VALUED FIELDS WITH CONTRACTIVE AUTOMORPHISM AND KAPLANSKY FIELDS

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1. Introduction

In this paper we investigate algebraic and model theoretic properties of valued difference fields whose distinguished automorphism interacts with the valuation (topology) as a contractive map. We also point out a connection of this study to certain valued fields of positive characteristic, namely the Kaplansky Fields.

We consider valued fields as three-sorted structures

$$\mathcal{K} = (K, \Gamma, \boldsymbol{k};\ v, \pi)$$

where K is the underlying field, Γ is an ordered abelian group (the *value group*), $\boldsymbol{k}$ is a field, $v : K^\times \to \Gamma$ is the valuation, with valuation ring

$$\mathcal{O}_v := \{a \in K :\ v(a) \geq 0\}$$

and maximal ideal $\mathfrak{m}_v := \{a \in K : v(a) > 0\}$ of $\mathcal{O}_v$, and $\pi : \mathcal{O}_v \to \boldsymbol{k}$ is a surjective ring morphism. Then π induces an isomorphism of fields,

$$a + \mathfrak{m} \mapsto \pi(a) :\ \mathcal{O}_v/\mathfrak{m} \to \boldsymbol{k}$$

and we identify the residue field $\mathcal{O}_v/\mathfrak{m}$ with $\boldsymbol{k}$ via this isomorphism. Accordingly, we refer to $\boldsymbol{k}$ as "the residue field". A *difference field* [1] is a field equipped with a distinguished automorphism. The distinguished automorphism of a difference field is denoted by σ, unless specified otherwise. A *valued difference field* is a valued field $\mathcal{K}$ as above where K is not just a field, but a difference field whose automorphism σ satisfies $\sigma(\mathcal{O}_v) = \mathcal{O}_v$. It follows that σ induces an automorphism of the residue field:

$$\pi(a) \mapsto \pi(\sigma(a)) :\ \boldsymbol{k} \to \boldsymbol{k}, \quad a \in \mathcal{O}_v.$$

We denote this automorphism by $\bar{\sigma}$, and $\boldsymbol{k}$ equipped with $\bar{\sigma}$ is called the *residue difference field of* $\mathcal{K}$. Likewise σ induces an automorphism of the value group, which we also denote by σ:

$$\gamma \mapsto \sigma(\gamma) := v(\sigma(a)) \text{ where } v(a) = \gamma.$$

The value group of $\mathcal{K}$ is construed as an ordered group equipped with the automorphism defined above, and we call it the *value difference group*. We say that σ is *contractive* if

$$v(\sigma(a)) > nv(a)$$

for every $n \in \mathbb{N}$, whenever $a \in K^\times$ with $v(a) > 0$.

In section 7 we present the main model theoretic results, which include Theorem 7.1:

[1] Note that traditionally a difference field means a field equipped with a distinguished *endomorphism*. We deviate from this convention because we only study valued fields equipped with a distinguished automorphism.

Theorem. *Let $\mathcal{K}$ and $\mathcal{K}'$ be σ-henselian valued difference fields with contractive distinguished automorphisms. Suppose $\mathcal{K}$ and $\mathcal{K}'$ have residue difference fields $\boldsymbol{k}$ and $\boldsymbol{k}'$, of characteristic 0, and value difference groups Γ and Γ' respectively. Then $\mathcal{K} \equiv \mathcal{K}'$ if and only if $\boldsymbol{k} \equiv \boldsymbol{k}'$, as difference fields, and $\Gamma \equiv \Gamma'$, as ordered difference groups.*

In other words, the elementary theory of $\mathcal{K}$ is determined by the elementary theories of its residue difference field and value difference group. This is an analogue of the classical result of Ax & Kochen and Ershov on henselian valued fields, see [9].

The notion of *σ-henselianity* (see Definition 4.5) is devised for valued difference fields, where σ is contractive, to replace the familiar notion of *henselianity*. For ordinary valued fields, the henselianity requirement does not restrict the residue field. In contrast, for valued difference fields, σ-henselianity implies that the residue difference field is *linear difference closed*. That is, if $\mathcal{K}$ is a σ-henselian valued difference field as in Definition 4.5, then for all $\alpha_0, \dots, \alpha_n \in \boldsymbol{k}$ with $\alpha_i \neq 0$ for some i, the equation

$$1 + \alpha_0 x + \alpha_1 \bar{\sigma}(x) + \cdots + \alpha_n \bar{\sigma}^n(x) = 0$$

has a solution in $\boldsymbol{k}$; see Lemma 4.6. Therefore the above theorem does not apply when $\bar{\sigma}$ is the identity on the residue field. In particular, the elementary theory of the field of logarithmic-exponential series equipped with its so-called right-shift operator, see [6], remains unknown. It is remarkable that similar problems with the residue difference field arise in the context of valued difference fields where the distinguished automorphism fixes the valuation of every element, see [3], [1]. There is a suitable notion of *σ-henselianity* in that context as well, and together with some mild assumptions it also implies that the residue difference field is linear difference closed.

On the algebraic side we study certain extensions of valued difference fields with contractive distinguished automorphism. A valued field extension $\mathcal{K}'$ of a valued field $\mathcal{K}$ is said to be *immediate* if the residue field and the value group of $\mathcal{K}'$ are the same as those of $\mathcal{K}$. A valued field is *maximal* if it has no proper immediate extensions and is *algebraically maximal* if it has no proper immediate algebraic extensions. The following result due to Ostrowski (in the case of archimedean value groups) and Kaplansky is a key to understanding the model theoretic properties valued fields with residue characteristic 0:

All algebraically maximal immediate algebraic extensions of a valued field $\mathcal{K}$, with residue characteristic 0, are isomorphic over $\mathcal{K}$; see [8].

In the presence of a difference operator, the role of algebraically maximal valued fields is replaced by that of *σ-algebraically maximal* valued difference fields. A σ-polynomial over a difference field is a polynomial over K in the distinct variables $x, \sigma(x), \sigma^2(x), \dots$. A difference field extension of a difference field K is *σ-algebraic* if for each element a in the extension there is a σ-polynomial F over K such that $F \notin K$ and $F(a) = 0$. By an *extension* of a valued difference field $\mathcal{K}$ we mean a valued difference field extension of $\mathcal{K}$. A valued difference field is *σ-algebraically maximal* if it has no proper σ-algebraic immediate extensions. It is easy to see that every valued difference field has σ-algebraically maximal immediate algebraic extensions. Now suppose that $\mathcal{K}$ is a valued difference field with residue characteristic 0 and that σ is contractive. A key question then becomes: When are all σ-algebraically maximal, immediate σ-algebraic extensions of a given valued difference field $\mathcal{K}$ isomorphic over $\mathcal{K}$? As the main algebraic step to obtain model theoretic conclusions, we prove the following Kaplansky-type result, see Theorem 5.8:

Theorem. *Let $\mathcal{K}$ be a valued difference field as such that*

- $\boldsymbol{k}$ *has characteristic* 0
- $v(\sigma(a)) > nv(a)$ *for all n whenever $a \in K^\times$ and $v(a) > 0$,*
- $\boldsymbol{k}$ *is linear difference closed.*

Then all σ-algebraically maximal, immediate σ-algebraic extensions of $\mathcal{K}$ are isomorphic over $\mathcal{K}$.

We also present an example, see 5.11, which illustrates that the linear difference closedness assumption on the residue difference field can not be dropped in the above theorem.

In Section 8, we formalize the idea of considering valued fields in equal characteristic $p > 0$ as valued difference fields equipped with the Frobenius endomorphism which in many aspects resemble valued difference fields equipped with a contractive automorphism. This does not lead to any new results on valued fields with positive characteristic but it proves to be a worthy instrument in the study of valued difference fields with positive characteristic, see Chapter 6 in [2]. In order to utilize this idea we need to carry the restrictions from the context of valued difference fields (with contractive automorphism), which translate in the case at hand to:

(1) the value group is p-divisible;
(2) for $\alpha_0, \dots, \alpha_n$ in the residue field, not all zero, the equation

$$1 + \alpha_0 x + \alpha_1 x^p + \cdots + \alpha_n x^{p^n} = 0$$

has a solution in the residue field.

Valued fields of characteristic $p > 0$ satisfying these conditions are known as *Kaplansky fields*. Inspired by σ-henselianity, we derive a notion of ϕ-henselianity for Kaplanksy fields (which is stronger than henselianity). We prove that a Kaplansky field is ϕ-henselian if and only if it is algebraically maximal, see Theorem 8.12. This gives yet another way of seeing why being algebraically maximal is a first order property for Kaplansky fields. These considerations also provide an alternative proof of the following well-known result from [8][2]:

All algebraically maximal immediate algebraic extensions of a Kaplansky field $\mathcal{K}$ are isomorphic over $\mathcal{K}$.

The elementary theory of an algebraically maximal Kaplansky field is determined by the elementary theories of its residue field and value group, see for example [5]. This relies on the above uniqueness result which does not hold when the assumptions on the residue field or the value group are dropped, see [8]. In 1990 F.-V. Kuhlmann classified elementary theories of a larger class of valued fields in characteristic $p > 0$, see [10]. Instead of Kaplansky fields, he considers valued fields with p-divisible value group and perfect residue field. For such valued fields the uniqueness of algebraically maximal immediate extensions fails and yet their elementary theory can be understood. This gives hope for understanding the elementary theory of valued difference fields with a contractive automorphism in a more general context. There are preliminary results towards this end, see Chapter 7 in [2].

[2] See also [11] for an account of these matters from a different point of view.

2. Preliminaries

Throughout, $\mathbb{N} = \{0, 1, 2, \dots\}$, and m, n range over $\mathbb{N}$. We let $K^\times = K \setminus \{0\}$ be the multiplicative group of a field K. We shall follow the notations and conventions of [1] on valued fields, valued difference fields, σ-polynomials, Taylor expansions of σ-polynomials etc.

Ordered difference groups. An *ordered difference group* is an ordered abelian group equipped with a distinguished (order-preserving) automorphism. We consider an ordered difference group in the obvious way as a structure for the language $\{0, +, -, <, \sigma\}$, where the unary function symbol σ is interpreted as the distinguished automorphism. Let $\Delta \subseteq \Gamma$ be an extension of ordered difference groups, and $\gamma \in \Gamma$. We define $\Delta\langle\gamma\rangle$ to be the smallest ordered difference subgroup of Γ containing Δ and γ. For $\boldsymbol{i} = (i_0, \dots, i_n) \in \mathbb{Z}^{n+1}$ we put

$$\boldsymbol{\sigma}^{\boldsymbol{i}}(\gamma) := \sum_{k=0}^{n} i_k \sigma^k(\gamma).$$

Consider the polynomial ring $\mathbb{Z}[\sigma]$ where σ is taken as an indeterminate. We construe the ordered difference group Γ as a $\mathbb{Z}[\sigma]$-module as follows: for

$$\tau = \sum_{k=0}^{n} i_k \sigma^k \in \mathbb{Z}[\sigma], \quad \gamma \in \Gamma,$$

we set $\tau(\gamma) := \boldsymbol{\sigma}^{\boldsymbol{i}}(\gamma)$ where $\boldsymbol{i} = (i_0, \dots, i_n) \in \mathbb{Z}^{n+1}$. We also consider each ordered difference subgroup of Γ as a $\mathbb{Z}[\sigma]$-submodule of Γ. Let $\gamma \in \Gamma \setminus \Delta$. We define the *annihilator of* γ *modulo* Δ to be

$$\operatorname{Ann}_{\Gamma/\Delta}(\gamma) := \{\tau \in \mathbb{Z}[\sigma] : \tau(\gamma) \in \Delta\},$$

which is an ideal of $\mathbb{Z}[\sigma]$. The ideal generated by $\tau_1, \dots, \tau_n \in \mathbb{Z}[\sigma]$ is denoted $(\tau_1, \dots, \tau_n)$.

Let $\mathcal{K}$ be a valued difference field. Then its underlying valued difference field K and value difference group Γ are considered naturally as $\mathbb{Z}[\sigma]$-modules. Furthermore

$$v(\boldsymbol{\sigma}(y)^{\boldsymbol{i}}) = \boldsymbol{\sigma}^{\boldsymbol{i}}(\gamma)$$

for all $y \in K^\times$ with $v(y) = \gamma$ and hence $v : K^\times \to \Gamma$ is a morphism of $\mathbb{Z}[\sigma]$-modules. If

$$v(\sigma(a)) > nv(a)$$

for all n whenever $a \in K$ with $v(a) > 0$, then Γ is a torsion-free $\mathbb{Z}[\sigma]$-module.

