumps for the event $1_{\{\tau^b = \tau \leq T\}}$. For given fixed time horizon T , there is at least one jump in order to incur default before T . If there is exactly one jump, then jump-to-default must occur. If there are two jumps, there are two possibilities. Either the jump-to-default occurs at the first jump, or at the second jump with the first jump at a level

that is above b . Following this argument, due to the independence of N_T and Y_i we have that:

$$\begin{aligned}
P(\tau^b = \tau \leq T) &= P(N_T = 1)P(Y_1 + x_0 \leq 0) \\
&\quad + P(N_T = 2)[P(Y_1 + x_0 \leq 0) + P(Y_1 + x_0 > b, Y_1 + Y_2 + x_0 \leq 0)] \\
&\quad + P(N_T = 3)[P(Y_1 + x_0 \leq 0) + P(Y_1 + x_0 > b, Y_1 + Y_2 + x_0 \leq 0) \\
&\quad\quad + P(Y_1 + x_0 > b, Y_1 + Y_2 + x_0 > b, Y_1 + Y_2 + Y_3 + x_0 \leq 0)] \\
&\quad + \dots \\
&= \sum_{n=1}^{\infty} G_Y^n(x_0, b) \frac{(\lambda T)^n e^{-\lambda T}}{n!}.
\end{aligned}$$

B.6 Proof of Example A.2

Since $Y_i \leq 0$ and $Z_n := -\sum_{i=1}^n Y_i \sim \text{Gamma}(n, 1/\beta)$, we have:

$$\begin{aligned}
F_Y^n(x_0) &= P(Y_1 + x_0 > 0, Y_1 + Y_2 + x_0 > 0, \dots, Y_1 + \dots + Y_n + x_0 > 0) \\
&= P(Y_1 + \dots + Y_n + x_0 > 0) = P(Z_n < x_0) = \frac{\gamma(n, \beta x_0)}{\Gamma(n)}.
\end{aligned}$$

Further, we have:

$$\begin{aligned}
&P(Y_1 + x_0 > b, \dots, Y_1 + \dots + Y_{n-1} + x_0 > b, Y_1 + \dots + Y_n + x_0 \leq 0) \\
&= P(Y_1 + \dots + Y_{n-1} + x_0 > b, Y_1 + \dots + Y_n + x_0 \leq 0) = P(x_0 + Y_n \leq Z_{n-1} < x_0 - b) \\
&= \int_0^{x_0 - b} \frac{\beta^{n-1} z^{n-2} e^{-\beta z}}{\Gamma(n-1)} e^{\beta(z-x_0)} dz = \frac{\beta^{n-1} e^{-\beta x_0} (x_0 - b)^{n-1}}{\Gamma(n)}.
\end{aligned}$$

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