Hahn difference fields. Let $\boldsymbol{k}$ be a field and Γ an ordered abelian group. This gives the Hahn field $\boldsymbol{k}((t^\Gamma))$ whose elements are the formal sums $a = \sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ with $a_\gamma \in \boldsymbol{k}$ for all γ, with well-ordered *support* $\{\gamma : a_\gamma \neq 0\} \subseteq \Gamma$. With a as above, we define the valuation $v : \boldsymbol{k}((t^\Gamma))^\times \to \Gamma$ by $v(a) := \min\{\gamma : a_\gamma \neq 0\}$, and the surjective ring morphism $\pi : \mathcal{O}_v \to \boldsymbol{k}$ by $\pi(a) := a_0$. In this way we obtain the (maximal) valued field $\mathcal{K} = (\boldsymbol{k}((t^\Gamma)), \Gamma, \boldsymbol{k}; v, \pi)$ to which we also just refer to as the *Hahn field* $\boldsymbol{k}((t^\Gamma))$.

Suppose that the field $\boldsymbol{k}$ is equipped with an automorphism $\bar{\sigma}$ and that the ordered group Γ is equipped with an order-preserving automorphism σ. Then

$$\sum_{\gamma} a_\gamma t^\gamma \mapsto \sum_{\gamma} \bar{\sigma}(a_\gamma) t^{\sigma(\gamma)}$$

is an automorphism, to be denoted by σ, of the field $\boldsymbol{k}((t^\Gamma))$, with $\sigma(\mathcal{O}_v) = \mathcal{O}_v$. We consider the three-sorted structure $(\boldsymbol{k}((t^\Gamma)), \Gamma, \boldsymbol{k};\ v, \pi)$, with the field $\boldsymbol{k}((t^\Gamma))$ equipped with the automorphism σ as above, as a valued difference field, and also refer to it as the *Hahn difference field* [3] $\boldsymbol{k}((t^\Gamma))$.

3. Extending the residue field and value group

Theorem 6.2, which leads to our main model theoretic conclusions, involves extending certain partial isomorphisms between valued difference fields to larger domains. In this chapter we establish the algebraic background necessary to understand the structure of valued difference field extensions where only the residue difference field or the value difference group is extended.

We let $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi)$ and $\mathcal{K}' = (K', \Gamma', \boldsymbol{k}'; v', \pi')$ be valued difference fields, put $\mathcal{O} := \mathcal{O}_v,\ \mathcal{O}' := \mathcal{O}_{v'}$, and let σ denote both the difference operator of $\mathcal{K}$ and of $\mathcal{K}'$. Let $\mathcal{E} = (E, \Gamma_E, \boldsymbol{k}_E; \dots)$ be a valued difference subfield of both $\mathcal{K}$ and $\mathcal{K}'$, that is, $\mathcal{E} \leq \mathcal{K}$ and $\mathcal{E} \leq \mathcal{K}'$. For residue difference field extensions we refer to Lemmas 2.5 and 2.6 from [1]:

Lemma 3.1. *Let $a \in \mathcal{O}$ and assume $\alpha = \bar{a}$ is $\bar{\sigma}$-transcendental over $\boldsymbol{k}_E$. Then*

(i) *$v(P(a)) = \min_{\boldsymbol{l}}\{v(b_{\boldsymbol{l}})\}$ for each σ-polynomial $P(x) = \sum b_{\boldsymbol{l}} \boldsymbol{\sigma}^{\boldsymbol{l}}(x)$ over E;*

(ii) *$v(E\langle a\rangle^\times) = v(E^\times) = \Gamma_E$, and $\mathcal{E}\langle a\rangle$ has residue field $\boldsymbol{k}_E\langle\alpha\rangle$;*

(iii) *if $b \in \mathcal{O}'$ is such that $\beta = \bar{b}$ is $\bar{\sigma}$-transcendental over $\boldsymbol{k}_E$, then there is a valued difference field isomorphism $\mathcal{E}\langle a\rangle \to \mathcal{E}\langle b\rangle$ over $\mathcal{E}$ sending a to b.*

Lemma 3.2. *Assume* $\mathrm{char}(\boldsymbol{k}) = 0$, *and let $G(x)$ be a nonconstant σ-polynomial over the valuation ring of E whose reduction $\bar{G}(x)$ has the same complexity as $G(x)$. Let $a \in \mathcal{O}$, $b \in \mathcal{O}'$, and assume that $G(a) = 0$, $G(b) = 0$, and that $\bar{G}(x)$ is a minimal $\bar{\sigma}$-polynomial of $\alpha := \bar{a}$ and of $\beta := \bar{b}$ over $\boldsymbol{k}_E$. Then*

(i) *$\mathcal{E}\langle a\rangle$ has value group $v(E^\times) = \Gamma_E$ and residue field $\boldsymbol{k}_E\langle\alpha\rangle$;*

(ii) *if there is a difference field isomorphism $\boldsymbol{k}_E\langle\alpha\rangle \to \boldsymbol{k}_E\langle\beta\rangle$ over $\boldsymbol{k}_E$ sending α to β, then there is a valued difference field isomorphism $\mathcal{E}\langle a\rangle \to \mathcal{E}\langle b\rangle$ over $\mathcal{E}$ sending a to b.*

In the rest of this chapter we assume that Γ is a torsion-free $\mathbb{Z}[\sigma]$-module. Under this assumption, we describe several types of valued difference field extensions where only the value difference group extends.

Lemma 3.3. *Let $\gamma \in \Gamma \setminus \Gamma_E$ and assume that $\mathrm{Ann}_{\Gamma/\Delta}(\gamma) = \{0\}$. Let $a \in K$ be such that $v(a) = \gamma$. Then*

(i) *$v(P(a)) = \min_{\boldsymbol{l}}\{v(b_{\boldsymbol{l}}) + \boldsymbol{\sigma}^{\boldsymbol{l}}(\gamma)\}$ for each σ-polynomial $P(x) = \sum b_{\boldsymbol{l}} \boldsymbol{\sigma}^{\boldsymbol{l}}(x)$ over E;*

(ii) *$v(E\langle a\rangle^\times) = \Gamma_E\langle\gamma\rangle$, and $\mathcal{E}\langle a\rangle$ has residue field $\boldsymbol{k}_E$;*

(iii) *if $a' \in K'$ is such that $\mathrm{Ann}_{\Gamma/\Delta}(\gamma') = \{0\}$ where $\gamma' = v(a')$, then there is a valued difference field isomorphism $\mathcal{E}\langle a\rangle \to \mathcal{E}\langle a'\rangle$ over $\mathcal{E}$ sending a to a'.*

[3] Note that this is a more general construction than the one used in [1].

Proof. Let $P(x) = \sum b_{\boldsymbol{l}} \boldsymbol{\sigma}^{\boldsymbol{l}}(x)$ be a σ-polynomial over E. For $\boldsymbol{i} \neq \boldsymbol{j}$,

$$v(b_{\boldsymbol{i}}) + \boldsymbol{\sigma}^{\boldsymbol{i}}(\gamma) \neq v(b_{\boldsymbol{j}}) + \boldsymbol{\sigma}^{\boldsymbol{j}}(\gamma)\}$$

since $\mathrm{Ann}_{\Gamma/\Delta}(\gamma) = \{0\}$. This proves (i) and clearly $v(E\langle a\rangle^\times) = \Gamma_E\langle\gamma\rangle$. If $v(P(a)) = 0$ for a σ-polynomial P over E as above then the constant term of P has valuation zero and the residue class of $P(a)$ is equal to the residue class of the constant term of P. Hence the residue field of $E\langle a\rangle$ is $\boldsymbol{k}_E$. It follows from (i) that a is σ-transcendental over E, (iii) follows from this fact together with (i). □

Lemma 3.4. *Let $\gamma \in \Gamma \setminus \Gamma_E$ be such that $\mathrm{Ann}_{\Gamma/\Delta}(\gamma) = (\tau)$ where*

$$\tau = \sum_{k=0}^{n} i_k \sigma^k \in \mathbb{Z}[\sigma]$$

with $i_n > 0$. Take $\boldsymbol{j}, \boldsymbol{l} \in \mathbb{N}^{n+1}$ such that

$$j_k = i_k \text{ if } i_k \geq 0 \text{ and } j_k = 0 \text{ otherwise,}$$

$$l_k = -i_k \text{ if } i_k < 0 \text{ and } l_k = 0 \text{ otherwise,}$$

for $k = 0, \dots, n$. Pick $b \in E$ with $v(b) = \tau(\gamma)$. Suppose that $a \in K$ is a zero of the σ-polynomial

$$F(x) = \boldsymbol{\sigma}(x)^{\boldsymbol{j}} - \boldsymbol{\sigma}(x)^{\boldsymbol{l}} b.$$

Then

(i) *F is a minimal σ-polynomial of a over E,*
(ii) *$E\langle a\rangle$ has value group $\Gamma_E\langle\gamma\rangle$ and residue field $\boldsymbol{k}_E$,*
(iii) *if $a' \in K'$ is a zero of F and $\mathrm{Ann}_{\Gamma'/\Delta}(\gamma') = \tau$ where $\gamma' = v(a')$ then there is a valued difference field isomorphism $\mathcal{E}\langle a\rangle \to \mathcal{E}\langle a'\rangle$ over $\mathcal{E}$ sending a to a'.*

Proof. First observe that $v(a) = \gamma$ since Γ is a torsion-free $\mathbb{Z}[\sigma]$-module and

$$\boldsymbol{\sigma}^{\boldsymbol{j}}(va) - \boldsymbol{\sigma}^{\boldsymbol{l}}(va) = v(b) = \tau(\gamma) = \boldsymbol{\sigma}^{\boldsymbol{j}}(\gamma) - \boldsymbol{\sigma}^{\boldsymbol{l}}(\gamma).$$

Consider a nonzero σ-polynomial G over E,

$$G(x) = \sum_{\boldsymbol{r} \in \mathbb{N}^{n+1}} a_{\boldsymbol{r}} \cdot \boldsymbol{\sigma}(x)^{\boldsymbol{r}} \qquad (\text{all } a_{\boldsymbol{r}} \in E).$$

Suppose that either the order of G is less than n, or the order of G is n and the σ^n-degree of G is less than i_n. Let $\boldsymbol{r}, \boldsymbol{r}' \in \mathbb{N}^{n+1}$ be such that $a_{\boldsymbol{r}}, a_{\boldsymbol{r}'} \neq 0$ and $\boldsymbol{r} \neq \boldsymbol{r}'$. Then

$$v(a_{\boldsymbol{r}} \boldsymbol{\sigma}(a)^{\boldsymbol{r}}) \neq v(a_{\boldsymbol{r}'} \boldsymbol{\sigma}(a)^{\boldsymbol{r}'})$$

because otherwise $\boldsymbol{\sigma}^{\boldsymbol{r}}(\gamma) - \boldsymbol{\sigma}^{\boldsymbol{r}'}(\gamma) \in \Delta$, contradicting the minimality of τ. Hence

$$v(G(a)) = \min_{\boldsymbol{r} \in \mathbb{N}^{n+1}} v(a_{\boldsymbol{r}}) + \boldsymbol{\sigma}^{\boldsymbol{r}}(\gamma),$$

and in particular $G(a) \neq 0$.

We now show that F is an irreducible σ-polynomial over E, this will establish (i). Assume towards a contradiction that $F(x) = G(x)H(x)$ for some nonconstant σ-polynomials G, H over E. Then either $G(a)$ or $H(a)$ is equal to zero, say $G(a) = 0$. Since $\boldsymbol{\sigma}^{\boldsymbol{j}}(x)$ and $\boldsymbol{\sigma}^{\boldsymbol{l}}(x)$ do not have nonconstant common divisors (in the ring of σ-polynomials over E) either the order of G is less than n, or the order of G is n and the σ^n-degree of G is less than i_n, contradiction.

For $k \geq 0$, let E_k be the subfield $E(a, \dots, \sigma^k(a))$ of K. The considerations above show that E_n has value group $\Delta(\gamma, \dots, \sigma^n(\gamma))$ and residue field $\boldsymbol{k}_E$. Note that there is $c \in E_n$ such that σ^{n+1} is a zero of the polynomial $f(x) = x^{i_n} - c$. Since $\mathrm{Ann}_{\Gamma/\Delta}(\gamma) = (\tau)$,

$$m\sigma^{n+1}(\gamma) \notin \Delta(\gamma, \dots, \sigma^n(\gamma))$$

for $m < i_n$. Therefore f is the minimum polynomial of $\sigma^{n+1}(a)$ over E_n. So E_{n+1} has value group $\Delta(\gamma, \dots, \sigma^{n+1}(\gamma))$ and residue field $\boldsymbol{k}_E$. Likewise, for $k > n$ the minimum polynomial of $\sigma^{k+1}(a)$ over E_k is

$$x^{i_n} - \sigma^{k-n-1}(c).$$

So $E(\sigma^k(a) : k \in \mathbb{N})$ has value group $\Delta(\sigma^k(\gamma) : k \in \mathbb{N})$ and residue field $\boldsymbol{k}_E$. Applying the valued field automorphism σ^{-m}, for $m \in \mathbb{N}$, we see that $E\langle a\rangle$ has value group $\Delta\langle\gamma\rangle$ and residue field $\boldsymbol{k}_E$.

Now let $a' \in K'$ be as in (iii). Then F is a minimal σ-polynomial of a' over E and furthermore the minimum polynomial of $\sigma^{k+1}(a')$ over $E(a', \dots, \sigma^k(a'))$ is

$$x^{i_n} - \sigma^{k-n-1}(c)$$

for $k > n$. Therefore there is valued field isomorphism between $E(\sigma^k(a) : k \in \mathbb{N})$ and $E(\sigma^k(a') : k \in \mathbb{N})$ which sends $\sigma^k(a)$ to $\sigma^k(a')$ for $k \in \mathbb{N}$. We obtain (iii) by applying σ^{-m}, for $m \in \mathbb{N}$. □

Lemma 3.5. *Let $\gamma \in \Gamma \setminus \Gamma_E$. Suppose that $\mathrm{Ann}_{\Gamma/\Delta}(\gamma) = (p, \tau)$ where p is a prime number and*

$$\tau = \sum_{k=0}^{n} i_k \sigma^k \in \mathbb{Z}[\sigma]$$

is monic. Take $\boldsymbol{j}, \boldsymbol{l} \in \mathbb{N}^{n+1}$ such that

$$j_k = i_k \text{ if } i_k \geq 0 \text{ and } j_k = 0 \text{ otherwise,}$$

$$l_k = -i_k \text{ if } i_k < 0 \text{ and } l_k = 0 \text{ otherwise,}$$

for $k = 0, \dots, n$. Let $b, c \in E$ be such that $v(b) = \tau(\gamma)$ and $v(c) = p\gamma$. Suppose that $a \in K$ is simultaneously a zero of the σ-polynomial

$$F(x) = \boldsymbol{\sigma}(x)^{\boldsymbol{j}} - \boldsymbol{\sigma}(x)^{\boldsymbol{l}} b$$

and a zero of $G(x) = x^p - c$. Then

(i) *G is the minimum polynomial of a over E,*
(ii) *$E\langle a\rangle$ has value group $\Gamma_E\langle\gamma\rangle$ and residue field $\boldsymbol{k}_E$,*
(iii) *if $a' \in K'$ is a zero of both F and G with $\mathrm{Ann}_{\Gamma'/\Delta}(\gamma') = \mathrm{Ann}_{\Gamma/\Delta}(\gamma)$ where $\gamma' = v(a')$ then there is a valued difference field isomorphism $\mathcal{E}\langle a\rangle \to \mathcal{E}\langle a'\rangle$ over $\mathcal{E}$ sending a to a'.*

Proof. Since $m\gamma \notin \Delta$ for $0 < m < p$, G is the minimum polynomial of a over E. Moreover

$$x^p - \sigma^k(c)$$

is the minimum polynomial of $\sigma^k(a)$ over $E(a, \dots, \sigma^{k-1}(a))$ for $0 < k < n$. Therefore $E(a, \dots, \sigma^{n-1}(a))$ has value group $\Delta(\gamma, \dots, \sigma^{n-1}(\gamma))$ and residue field $\boldsymbol{k}_E$.

Since τ is monic, $\sigma^n(a) \in E(a, \dots, \sigma^{n-1}(a))$ and hence $E(\sigma^k(a) : k \in \mathbb{N})$ has value group $\Delta(\sigma^k(\gamma) : k \in \mathbb{N})$ and residue field $\boldsymbol{k}_E$. Applying σ^{-m}, for $m \in \mathbb{N}$, as usual we conclude that $E\langle a\rangle$ has value group $\Delta\langle\gamma\rangle$ and residue field $\boldsymbol{k}_E$.

Let $a' \in K'$ be as in (iii). Then

$$x^p - \sigma^k(c)$$

is the minimum polynomial of $\sigma^k(a')$ over $E(a', \dots, \sigma^{k-1}(a'))$ for $0 < k < n$. So there is a valued field isomorphism between $E(a, \dots, \sigma^{n-1}(a))$ and $E(a', \dots, \sigma^{n-1}(a'))$ that sends $\sigma^k(a)$ to $\sigma^k(a')$ for $0 \le k < n$. Since $F(a') = 0$ we have an isomorphism of valued fields between $E(\sigma^k(a) : k \in \mathbb{N})$ and $E(\sigma^k(a') : k \in \mathbb{N})$ that sends $\sigma^k(a)$ to $\sigma^k(a')$ for $k \in \mathbb{N}$. Using σ^{-m}, for $m \in \mathbb{N}$, we obtain (iii). □

4. Pseudo-continuity and approximating zeroes σ-polynomials

Let $\mathcal{K}$ be a valued difference field, and consider the following condition on $\mathcal{K}$.

Axiom 1. For all $a \in K^\times$ with $v(a) > 0$, $v(\sigma(a)) > nv(a)$ for all n.

Suppose Γ is an ordered abelian group equipped with an order preserving automorphism σ such that $\sigma(\gamma) > n\gamma$ for all n whenever $\gamma > 0$. Let $\boldsymbol{k}$ be a difference field, then the Hahn difference field $\boldsymbol{k}((t^\Gamma))$ satisfies Axiom 1. If $\mathcal{K}$ satisfies Axiom 1, so does any valued difference subfield of $\mathcal{K}$, and any extension of $\mathcal{K}$ with the same value group.

From now on we assume that all our valued difference fields satisfy Axiom 1. By this convention, whenever we refer to an extension of a valued difference field, this extension is also assumed to satisfy Axiom 1. We show that the *pseudo-continuity* of polynomials over valued fields holds also for difference polynomials in this context:

Lemma 4.1. *Let $\mathcal{K}$ be a valued difference field. Let $\{a_\rho\}$ be a pc-sequence in K and G a non-constant σ-polynomial over K of order n. Assume a is a pseudo-limit of $\{a_\rho\}$ in some extension. Then $G(a_\rho) \rightsquigarrow G(a)$.*

Proof. Let $\gamma_\rho = v(a - a_\rho)$, and let $\boldsymbol{i}, \boldsymbol{j}$ range over $\mathbb{N}^{n+1}$. For each ρ we have

$$G(a) - G(a_\rho) = \sum_{\boldsymbol{i} \neq 0} G_{(\boldsymbol{i})}(a) \cdot \boldsymbol{\sigma}(a - a_\rho)^{\boldsymbol{i}}.$$

As G is non-constant, $G_{(\boldsymbol{i})}(a) \neq 0$ for some nonzero $\boldsymbol{i}$. For each nonzero $\boldsymbol{i}$ with $G_{(\boldsymbol{i})}(a) \neq 0$, consider the function

$$\gamma \mapsto v(G_{(\boldsymbol{i})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{i}}(\gamma) : \ \Delta \to \Delta$$

where Δ is the value group of an extension of K that contains a. By applying Lemma 2.2 of [1] to these functions we obtain nonzero $\boldsymbol{i}$ such that for all nonzero $\boldsymbol{j} \neq \boldsymbol{i}$,

$$v(G_{(\boldsymbol{i})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{i}}(\gamma_\rho) < v(G_{(\boldsymbol{j})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{j}}(\gamma_\rho), \quad \text{eventually, so}$$
$$v(G(a) - G(a_\rho)) = v(G_{(\boldsymbol{i})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{i}}(\gamma_\rho), \quad \text{eventually,}$$

and thus $G(a_\rho) \rightsquigarrow G(a)$. □

Let $\mathcal{K}$ be a valued difference field. Let $G(x)$ be a σ-polynomial over K of order $\le n$ and $a \in K$. Let $\boldsymbol{i}$ range over tuples $(i_0, \dots, i_n) \in \mathbb{N}^{n+1}$, and likewise with $\boldsymbol{j} = (j_0, \dots, j_n)$, $\boldsymbol{l} = (l_0, \dots, l_n)$.

Definition 4.2. *We say (G, a) is in* σ-hensel configuration *if $G \notin K$ and there is $\boldsymbol{i}$ with $|\boldsymbol{i}| = 1$ and $\gamma \in \Gamma$ such that*

(*i*) $v(G(a)) = v(G_{(\boldsymbol{i})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{i}}\gamma \le v(G_{(\boldsymbol{j})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{j}}\gamma$ *whenever* $|\boldsymbol{j}| = 1$,

(ii) $v(G_{(\boldsymbol{j})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{j}}\gamma < v(G_{(\boldsymbol{j}+\boldsymbol{l})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{j}+\boldsymbol{l}}\gamma$ *whenever* $\boldsymbol{j}, \boldsymbol{l} \neq 0$ *and* $G_{(\boldsymbol{j})} \neq 0$.

If (G, a) is in σ-hensel configuration, then $G_{(\boldsymbol{j})}(a) \neq 0$ whenever $\boldsymbol{j} \neq 0$ and $G_{(\boldsymbol{j})} \neq 0$, so $G(a) \neq 0$, and therefore γ as above satisfies

$$v(G(a)) = \min_{|\boldsymbol{j}|=1} v(G_{(\boldsymbol{j})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{j}}\gamma,$$

so is unique, and we set $\gamma(G, a) := \gamma$. If (G, a) is not in σ-hensel configuration, we set $\gamma(G, a) := \infty$.

Remark 4.3. *Suppose G is nonconstant, $G(a) \neq 0$, $v(G(a)) > 0$ and $v(G_{(\boldsymbol{i})}(a)) = 0$ for all $\boldsymbol{i} \neq 0$ with $G_{(\boldsymbol{i})} \neq 0$. Then (G, a) is in σ-hensel configuration with $\gamma(G, a) > 0$.*

Given (G, a) in σ-hensel configuration we aim to find $b \in K$ such that $v(G(b)) > v(G(a))$ and (G, b) is in σ-hensel configuration. This requires an additional assumption on the residue difference field.

Axiom 2_n. If $\alpha_0, \dots, \alpha_n \in \boldsymbol{k}$ are not all 0, then the equation

$$1 + \alpha_0 x + \cdots + \alpha_n \bar{\sigma}^n(x) = 0$$

has a solution in $\boldsymbol{k}$.

Lemma 4.4. *Suppose that $\mathcal{K}$ satisfies Axiom 2_n, and (G, a) is in σ-hensel configuration. Then there is $b \in K$ such that*

(1) $v(b - a) \geq \gamma(G, a), \quad v(G(b)) > v(G(a))$,
(2) *either $G(b) = 0$, or (G, b) is in σ-hensel configuration.*

For any such b we have $v(b - a) = \gamma(G, a)$ and $\gamma(G, b) > \gamma(G, a)$.

Proof. Let $\gamma = \gamma(G, a)$, pick $\epsilon \in K$ with $v(\epsilon) = \gamma$. Let $b = a + \epsilon u$ where $u \in K$ is to be determined later; we only impose $v(u) \geq 0$ for now. Consider

$$G(b) = G(a) + \sum_{|\boldsymbol{i}| \geq 1} G_{(\boldsymbol{i})}(a) \cdot \boldsymbol{\sigma}(b - a)^{\boldsymbol{i}}.$$

Therefore $G(b) = G(a) \cdot (1 + \sum_{|\boldsymbol{i}| \geq 1} c_{\boldsymbol{i}} \cdot \boldsymbol{\sigma}(u)^{\boldsymbol{i}})$, where

$$c_{\boldsymbol{i}} = \frac{G_{(\boldsymbol{i})}(a) \cdot \boldsymbol{\sigma}(\epsilon)^{\boldsymbol{i}}}{G(a)}.$$

From $v(\epsilon) = \gamma$ we obtain $\min_{|\boldsymbol{i}|=1} v(c_{\boldsymbol{i}}) = 0$ and $v(c_{\boldsymbol{j}}) > 0$ for $|\boldsymbol{j}| > 1$. Then imposing $v(G(b)) > v(G(a))$ forces $\bar{u}$ to be a solution of the equation

$$1 + \sum_{|\boldsymbol{i}|=1} \bar{c}_{\boldsymbol{i}} \cdot \bar{\boldsymbol{\sigma}}(x)^{\boldsymbol{i}} = 0.$$

By Axiom 2_n we can take u with this property, and then $v(u) = 0$, so $v(b - a) = \gamma(G, a)$ and $v(G(b)) > v(G(a))$.

Assume that $G(b) \neq 0$. It remains to show that then (G, b) is in σ-hensel configuration with $\gamma(G, b) > \gamma$. Let $\boldsymbol{j} \neq 0$, $G_{(\boldsymbol{j})} \neq 0$ and consider

$$G_{(\boldsymbol{j})}(b) = G_{(\boldsymbol{j})}(a) + \sum_{\boldsymbol{l} \neq 0} G_{(\boldsymbol{j})(\boldsymbol{l})}(a) \cdot \boldsymbol{\sigma}(b - a)^{\boldsymbol{l}}.$$

Note that $G_{(\boldsymbol{j})}(a) \neq 0$. Since $\mathcal{K}$ is of equal characteristic 0, $v(G_{(\boldsymbol{j})(\boldsymbol{l})}(a)) = v(G_{(\boldsymbol{j}+\boldsymbol{l})}(a))$. Therefore, for all $\boldsymbol{l} \neq 0$,

$$v\big(G_{(\boldsymbol{j})(\boldsymbol{l})}(a) \cdot \boldsymbol{\sigma}(b-a)^{\boldsymbol{l}}\big) > v(G_{(\boldsymbol{j})}(a)),$$

hence $v(G_{(\boldsymbol{j})}(b)) = v(G_{(\boldsymbol{j})}(a))$. If $|\boldsymbol{i}| = 1$, then $\theta \mapsto \boldsymbol{\sigma}^{\boldsymbol{i}}(\theta)$ is an automorphism of Γ. Since $G(b) \neq 0$, it follows that we can pick $\gamma_1 \in \Gamma$ such that

$$G(b) = \min_{|\boldsymbol{i}|=1} v(G_{(\boldsymbol{i})}(b)) + \boldsymbol{\sigma}^{\boldsymbol{i}}\gamma_1.$$

Note that $\gamma_1 > \gamma$ because $v(G(b)) > v(G(a))$ and $v(G_{(\boldsymbol{i})}(b)) = v(G_{(\boldsymbol{i})}(a))$ for $\boldsymbol{i} \neq 0$. Also for $\boldsymbol{i}, \boldsymbol{j} \neq 0$ and $\theta \in \Gamma$ with $\theta > 0$ we have $\boldsymbol{\sigma}^{\boldsymbol{i}}\theta < \boldsymbol{\sigma}^{\boldsymbol{i}+\boldsymbol{j}}\theta$. Now the inequality

$$v(G_{(\boldsymbol{i})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{i}}\gamma < v(G_{(\boldsymbol{i}+\boldsymbol{j})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{i}+\boldsymbol{j}}\gamma$$

together with $\gamma_1 > \gamma$ leads to

$$v(G_{(\boldsymbol{i})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{i}}\gamma_1 < v(G_{(\boldsymbol{i}+\boldsymbol{j})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{i}+\boldsymbol{j}}\gamma_1.$$

Hence (G, b) is in σ-hensel configuration with $\gamma_1 = \gamma(G, b)$. □

Definition 4.5. *A valued difference field $\mathcal{K}$ is σ-*henselian *if for all (G, a) in σ-hensel configuration there is $b \in K$ such that $v(b-a) = \gamma(G, a)$ and $G(b) = 0$.*

By **Axiom** 2 we mean the set $\{\text{Axiom}\,2_n : \ n = 0, 1, 2, \dots\}$. So Axiom 2 is really an axiom scheme and $\mathcal{K}$ satisfies Axiom 2 if and only if $\boldsymbol{k}$ is linear difference closed.

Lemma 4.6. *If $\mathcal{K}$ is σ-henselian, then $\mathcal{K}$ satisfies Axiom* 2*.*

Proof. Assume that $\mathcal{K}$ is σ-henselian and let $\alpha_0, \dots, \alpha_n \in \boldsymbol{k}$, not all zero. Let

$$G(x) = 1 + a_0 x + \cdots + a_n \sigma^n(x) \qquad (\text{all } a_i \in K),$$

where $a_i = 0$ if $\alpha_i = 0$, and $v(a_i) = 0$ with $\bar{a}_i = \alpha_i$ if $\alpha_i \neq 0$, for $i = 0, \dots, n$. It is easy to see that $(G, 0)$ is in σ-hensel configuration with $\gamma(G, a) = 0$. This gives $a \in K$ such that $v(a) = 0$ and $G(a) = 0$. Then $\bar{a}$ is a solution of

$$1 + \alpha_0 x + \cdots + \alpha_n \bar{\sigma}^n(x) = 0.$$

□

Every henselian valued field of residue characteristic zero has a *lifting* of the residue field, the following is an analogue of this result for σ-henselian valued difference fields with residue characteristic zero.

Theorem 4.7. *Suppose that $\mathcal{K}$ is σ-henselian. Let $K_0 \subseteq \mathcal{O}_v$ be a σ-subfield of K. Then there is a σ-subfield K_1 of K such that $K_0 \subseteq K_1 \subseteq \mathcal{O}_v$ and $\bar{K}_1 = \boldsymbol{k}$.*

Proof. Suppose that $\bar{K}_0 \neq \boldsymbol{k}$. Take $a \in \mathcal{O}_v$ such that $\bar{a} \notin \bar{K}_0$. If $v(G(a)) = 0$ for all nonzero $G(x)$ over K_0, then $K_0\langle a \rangle$ is a proper σ-field extension of K_0 contained in $\mathcal{O}_v$. Next, consider the case that $v\big(G(a)\big) > 0$ for some nonzero $G(x)$ over K_0. Pick such G of minimal complexity. So $v(H(a)) = 0$ for all nonzero $H(x)$ over K_0 of lower complexity. The remark following the definition of σ-hensel configuration shows that G is σ-henselian at a. So there is $b \in \mathcal{O}_v$ with $G(b) = 0$ and $v(a-b) = v(G(a))$, so $\bar{a} = \bar{b}$. Now, by Lemma 3.2, $K_0\langle b \rangle$ is a proper σ-field extension of K_0 contained in $\mathcal{O}_v$.

We finish the proof by Zorn's Lemma. □

Example 4.8. Consider the ordered field $\mathbb{Q}(\theta)$ where θ is a polynomial indeterminate and $\theta > \mathbb{Q}$. Let Γ be the ordered additive subgroup $\mathbb{Z}[\theta, \theta^{-1}]$ of $\mathbb{Q}(\theta)$, so for nonzero $\gamma \in \Gamma$ we have

$$\gamma = \sum_{j=-m}^{n} k_j \theta^j, \qquad \text{all } k_j \in \mathbb{Z}, \quad k_n \neq 0,$$

and then $\gamma > 0$ if and only if $k_n > 0$. Let σ be the ordered group automorphism of Γ that sends θ^i to θ^{i+1} for all $i \in \mathbb{Z}$. Then Γ equipped with σ is an ordered difference group such that $\sigma(\gamma) > n\gamma$ for all n and all $\gamma > 0$ in Γ.

Let $\boldsymbol{k}$ be a difference field of characteristic 0 which is linear difference closed. Let $\mathcal{K}$ be the Hahn difference field $(\boldsymbol{k}((t^\Gamma)), \Gamma, \boldsymbol{k}; v, \pi)$. Then $\mathcal{K}$ is a valued difference field of equal characteristic 0, satisfying Axioms 1, 2. By Corollary 5.6 of the following section, $\mathcal{K}$ is σ-henselian.

Remark 4.9. *Valued difference fields as above are considered in* [7].

Definition 4.10. *Let $\{a_\rho\}$ be a pc-sequence from K. We say that $\{a_\rho\}$ is of* σ-algebraic type *over K if there is a σ-polynomial $G(x)$ over K such that $G(a_\rho) \rightsquigarrow 0$. Otherwise we say that $\{a_\rho\}$ is of* σ-transcendental type *over K. A* minimal σ-polynomial *of $\{a_\rho\}$ over K is a σ-polynomial G over K such that*

(*i*) *$G(a_\rho) \rightsquigarrow 0$ (in particular, $G \notin K$),*
(*ii*) *$H(a_\rho) \not\rightsquigarrow 0$ for all σ-polynomials H over K of lower complexity than G.*

Lemma 4.11. *Let $\mathcal{K}$ be a valued difference field. Let $\{a_\rho\}$ from K be a pc-sequence of σ-algebraic type over K with minimal σ-polynomial $G(x)$ over K and pseudolimit a in some extension. Then, with $\gamma_\rho := v(a - a_\rho)$:*

1. *(G, a_ρ) is in σ-hensel configuration with $\gamma(G, a_\rho) = \gamma_\rho$, eventually;*
2. *if $G(a) \neq 0$, then (G, a) is in σ-hensel configuration and $\gamma(G, a) > \gamma_\rho$, eventually.*

Proof. The proof of Lemma 4.1 yields nonzero $\boldsymbol{i}$ such that for all nonzero $\boldsymbol{j} \neq \boldsymbol{i}$,

$$v\big(G(a_\rho) - G(a)\big) = v\big(G_{(\boldsymbol{i})}(a)\big) + \boldsymbol{\sigma}^{\boldsymbol{i}}\gamma_\rho < v\big(G_{(\boldsymbol{j})}(a)\big) + \boldsymbol{\sigma}^{\boldsymbol{j}}\gamma_\rho, \qquad \text{eventually.}$$

Now $G(a_\rho) \rightsquigarrow 0$ and $G(a_\rho) \rightsquigarrow G(a)$, so $v\big(G(a)\big) > v\big(G(a_\rho)\big)$ eventually, and hence for all nonzero $\boldsymbol{j} \neq \boldsymbol{i}$:

$$(*) \quad v\big(G(a_\rho)\big) = v\big(G_{(\boldsymbol{i})}(a)\big) + \boldsymbol{\sigma}^{\boldsymbol{i}}\gamma_\rho < v\big(G_{(\boldsymbol{j})}(a)\big) + \boldsymbol{\sigma}^{\boldsymbol{j}}\gamma_\rho, \quad \text{eventually.}$$

But G is a minimal σ-polynomial of $\{a_\rho\}$, so for each nonzero $\boldsymbol{j}$ we have $v(G_{(\boldsymbol{j})}(a_\rho)) = v(G_{(\boldsymbol{j})}(a))$, eventually, hence for all nonzero $\boldsymbol{j} \neq \boldsymbol{i}$,

$$(**) \quad v\big(G(a_\rho)\big) = v\big(G_{(\boldsymbol{i})}(a_\rho)\big) + \boldsymbol{\sigma}^{\boldsymbol{i}}\gamma_\rho < v\big(G_{(\boldsymbol{j})}(a_\rho)\big) + \boldsymbol{\sigma}^{\boldsymbol{j}}\gamma_\rho, \quad \text{eventually.}$$

Suppose also that for all $\boldsymbol{j}, \boldsymbol{l} \neq 0$ with $G_{(\boldsymbol{j})} \neq 0$ we have

$$(***) \qquad v(G_{(\boldsymbol{j})}(a_\rho)) < v\big(G_{(\boldsymbol{j}+\boldsymbol{l})}(a_\rho)\big) + \boldsymbol{\sigma}^{\boldsymbol{l}}\gamma_\rho, \quad \text{eventually.}$$

Then $|\boldsymbol{i}| = 1$: otherwise $\boldsymbol{i} = \boldsymbol{j} + \boldsymbol{l}$ with $|\boldsymbol{j}| = 1$ and $\boldsymbol{l} \neq 0$, so $G_{(\boldsymbol{i})} \neq 0$ yields $G_{(\boldsymbol{j})} \neq 0$, and so (**) and (***) yield a contradiction. From $|\boldsymbol{i}| = 1$ and (**) and (***) we obtain (1).

Let $\boldsymbol{j}, \boldsymbol{l} \neq 0$ with $G_{(\boldsymbol{j})} \neq 0$. It remains to prove that then (***) holds. If $G_{(\boldsymbol{j})} \in K$, then clearly (***) holds, so assume $G_{(\boldsymbol{j})} \notin K$. Lemma 4.1 and its proof, with $G_{(\boldsymbol{j})}$ in the role of G, shows that

$$v\big(G_{(\boldsymbol{j})}(a_\rho) - G_{(\boldsymbol{j})}(a)\big) \leq v\big(G_{(\boldsymbol{j})(\boldsymbol{l})}(a)\big) + \boldsymbol{\sigma}^{\boldsymbol{l}}\gamma_\rho, \quad \text{eventually.}$$

Since $v(G_{(\boldsymbol{j})(\boldsymbol{l})}(a)) = v(G_{(\boldsymbol{j}+\boldsymbol{l})}(a))$, we get,

$$v\big(G_{(\boldsymbol{j})}(a_\rho) - G_{(\boldsymbol{j})}(a)\big) \leq v\big(G_{(\boldsymbol{j}+\boldsymbol{l})}(a)\big) + \boldsymbol{\sigma}^{\boldsymbol{l}}\gamma_\rho, \quad \text{eventually.}$$

Also $v(G_{(\boldsymbol{j})}(a_\rho)) = v(G_{(\boldsymbol{j})}(a))$ and $v(G_{(\boldsymbol{j}+\boldsymbol{l})}(a_\rho)) = v(G_{(\boldsymbol{j}+\boldsymbol{l})}(a))$, eventually, so

$$v(G_{(\boldsymbol{j})}(a_\rho)) < v\big(G_{(\boldsymbol{j})}(a_\rho) - G_{(\boldsymbol{j})}(a)\big) \leq v\big(G_{(\boldsymbol{j}+\boldsymbol{l})}(a_\rho)\big) + \boldsymbol{\sigma}^{\boldsymbol{l}}\gamma_\rho, \quad \text{eventually,}$$

and so we have established (***), and thus (1).

For (2), assume $G(a) \neq 0$. Note that $G_{(\boldsymbol{i})}(a) \neq 0$. Take an extension of $\mathcal{K}$ that contains a and let Δ be the value group of this extension. For each $\boldsymbol{j}$ with $|\boldsymbol{j}| = 1$ and $G_{(\boldsymbol{j})}(a) \neq 0$, the map

$$\gamma \mapsto v(G_{(\boldsymbol{j})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{j}}\gamma : \ \Delta \to \Delta$$

is an order preserving bijection and satisfies

$$v(G(a)) > v(G_{(\boldsymbol{i})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{i}}\gamma_\rho \leq v(G_{(\boldsymbol{j})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{j}}\gamma_\rho, \qquad \text{eventually.}$$

It follows that we can take the unique $\gamma \in \Delta$ such that

$$v(G(a)) = \min_{|\boldsymbol{j}|=1} v(G_{(\boldsymbol{j})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{j}}\gamma.$$

Then $\gamma > \gamma_\rho$, eventually. Therefore, G being a minimal σ-polynomial of $\{a_\rho\}$, if $\boldsymbol{j} \neq 0$ and $G_{(\boldsymbol{j})} \neq 0$, then $v(G_{(\boldsymbol{j})}(a_\rho)) = v(G_{(\boldsymbol{j})}(a))$, eventually, and thus for each $\boldsymbol{l} \neq 0$,

$$v(G_{(\boldsymbol{j})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{j}}\gamma < v(G_{(\boldsymbol{j}+\boldsymbol{l})}(a)) + \boldsymbol{\sigma}^{\boldsymbol{j}+\boldsymbol{l}}\gamma.$$

This gives (2). □

5. Immediate Extensions

Let $\mathcal{K}$ be a valued difference field, satisfying Axiom 1 as usual. The next two lemmas are the analogues of familiar results on valued fields.

Lemma 5.1. *Let $\{a_\rho\}$ from K be pc of σ-transcendental type over K. Then $\mathcal{K}$ has an immediate extension $(K\langle a\rangle, \Gamma, \boldsymbol{k}; v_a, \pi_a)$ such that:*

1. *a is σ-transcendental over K and $a_\rho \rightsquigarrow a$;*
2. *for any extension $(K_1, \Gamma_1, \boldsymbol{k}_1; v_1, \pi_1)$ of $\mathcal{K}$ and any $b \in K_1$ with $a_\rho \rightsquigarrow b$ there is a unique embedding*

$$(K\langle a\rangle, \Gamma, \boldsymbol{k}; v_a, \pi_a) \ \longrightarrow \ (K_1, \Gamma_1, \boldsymbol{k}_1; v_1, \pi_1)$$

over $\mathcal{K}$ that sends a to b.

Proof. Let $\mathcal{K}'$ be an elementary extension of $\mathcal{K}$ containing a pseudolimit a of $\{a_\rho\}$. Let $(K\langle a\rangle, \Gamma_a, \boldsymbol{k}_a; v_a, \pi_a)$ be the valued difference field generated by a over $\mathcal{K}$. To show $\Gamma_a = \Gamma$, consider a nonconstant σ-polynomial $G(x)$ over K. Then $G(a_\rho) \rightsquigarrow G(a)$, but $\{a_\rho\}$ is of σ-transcendental type, so $G(a_\rho) \not\rightsquigarrow 0$, hence

$$v_a(G(a)) = \text{eventual value of } v(G(a_\rho)) \in \Gamma.$$

Thus $\Gamma_a = \Gamma$ and a is σ-transcendental over K. A similar argument shows that $\boldsymbol{k}_a = \boldsymbol{k}$. Let b be as in (2). Then for each nonconstant σ-polynomial $G(x)$ over K we have $G(a_\rho) \rightsquigarrow G(b)$, hence $v_a(G(a)) = v_1(G(b)) \in \Gamma$. □

Corollary 5.2. *Let a from some extension of $\mathcal{K}$ be σ-algebraic over K and let $\{a_\rho\}$ be a pc-sequence in K such that $a_\rho \rightsquigarrow a$. Then $\{a_\rho\}$ is of σ-algebraic type over K.*

Lemma 5.3. *Let $\{a_\rho\}$ from K be pc of σ-algebraic type over K, with no pseudolimit in K. Let $G(x)$ be a minimal σ-polynomial of $\{a_\rho\}$ over K. Then $\mathcal{K}$ has an immediate extension $(K\langle a\rangle, \Gamma, \boldsymbol{k}; v_a, \pi_a)$ such that*

1. *$G(a) = 0$ and $a_\rho \rightsquigarrow a$;*
2. *for any extension $(K_1, \Gamma_1, \boldsymbol{k}_1; v_1, \pi_1)$ of $\mathcal{K}$ and any $b \in K_1$ with $G(b) = 0$ and $a_\rho \rightsquigarrow b$ there is a unique embedding*

$$(K\langle a\rangle, \Gamma, \boldsymbol{k}; v_a, \pi_a) \longrightarrow (K_1, \Gamma_1, \boldsymbol{k}_1; v_1, \pi_1)$$

over $\mathcal{K}$ that sends a to b.

Proof. Let $G(x) = F(\boldsymbol{\sigma}(x))$ with $F(x_0, \dots, x_n) \in K[x_0, \dots, x_n]$, $n = \text{order}(G)$. Clearly F is irreducible in $K[x_0, \dots, x_n]$.

Consider the domain $K[\xi_0, \dots, \xi_n] := K[x_0, \dots, x_n]/(F)$ with $\xi_i := x_i + (F)$ and let $L = K(\xi_0, \dots, \xi_n)$ be its field of fractions. We extend the valuation v on K to a valuation $v : L^\times \to \Gamma$ as follows. Pick a pseudolimit e of $\{a_\rho\}$ in some extension of $(K, \Gamma, \boldsymbol{k}; v, \pi)$ whose valuation we also denote by v. Let $\phi \in L$, $\phi \neq 0$, so $\phi = f(\xi_0, \dots, \xi_n)/g(\xi_0, \dots, \xi_{n-1})$ with $f \in K[x_0, \dots, x_n]$ of lower x_n-degree than F and $g \in K[x_0, \dots, x_{n-1}]$, $g \neq 0$. We claim: $v\big(f(\boldsymbol{\sigma}(e))\big), v\big(g(\boldsymbol{\sigma}(e))\big) \in \Gamma$, and $v\big(f(\boldsymbol{\sigma}(e))\big) - v\big(g(\boldsymbol{\sigma}(e))\big)$ depends only on ϕ and not on the choice of (f, g). To see why this claim is true, suppose that also $\phi = f_1(\xi_0, \dots, \xi_n)/g_1(\xi_0, \dots, \xi_{n-1})$ with $f_1 \in K[x_0, \dots, x_n]$ of lower x_n-degree than F and $g_1 \in K[x_0, \dots, x_{n-1}]$, $g_1 \neq 0$. Then $fg_1 \equiv f_1 g \mod F$ in $K[x_0, \dots, x_n]$, and thus $fg_1 = f_1 g$ since fg_1 and $f_1 g$ have lower degree in x_n than F. To avoid some tedious case distinctions we assume that f, g, f_1, g_1 are all nonconstant. (In the other cases the arguments below need some trivial modifications.) By the minimality of G, $f(\boldsymbol{\sigma}(a_\rho)) \not\rightsquigarrow 0$, so

$$v(f(\boldsymbol{\sigma}(e))) = \text{eventual value of } v(f(\boldsymbol{\sigma}(a_\rho))),$$

in particular, $v(f(\boldsymbol{\sigma}(e))) \in \Gamma$, and likewise with g, f_1 and g_1 instead of f. The identity $fg_1 = f_1 g$ now yields

$$v\big(f(\boldsymbol{\sigma}(e))\big) - v\big(g(\boldsymbol{\sigma}(e))\big) = v\big(f_1(\boldsymbol{\sigma}(e))\big) - v\big(g_1(\boldsymbol{\sigma}(e))\big).$$

This proves the claim and allows us to define $v : L^\times \to \Gamma$ by

$$v(\phi) := v\big(f(\boldsymbol{\sigma}(e))\big) - v\big(g(\boldsymbol{\sigma}(e))\big).$$

It is routine to check that this map v is a valuation on the field L that extends the valuation v on K. Likewise one shows that (L, v) has the same residue field as (K, v).

It is clear that $K(\xi_0, \dots, \xi_{n-1})$ is purely transcendental over K of transcendence degree n. The same is true for $K(\xi_1, \dots, \xi_n)$: by the minimality of G the variable x_0 must occur in F, so ξ_0 is algebraic over $K(\xi_1, \dots, \xi_n)$. This yields an isomorphism

$$K(\xi_0, \dots, \xi_{n-1}) \xrightarrow{\sigma} K(\xi_1, \dots, \xi_n), \quad \sigma(\xi_i) = \xi_{i+1} \text{ for } 0 \le i \le n-1,$$

between subfields of L that extends σ on K. We consider these subfields as equipped with the valuation induced by that of L. Let $c \in K[\xi_0, \dots, \xi_{n-1}]$, $c \neq 0$; we claim that if $v(c) = \gamma$ then $v(\sigma(c)) = \sigma(\gamma)$. We have

$$c = h(\xi_0, \dots, \xi_{n-1}), \quad h \in K[x_0, \dots, x_{n-1}].$$

Let $h^\sigma \in K[x_0, \dots, x_{n-1}]$ be obtained by applying σ to the coefficients of h. Then $\sigma(c) = h^\sigma(\xi_1, \dots, \xi_n)$. Also $\sigma(c) = f(\xi_0, \dots, \xi_n)/g(\xi_0, \dots, \xi_{n-1})$ with $f \in$

$K[x_0,\dots,x_n]$ of lower x_n-degree than F and $g \in K[x_0,\dots,x_{n-1}]$, $g \neq 0$. Thus

$$g(x_0,\dots,x_{n-1})h^\sigma(x_1,\dots,x_n) - f(x_0,\dots,x_n) = qF$$

with $q \in K[x_0,\dots,x_n]$. Put $\alpha := v(f(\xi_0,\dots,\xi_n))$, $\beta = v(g(\xi_0,\dots,\xi_{n-1}))$ and $\gamma = v(h(\xi_0,\dots,\xi_{n-1})) = v(c)$, so $\alpha,\beta,\gamma \in \Gamma$ and $v(\sigma(c)) = \alpha - \beta$. Then eventually,

$$v\big(f(\boldsymbol{\sigma}(a_\rho))\big) = \alpha, \quad v\big(g(\boldsymbol{\sigma}(a_\rho))\big) = \beta, \quad v\big(h(\boldsymbol{\sigma}(a_\rho))\big) = \gamma$$

and also $\{v\big((qF)(\boldsymbol{\sigma}(a_\rho))\big)\}$ is either eventually strictly increasing, or eventually equal to ∞. Now, to be explicit, $h(\boldsymbol{\sigma}(a_\rho)) = h(a_\rho,\dots,\sigma^{n-1}(a_\rho))$, so $\sigma\big(h(\boldsymbol{\sigma}(a_\rho))\big) = h^\sigma(\sigma(a_\rho),\dots,\sigma^n(a_\rho))$, and thus $v\big(h^\sigma(\sigma(a_\rho),\dots,\sigma^n(a_\rho))\big) = \sigma(\gamma)$, eventually. Since $\{v\big((qF)(\boldsymbol{\sigma}(a_\rho))\big)\}$ is either eventually strictly increasing, or eventually equal to ∞, it follows that eventually

$$\beta + \sigma(\gamma) = v\left(g\big(\boldsymbol{\sigma}(a_\rho)\big)\cdot h^\sigma\big(\sigma(a_\rho),\dots,\sigma^n(a_\rho)\big)\right) = v\left(f\big(\boldsymbol{\sigma}(a_\rho)\big)\right) = \alpha,$$

so $\alpha - \beta = \sigma(\gamma)$. Thus $v(\sigma(c)) = \sigma(\gamma)$. This proves our claim. In particular it follows that σ is an isomorphism of valued fields.

Consider the inclusion diagram of valued fields (with L^h the henselisation of L):

$$\begin{array}{ccccc} & & L^h & & \\ & & \uparrow & & \\ & & L & & \\ & \nearrow & & \nwarrow & \\ K(\xi_0,\dots,\xi_{n-1}) & & & & K(\xi_1,\dots,\xi_n) \\ & \nwarrow & & \nearrow & \\ & & K & & \end{array}$$

Note that L is an algebraic immediate extension of both $K(\xi_0,\dots,\xi_{n-1})$ and $K(\xi_1,\dots,\xi_n)$, so the same is true for L^h instead of L. This gives

$$K(\xi_0,\dots,\xi_{n-1})^h = K(\xi_1,\dots,\xi_n)^h = L^h$$

(since $\mathcal{K}$ is of equal characteristic 0) where we take the henselisations inside L^h. So σ extends uniquely to an automorphism σ of the *valued* field L^h. Put $a := \xi_0$ and let $(K\langle a\rangle, \Gamma, \boldsymbol{k}; v_a, \pi_a)$ be the valued σ-subfield of L^h generated by a over K. Note that then $G(a) = 0$ and $a_\rho \rightsquigarrow a$, since $v_a(a_\rho - a) = v(a_\rho - a)$.

To verify (2), the first display in the proof shows that the valuation on L defined above does not depend on the choice of the pseudolimit e. So we can take for e an element b as in the hypothesis of (2), from which the conclusion of (2) follows. $\square$

We note the following consequence.

Corollary 5.4. *K as a valued field has a proper immediate extension if and only if $\mathcal{K}$ as a valued difference field has a proper immediate extension.*

Lemma 5.5. *Suppose $\mathcal{K}$ satisfies Axiom 2. Let $G(x)$ over K and $a \in K$ be such that (G,a) is in σ-hensel configuration. Suppose also that there is no $b \in K$ with $G(b) = 0$ and $v(a-b) \geq \gamma(G,a)$. Then there is a pc-sequence $\{a_\rho\}$ in K without pseudolimit in K such that $G(a_\rho) \rightsquigarrow 0$.*

Proof. Let $\{a_\rho\}_{\rho<\lambda}$ be a sequence in K with λ an ordinal > 0, $a_0 = a$, and

(*i*) (G, a_ρ) is in σ-hensel configuration for all $\rho < \lambda$,

(*ii*) $v(a_{\rho'} - a_\rho) = \gamma(G, a_\rho)$ whenever $\rho < \rho' < \lambda$,

(*iii*) $v(G(a_{\rho'})) > v(G(a_\rho))$ and $\gamma(G, a_{\rho'}) > \gamma(G, a_\rho)$ whenever $\rho < \rho' < \lambda$.

Suppose $\lambda = \mu + 1$ is a successor ordinal. Then Lemma 4.4 yields $a_\lambda \in K$ such that the extended sequence $\{a_\rho\}_{\rho<\lambda+1}$ has the above properties with $\lambda + 1$ instead of λ.

Suppose λ is a limit ordinal. Then $\{a_\rho\}$ is a pc-sequence and $G(a_\rho) \rightsquigarrow 0$. If $\{a_\rho\}$ has no pseudolimit in K we are done. Assume otherwise, and take a pseudolimit $a_\lambda \in K$ of $\{a_\rho\}$. The extended sequence $\{a_\rho\}_{\rho<\lambda+1}$ clearly satisfies (ii) with $\lambda + 1$ instead of λ. Since $G(a_\rho) \rightsquigarrow 0$ and $G(a_\rho) \rightsquigarrow G(a_\lambda)$,

$$v(G(a_\lambda)) > v(G(a_\rho))$$

for all $\rho < \lambda$. Let $\boldsymbol{i} \neq 0$ be such that $G_{(\boldsymbol{i})} \neq 0$. Then

$$v\big(G_{(\boldsymbol{i})}(a_\lambda)\big) = v\Big(G_{(\boldsymbol{i})}(a_\rho) + \sum_{\boldsymbol{j}\neq 0} G_{(\boldsymbol{i})(\boldsymbol{j})}(a_\rho) \cdot \boldsymbol{\sigma}(a_\lambda - a_\rho)^{\boldsymbol{j}}\Big)$$

for all $\rho < \lambda$. Since $v(G_{(\boldsymbol{i})(\boldsymbol{j})}(a_\rho)) = v(G_{(\boldsymbol{i}+\boldsymbol{j})}(a_\rho))$ for all $\boldsymbol{j} \neq 0$ and (G, a_ρ) is in σ-hensel configuration with $\gamma(G, a_\rho) = v(a_\lambda - a_\rho)$, we have

$$v\big(G_{(\boldsymbol{i})(\boldsymbol{j})}(a_\rho) \cdot \boldsymbol{\sigma}(a_\lambda - a_\rho)^{\boldsymbol{j}}\big) > v(G_{(\boldsymbol{i})}(a_\rho))$$

for all $\boldsymbol{j} \neq 0$. Therefore $v(G_{(\boldsymbol{i})}(a_\lambda)) = v(G_{(\boldsymbol{i})}(a_\rho))$ for all $\rho < \lambda$. It follows that (i) and (iii) holds with $\lambda + 1$ instead of λ.

This building process must come to an end. □

We say that $\mathcal{K}$ is *σ-algebraically maximal* if it has no proper immediate σ-algebraic extension, and we say it is *maximal* if it has no proper immediate extension. Corollary 5.2 and Lemmas 5.3 and 5.5 yield:

Corollary 5.6. (1) *$\mathcal{K}$ is σ-algebraically maximal if and only if each pc-sequence in K of σ-algebraic type over K has a pseudolimit in K;*

(2) *Suppose $\mathcal{K}$ satisfies Axiom 2. If $\mathcal{K}$ is σ-algebraically maximal, then $\mathcal{K}$ is σ-henselian.*

It is clear that $\mathcal{K}$ has σ-algebraically maximal immediate σ-algebraic extensions, and also maximal immediate extensions. Provided $\mathcal{K}$ satisfies Axiom 2, both kinds of extensions are unique up to isomorphism, but for this we need one more lemma:

Lemma 5.7. *Let $\mathcal{K}'$ be a σ-algebraically maximal extension of $\mathcal{K}$ such that $\mathcal{K}'$ satisfies Axiom 2. Let $\{a_\rho\}$ from K be a pc-sequence of σ-algebraic type over K, with no pseudolimit in K and with minimal σ-polynomial $G(x)$ over K. Then there exists $b \in K'$ such that $a_\rho \rightsquigarrow b$ and $G(b) = 0$.*

Proof. The previous corollary provides a pseudolimit $a \in K'$ of $\{a_\rho\}$. If $G(a) = 0$, we are done, so assume $G(a) \neq 0$. Then (G, a) is in σ-hensel configuration with $\gamma(G, a) > v(a - a_\rho)$ eventually, by Lemma 4.11. Since $\mathcal{K}'$ is σ-algebraically maximal and satisfies Axiom 1, it is σ-henselian. Therefore, there is $b \in K'$ such that $v'(a - b) = \gamma(G, a)$ and $G(b) = 0$. Note that $a_\rho \rightsquigarrow b$ since $\gamma(G, a) > v(a - a_\rho)$ eventually. □

Together with Lemmas 5.1 and 5.3 this yields:

Theorem 5.8. *Suppose $\mathcal{K}$ satisfies Axiom 2. Then all its maximal immediate extensions are isomorphic over $\mathcal{K}$, and all its σ-algebraically maximal immediate σ-algebraic extensions are isomorphic over $\mathcal{K}$.*

We now state minor variants of the last two results using the notion of saturation from model theory.

Lemma 5.9. *Let $\mathcal{K}'$ be a $|\Gamma|^+$-saturated σ-henselian extension of $\mathcal{K}$. Let $\{a_\rho\}$ from K be a pc-sequence of σ-algebraic type over K, with no pseudolimit in K, and with minimal σ-polynomial $G(x)$ over K. Then there exists $b \in K'$ such that $\{a_\rho\} \rightsquigarrow b$ and $G(b) = 0$.*

Proof. By the saturation assumption we have a pseudolimit $a \in K'$ of $\{a_\rho\}$. If $G(a) = 0$, we are done, so assume $G(a) \neq 0$. Then (G, a) is in σ-hensel configuration with $\gamma(G, a) > v(a - a_\rho)$ eventually, by Lemma 4.11. Since $\mathcal{K}'$ is σ-henselian there is $b \in K'$ such that $v(a - b) = \gamma(G, a)$ and $G(b) = 0$. Note that $a_\rho \rightsquigarrow b$ since $v(a - b) = \gamma(G, a) > \gamma_\rho$ eventually. □

In combination with Lemmas 5.1 and 5.3 this yields:

Corollary 5.10. *Suppose that $\mathcal{K}$ satisfies Axiom 2 and $\mathcal{K}'$ is a $|\Gamma|^+$-saturated σ-henselian extension of $\mathcal{K}$. Let $\mathcal{K}^*$ be a maximal immediate extension of $\mathcal{K}$. Then $\mathcal{K}^*$ can be embedded in $\mathcal{K}'$ over $\mathcal{K}$.*

This result plays a central role in the proof of the embedding theorem in the next section. We now present an example which illustrates why Axiom 2 can't be dropped from the assumptions of Theorem 5.8.

Example 5.11. Let $\Gamma = \mathbb{Z}[\theta, \theta^{-1}]$ be the ordered difference group introduced in Example 4.8 where $\mathbb{Z} < \theta$ and $\sigma(\theta^i) = \theta^{i+1}$ for all $i \in \mathbb{Z}$. Let $\boldsymbol{k}$ be any field of characteristic 0 construed as a difference field equipped with its identity automorphism and let $\mathcal{K}$ be the Hahn difference field $(\boldsymbol{k}((t^\Gamma), \Gamma, \boldsymbol{k}; v, \pi)$.

For each n let $\Gamma_n := \theta^{-n}\mathbb{Z}[\theta]$ and let $\mathcal{K}_n$ be the Hahn field $(\boldsymbol{k}((t^{\Gamma_n})), \Gamma_n, \boldsymbol{k}; v, \pi)$. Let

$$\mathcal{K}_\infty = \big(\bigcup_n \boldsymbol{k}((t^{\Gamma_n})), \Gamma, \boldsymbol{k}; v, \pi\big).$$

Then $\mathcal{K}_\infty$ equipped with the restriction of σ, is a valued difference subfield of $\mathcal{K}$ and σ is contractive. We have a pc-sequence $\{a_n\}$,

$$a_n = \sum_{i=1}^{n} t^{-\theta^{-i}}$$

from $\mathcal{K}_\infty$ which has no pseudo-limit in $\mathcal{K}_\infty$. For

$$G(x) = \sigma(x) - x - t^{-1}$$

$G(a_n) \rightsquigarrow 0$, and hence $\{a_n\}$ is of σ-algebraic type over $\mathcal{K}_\infty$.

Note that $\mathcal{K}_\infty$ is henselian as a valued field since it is a union of henselian valued fields, and as it is of equal characteristic 0, $\mathcal{K}_\infty$ is algebraically maximal. Therefore $G(x)$ is a minimal σ-polynomial of $\{a_n\}$ over $\mathcal{K}_\infty$. Also

$$G(a_n) + 1 \rightsquigarrow 0,$$

and so $G(x) + 1$ is a minimal σ-polynomial of a_n over $\mathcal{K}_\infty$ as well. By Lemma 5.3 there are immediate extensions $\mathcal{K}_\infty\langle a\rangle$, $\mathcal{K}_\infty\langle a'\rangle$ of $\mathcal{K}$ such that $a_n \rightsquigarrow a$, $G(a) = 0$ and $a_n \rightsquigarrow a'$, $G(a') + 1 = 0$. Let $\mathcal{L}_1$ be a σ-algebraically maximal, immediate, σ-algebraic extension of $\mathcal{K}_\infty\langle a\rangle$. Let $\mathcal{L}_2$ be a σ-algebraically maximal, immediate, σ-algebraic extension of $\mathcal{K}_\infty\langle a'\rangle$.

Now we claim that $\mathcal{L}_1$ and $\mathcal{L}_2$ are *not* isomorphic over $\mathcal{K}_\infty$. Assume towards a contradiction that they are isomorphic over $\mathcal{K}_\infty$. Then there is $b \in \mathcal{L}_1$ such that $G(b)+1=0$. Since $G(a)=0$ we have

$$\sigma(b-a)-(b-a)+1=0.$$

However, this is only possible when $v(b-a)=0$ and $\overline{b-a}$ is a solution of

$$\bar{\sigma}(x)-x+1=0,$$

contradiction.

This example is obtained by parallelling the counter-examples of Kaplansky in [8]. Here we considered a particular instance of failure of Axiom 2; namely when $\bar{\sigma}$ is the identity the above $\bar{\sigma}$-linear equation does not have a solution in $\boldsymbol{k}$. However, one can easily produce a similar construction for any nondegenerate inhomogeneous $\bar{\sigma}$-linear equation which does not have a solution in $\boldsymbol{k}$.

6. The Embedding Theorem

In this section we prove the main result of the paper, Theorem 6.2, from which we derive various model theoretic consequences for valued difference fields of equal characteristic 0. The presence of a cross-section which is compatible with σ is useful in proving this result. It can be discarded afterwards except for the relative quantifier elimination, Theorem 7.2.

Let $\mathcal{K}=(K,\Gamma,\boldsymbol{k};\ v,\pi)$ be a valued difference field. A *cross-section* on $\mathcal{K}$ is a cross-section s on $\mathcal{K}$ as valued field such that for all $\gamma\in\Gamma$ and $\tau=\sum_{i=0}^n a_i\sigma^i\in\mathbb{Z}[\sigma]$,

$$\mathrm{s}(\tau(\gamma))=\mathrm{s}(\gamma)^{\boldsymbol{i}}$$

where $\boldsymbol{i}=(i_0,\dots,i_n)$. For Hahn difference fields $\boldsymbol{k}((t^\Gamma))$ we have a cross-section given by $\mathrm{s}(\gamma)=t^\gamma$.

Lemma 6.1. *Each valued difference field $\mathcal{K}=(K,\Gamma,\boldsymbol{k};\ v,\pi)$ has an elementary extension which can be equipped with a cross-section.*

Proof. A cross-section $\mathrm{s}:\Gamma\to K$ on a valued difference field $\mathcal{K}$ corresponds to a splitting of the exact sequence of $\mathbb{Z}[\sigma]$-modules

$$1\ \to\ U(\mathcal{O})\ \to\ K^\times\ \to\ \Gamma\ \to\ 0$$

where $U(\mathcal{O})$ is the multiplicative group of units of the valuation ring $\mathcal{O}$. Therefore, a cross-section on $\mathcal{K}$ exists whenever $U(\mathcal{O})$ is pure-injective as a $\mathbb{Z}[\sigma]$-module. As $\aleph_1$-saturated $\mathbb{Z}[\sigma]$-modules are pure-injective, see [4] p.171, each $\aleph_1$-saturated elementary extension ff $\mathcal{K}$ can be expanded to a $\times$-valued difference field. □

In the rest of this section we consider 3-sorted structures

$$\mathcal{K}=\big(K,\Gamma,\boldsymbol{k};v,\pi,\mathrm{s}\big)$$

such that $(K,\Gamma,\boldsymbol{k};v,\pi)$ is a valued difference field (satisfying Axiom 1 as usual) and $\mathrm{s}:K^\times\to\boldsymbol{k}^\times$ is a cross-section on $(K,\Gamma,\boldsymbol{k};v,\pi)$. Such a structure will be called a $\times$*-valued difference field*. We let $\mathcal{O}:=\mathcal{O}_v$ denote the valuation ring and any subfield E of K is viewed as a valued subfield of $\mathcal{K}$ with valuation ring $\mathcal{O}_E:=\mathcal{O}\cap E$

A *good substructure* of $\mathcal{K}=(K,\Gamma,\boldsymbol{k};v,\pi,\mathrm{s})$ is a triple $\mathcal{E}=(E,\Gamma_\mathcal{E},\boldsymbol{k}_\mathcal{E})$ such that

(1) E is a difference subfield of K,
(2) $\Gamma_\mathcal{E}$ is an ordered difference subgroup of Γ with $v(E^\times)=\Gamma_\mathcal{E}$ and $\mathrm{s}(\Gamma_\mathcal{E})\subseteq E$,

(3) $\boldsymbol{k}_{\mathcal{E}}$ is a difference subfield of $\boldsymbol{k}$ with $\pi(\mathcal{O}_E) \subseteq \boldsymbol{k}_{\mathcal{E}}$.

For good substructures $\mathcal{E}_1 = (E_1, \Gamma_1, \boldsymbol{k}_1)$ and $\mathcal{E}_2 = (E_2, \Gamma_2, \boldsymbol{k}_2)$ of $\mathcal{K}$, we define $\mathcal{E}_1 \subseteq \mathcal{E}_2$ to mean that $E_1 \subseteq E_2$, $\Gamma_1 \subseteq \Gamma_2$, $\boldsymbol{k}_1 \subseteq \boldsymbol{k}_2$. If E is a difference subfield of K with $s(v(E^\times)) \subseteq E$ then $\big(E, v(E^\times), \pi(\mathcal{O}_E)\big)$ is a good substructure of $\mathcal{K}$.

We say that a good substructure $\mathcal{E}$ satisfies Axiom 2 if the valued difference subfield $(E, v(E^\times), \pi(\mathcal{O}_E); \dots)$ of $\mathcal{K}$ does. Likewise, we say that $\mathcal{E}$ is σ-henselian if this valued difference subfield of $\mathcal{K}$ is.

Throughout this subsection

$$\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi, \mathrm{s}), \qquad \mathcal{K}' = (K', \Gamma', \boldsymbol{k}'; v', \pi', \mathrm{s}')$$

are $\times$-valued difference fields, with valuation rings $\mathcal{O}$ and $\mathcal{O}'$, and

$$\mathcal{E} = (E, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}}), \qquad \mathcal{E}' = (E', \Gamma_{\mathcal{E}'}, \boldsymbol{k}_{\mathcal{E}'})$$

are good substructures of $\mathcal{K}$, $\mathcal{K}'$ respectively. To avoid too many accents we let σ denote the difference operator of each of K, K', E, E', and put $\mathcal{O}_{E'} := \mathcal{O}' \cap E'$.

A *good map* $\mathbf{f} : \mathcal{E} \to \mathcal{E}'$ is a triple $\mathbf{f} = (f, f_{\mathrm{val}}, f_{\mathrm{res}})$ consisting of a difference field isomorphism $f : E \to E'$, an ordered difference group isomorphism $f_{\mathrm{val}} : \Gamma_{\mathcal{E}} \to \Gamma_{\mathcal{E}'}$ and a difference field isomorphism $f_{\mathrm{res}} : \boldsymbol{k}_{\mathcal{E}} \to \boldsymbol{k}_{\mathcal{E}'}$ such that

(i) $f_{\mathrm{val}}(v(a)) = v'(f(a))$ for all $a \in E^\times$, $f(\mathrm{s}(\gamma)) = \mathrm{s}'(f_{\mathrm{val}}(\gamma))$ for all $\gamma \in \Gamma_{\mathcal{E}}$, and f_{val} is elementary as a partial map between the ordered difference groups Γ and Γ';

(ii) $f_{\mathrm{res}}(\pi(a)) = \pi'(f(a))$ for all $a \in \mathcal{O}$, and f_{res} is elementary as a partial map between the difference fields $\boldsymbol{k}$ and $\boldsymbol{k}'$.

Let $\mathbf{f} : \mathcal{E} \to \mathcal{E}'$ be a good map as above. Then the field part $f : E \to E'$ of $\mathbf{f}$ is a valued difference field isomorphism, and f_{val} and f_{res} agree on $\Gamma_{\mathcal{E}}$ and $\pi(\mathcal{O}_E)$ with the maps $\Gamma_{\mathcal{E}} \to \Gamma_{\mathcal{E}'}$ and $\pi(\mathcal{O}_E) \to \pi'(\mathcal{O}_{E'})$ induced by f. We say that a good map $\mathbf{g} = (g, g_{\mathrm{val}}, g_{\mathrm{res}}) : \mathcal{F} \to \mathcal{F}'$ *extends* $\mathbf{f}$ if $\mathcal{E} \subseteq \mathcal{F}$, $\mathcal{E}' \subseteq \mathcal{F}'$, and g, g_{val}, g_{res} extend f, f_{val}, f_{res}, respectively. The *domain* of $\mathbf{f}$ is $\mathcal{E}$.

Theorem 6.2. *Suppose* $\mathrm{char}(\boldsymbol{k}) = 0$ *and* $\mathcal{K}$, $\mathcal{K}'$ *are* σ*-henselian. Then any good map* $\mathcal{E} \to \mathcal{E}'$ *is a partial elementary map between* $\mathcal{K}$ *and* $\mathcal{K}'$.

Proof. The case $\Gamma = \{0\}$ is trivial so we assume that $\Gamma \neq \{0\}$. Let $\mathbf{f} = (f, f_{\mathrm{val}}, f_{\mathrm{res}}) : \mathcal{E} \to \mathcal{E}'$ be a good map. By passing to elementary extensions of $\mathcal{K}$ and $\mathcal{K}'$ we can arrange that $\mathcal{K}$ and $\mathcal{K}'$ are κ-saturated, where κ is an uncountable cardinal such that $|\boldsymbol{k}_{\mathcal{E}}|$, $|\Gamma_{\mathcal{E}}| < \kappa$. Call a good substructure $\mathcal{E}_1 = (E_1, \boldsymbol{k}_1, \Gamma_1)$ of $\mathcal{K}$ *small* if $|\boldsymbol{k}_1|$, $|\Gamma_1| < \kappa$. We shall prove that the good maps with small domain form a back-and-forth system between $\mathcal{K}$ and $\mathcal{K}'$ which suffices to obtain the theorem. So we need to prove that for each $a \in K$ there is a good map $\mathbf{g}$ extending $\mathbf{f}$ such that $\mathbf{g}$ has small domain $\mathcal{F} = (F, \dots)$ with $a \in F$. We do this by an appropriate iteration of Corollary 5.10 together with the following extension procedures.

(1) *Arranging* $\boldsymbol{k}_{\mathcal{E}} = \pi(\mathcal{O}_E)$. Suppose $\alpha \in \boldsymbol{k}_{\mathcal{E}}$, $\alpha \notin \pi(\mathcal{O}_E)$; set $\alpha' := f_{\mathrm{res}}(\alpha)$.

If α is $\bar{\sigma}$-transcendental over $\pi(\mathcal{O}_E)$, we pick $a \in \mathcal{O}$ and $a' \in \mathcal{O}'$ such that $\bar{a} = \alpha$ and $\bar{a'} = \alpha'$. Then Lemma 3.1 yields a good map $\mathbf{g} = (g, f_{\mathrm{val}}, f_{\mathrm{res}})$ with small domain $(E\langle a\rangle, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}})$ such that $\mathbf{g}$ extends $\mathbf{f}$ and $g(a) = a'$.

Next, assume that α is $\bar{\sigma}$-algebraic over $\pi(\mathcal{O}_E)$. Let $G(x)$ be a σ-polynomial over $\mathcal{O}_E$ such that $\bar{G}(x)$ is a minimal $\bar{\sigma}$-polynomial of α over $\pi(\mathcal{O}_E)$ and has the same complexity as $G(x)$. Pick $a \in \mathcal{O}$ such that $\bar{a} = \alpha$. Then (G, a) is in σ-hensel configuration with $\gamma(G, a) > 0$. So we have $b \in \mathcal{O}$ such that $G(b) = 0$ and $\bar{b} = \bar{a} = \alpha$.

Likewise, we obtain $b' \in \mathcal{O}'$ such that $f(G)(b') = 0$ and $\bar{b'} = \alpha'$, where $f(G)$ is the difference polynomial over E' that corresponds to G under f. By Lemma 3.2 we obtain a good map extending $\mathbf{f}$ with small domain $(E\langle b\rangle, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}})$ and sending b to b'.

By iterating (1) we can arrange $\boldsymbol{k}_{\mathcal{E}} = \pi(\mathcal{O}_E)$. This condition is preserved in each of the extension procedures (2)–(5) below. We assume in the rest of the proof that $\boldsymbol{k}_{\mathcal{E}} = \pi(\mathcal{O}_E)$, and we refer from now on to $\boldsymbol{k}_{\mathcal{E}}$ as the *residue difference field* of E.

(2) *Extending the residue field*. Let $\alpha \in \boldsymbol{k} \setminus \boldsymbol{k}_E$; take $a \in \mathcal{O}$ with $\bar{a} = \alpha$.

Using saturation we can take $\alpha' \in \boldsymbol{k}'$ and a difference field isomorphism $\boldsymbol{k}_{\mathcal{E}}\langle\alpha\rangle \cong \boldsymbol{k}_{\mathcal{E}'}\langle\alpha'\rangle$ extending f_{res} and sending α to α' that is elementary as a partial map between the difference fields $\boldsymbol{k}$ and $\boldsymbol{k}'$.

If α is $\bar{\sigma}$-transcendental over $\boldsymbol{k}_{\mathcal{E}}$, pick $a' \in \mathcal{O}'$ such that $\bar{a'} = \alpha'$. Then by Lemma 3.1 we obtain a good map $\mathbf{g} = (g, \dots)$ extending $\mathbf{f}$ with small domain $(E\langle a\rangle, \Gamma_{\mathcal{E}}, \boldsymbol{k}_E\langle\alpha\rangle)$ and $g(a) = a'$.

If α is $\bar{\sigma}$-algebraic over $\boldsymbol{k}_E$. Let $G(x)$ be a σ-polynomial over $\mathcal{O}_E$ such that $\bar{G}(x)$ is a minimal $\bar{\sigma}$-polynomial of α and has the same complexity as $G(x)$. Then (G, a) is in σ-hensel configuration with $\gamma(G, a) > 0$. So there is $b \in \mathcal{O}$ such that $G(b) = 0$ and $\bar{b} = \bar{a} = \alpha$. Likewise, there is $b' \in \mathcal{O}'$ such that $f(G)(b') = 0$ and $\bar{b'} = \alpha'$, where $f(G)$ is the difference polynomial over E' that corresponds to G under f. By Lemma 3.2 there is a good map $\mathbf{g}$ extending $\mathbf{f}$ with small domain $(E\langle b\rangle, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}}\langle\alpha\rangle)$ sending b to b'.

For the extension procedures (3)–(5), we let $\gamma \in \Gamma \setminus \Gamma_{\mathcal{E}}$ and let $\text{Ann}(\gamma)$ denote the annihilator of $\gamma \in \Gamma$ modulo $\Gamma_{\mathcal{E}}$. We shall describe extension procedures depending on $\text{Ann}(\gamma)$. In each of (3)–(5), using saturation we can pick $\gamma' \in \Gamma'$ and an ordered difference group isomorphism

$$g_{\text{val}} : \Gamma_{\mathcal{E}}\langle\gamma\rangle \to \Gamma_{\mathcal{E}'}\langle\gamma'\rangle$$

that extends f_{val}, sends γ to γ' and is elementary as a partial map between the ordered difference groups Γ and Γ'. Also let $a := \text{s}(\gamma)$, $a' := \text{s}(\gamma')$ and let $\text{Ann}(\gamma')$ denote the annihilator of γ' modulo $\Gamma_{\mathcal{E}'}$.

(3) *Extending the value group, case 1.* Assume that $\text{Ann}(\gamma) = \{0\}$.

Then $\text{Ann}(\gamma') = \{0\}$ and by Lemma 3.3 we have an isomorphism of valued difference fields

$$g : E\langle a\rangle \to E'\langle a'\rangle$$

that extends f and sends a to a'. Then $(g, g_{\text{val}}, f_{\text{res}})$ is a good map with small domain $(E\langle a\rangle, \Gamma_{\mathcal{E}}\langle\gamma\rangle, \boldsymbol{k}_{\mathcal{E}})$.

(4) *Extending the value group, case 2.* Assume that $\text{Ann}(\gamma) = (\tau)$ where $\tau = \sum_{k=0}^{n} i_k\sigma^k \in \mathbb{Z}[\sigma]$.

Note that $\text{Ann}(\gamma') = (\tau)$ as well. Let $b = \text{s}(\tau(\gamma))$, $b' = \text{s}'(\tau(\gamma'))$. Let $\boldsymbol{j}, \boldsymbol{l} \in \mathbb{N}^{n+1}$ be such that

$$j_k = i_k \text{ if } i_k \geq 0 \text{ and } j_k = 0 \text{ otherwise,}$$

$$l_k = -i_k \text{ if } i_k < 0 \text{ and } l_k = 0 \text{ otherwise,}$$

for $k = 0, \dots, n$. Then a is a zero of the σ-polynomial

$$F(x) = \sigma(x)^{\boldsymbol{j}} - \sigma(x)^{\boldsymbol{l}} b$$

and a' is a zero of the σ-polynomial

$$G(x) = \sigma(x)^{\boldsymbol{j}} - \sigma(x)^{\boldsymbol{l}} b'.$$

Hence by Lemma 3.4 we have an isomorphism of valued difference fields

$$g : E\langle a\rangle \to E'\langle a'\rangle$$

that extends f and sends a to a'. Then $(g, g_{\mathrm{val}}, f_{\mathrm{res}})$ is a good map with small domain $(E\langle a\rangle, \Gamma_{\mathcal{E}}\langle\gamma\rangle, \boldsymbol{k}_{\mathcal{E}})$.

(5) *Extending the value group, case 3.* Assume that $\mathrm{Ann}(\gamma)$ is a maximal ideal of $\mathbb{Z}[\sigma]$.

Note that $\mathrm{Ann}(\gamma') = \mathrm{Ann}(\gamma)$. Maximal ideals of $\mathbb{Z}[\sigma]$ are of the form (p, τ) where $p \in \mathbb{Z}$ is a prime number and τ is irreducible modulo $p\mathbb{Z}[\sigma]$. So we assume that $\mathrm{Ann}(\gamma) = \mathrm{Ann}(\gamma') = (p, \tau)$ where p is prime, $\tau = \sum_{i=0}^{n} a_i \sigma^i$ is monic and irreducible modulo $p\mathbb{Z}[\sigma]$.

Let $b = \mathrm{s}(p\gamma)$, $b' = \mathrm{s}'(p\gamma')$, $c = \mathrm{s}(\tau(\gamma))$, $c' = \mathrm{s}'(\tau(\gamma'))$. Let $\boldsymbol{j}, \boldsymbol{l} \in \mathbb{N}^{n+1}$ be such that

$$j_k = i_k \text{ if } i_k \geq 0 \text{ and } j_k = 0 \text{ otherwise},$$
$$l_k = -i_k \text{ if } i_k < 0 \text{ and } l_k = 0 \text{ otherwise},$$

for $k = 0, \ldots, n$. Then a is a zero of $x^p - b$, a' is a zero of $x^p - b'$. Furthermore a is a zero of the σ-polynomial

$$F(x) = \sigma(x)^{\boldsymbol{j}} - \sigma(x)^{\boldsymbol{l}} c$$

and a' is a zero of the σ-polynomial

$$G(x) = \sigma(x)^{\boldsymbol{j}} - \sigma(x)^{\boldsymbol{l}} c'.$$

By Lemma 3.5 there is an isomorphism of valued difference fields

$$g : E\langle a\rangle \to E'\langle a'\rangle$$

that extends f and sends a to a'. Then $(g, g_{\mathrm{val}}, f_{\mathrm{res}})$ is a good map with small domain $(E\langle a\rangle, \Gamma_{\mathcal{E}}\langle\gamma\rangle, \boldsymbol{k}_{\mathcal{E}})$.

Assume now that $\gamma \in \Gamma_{\mathcal{E}}$ is given. Let $\mathrm{Ann}(\gamma)$ denote the annihilator of γ modulo $\Gamma_{\mathcal{E}}$ and $a = \mathrm{s}(\gamma)$. We claim that we can extend $\mathbf{f}$ to a good map whose domain is small and contains a. Using the extension procedures of (3) and (4) we reduce to the case that $\mathrm{Ann}(\gamma)$ is non-principal ideal and is not equal to $\{0\}$. Note that if the claim is established when when $\mathrm{Ann}(\gamma)$ is a primitive ideal of $\mathbb{Z}[\sigma]$ (an ideal whose elements have no common divisors other than 1 and -1) then this fact combined with the extension procedure of (4) shows that the claim holds in general. So we assume that $\mathrm{Ann}(\gamma)$ is a non-principal, primitive ideal which is not equal to $\{0\}$. It is well-known that such ideals of $\mathbb{Z}[\sigma]$ contain a natural number, see for example [12]. In that case we can iterate the extension procedures of (4) and (5) to establish the claim.

Now let $a \in K$ be given. We want to extend $\mathbf{f}$ to a good map whose domain is small and contains a. Since K is σ-henselian it satisfies Axiom 2. By the claim above, the extension procedures (2)–(5) can be applied to reduce to the case that $\mathcal{E}$ satisfies Axiom 2 and $E\langle a\rangle$ is an immediate extension of E where both fields are equipped with the valuation induced by $\mathcal{K}$. Let $\mathcal{E}\langle a\rangle$ be the valued difference subfield of $\mathcal{K}$ that has $E\langle a\rangle$ as underlying difference field. By Corollary 5.10, $\mathcal{E}\langle a\rangle$

has a maximal immediate valued difference field extension $\mathcal{E}_1 \leq \mathcal{K}$ which is a $\times$-valued difference field. Then $\mathcal{E}_1$ is a maximal immediate extension of $\mathcal{E}$ as well. Applying Corollary 5.10 to $\mathcal{E}'$ and using Theorem 5.8, we can extend $\mathbf{f}$ to a good map with domain $\mathcal{E}_1$, which we see as a good substructure of $\mathcal{K}$ in the obvious way. It remains to note that a is in the underlying difference field of $\mathcal{E}_1$.

□

7. Completeness and Quantifier Elimination

In this section we state model theoretic consequences of Theorem 6.2. We use $\equiv$ to denote elementary equivalence. Let $\mathcal{L}$ be the 3-sorted language of valued fields and view a valued field $(K, \Gamma, \boldsymbol{k}; v, \pi)$ as an $\mathcal{L}$-structure in the obvious way with K as the *field sort*, Γ as the *value sort* and $\boldsymbol{k}$ as the *residue sort*. Adding a function symbol σ which goes from the field sort to itself gives the language $\mathcal{L}(\sigma)$ of valued difference fields. We obtain the language of $\times$-valued difference fields $\mathcal{L}(\sigma, \mathrm{s})$ by adding a function symbol s which goes from the value sort to the field sort. Throughout this section

$$\mathcal{K} = (K, \Gamma, \boldsymbol{k}; \dots), \quad \mathcal{K}' = (K', \Gamma', \boldsymbol{k}'; \dots)$$

are σ-henselian $\times$-valued difference fields with residue characteristic 0 and they are considered as $\mathcal{L}(\sigma, \mathrm{s})$-structures. It is routine to obtain the following completeness result from Theorem 6.2, see for example [1].

Theorem 7.1. *$\mathcal{K} \equiv \mathcal{K}'$ if and only if $\boldsymbol{k} \equiv \boldsymbol{k}'$ as difference fields and $\Gamma \equiv \Gamma'$ as ordered difference groups.*

By Lemma 6.1, the above result goes through for valued difference fields instead of $\times$-valued difference fields. Thus any σ-henselian valued difference field is elementarily equivalent to the Hahn difference field $\boldsymbol{k}((t^{\Gamma}))$.

Theorem 7.2. *Let T be the $\mathcal{L}(\sigma, \mathrm{s})$-theory of σ-henselian $\times$-valued difference fields of residue characteristic 0 and $\phi(\mathbf{x})$ an $\mathcal{L}(\sigma, \mathrm{s})$-formula. Then there is an $\mathcal{L}(\sigma, \mathrm{s})$-formula $\psi(\mathbf{x})$ in which all occurrences of field variables are free, such that*

$$T \vdash \phi(\mathbf{x}) \longleftrightarrow \psi(\mathbf{x}).$$

Proof. Let ψ range over $\mathcal{L}(\sigma, s)$ formulas in which all occurrences of field variables are free. For a model $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; \dots)$ of T and $a \in K^l$, $\gamma \in \Gamma^m$, $r \in \boldsymbol{k}^n$, let

$$\mathrm{rqftp}^{\mathcal{K}}(a, \gamma, r) := \{\psi : \mathcal{K} \models \psi(a, \gamma, r)\}.$$

Let $\mathcal{K}$, $\mathcal{K}'$ be models of T and suppose

$$(a, \gamma, r) \in K^l \times \Gamma^m \times \boldsymbol{k}^n, \quad (a', \gamma', r') \in K'^l \times \Gamma'^m \times \boldsymbol{k}'^n$$

are such that $\mathrm{rqftp}^{\mathcal{K}}(a, \gamma, r) = \mathrm{rqftp}^{\mathcal{K}'}(a', \gamma', r')$. It suffices to show that

$$\mathrm{tp}^{\mathcal{K}}(a, \gamma, r) = \mathrm{tp}^{\mathcal{K}'}(a', \gamma', r').$$

Let E be the $\times$-valued difference subfield of $\mathcal{K}$ generated by a and γ and consider the good substructure $\mathcal{E} = (E, v(E^{\times}), \boldsymbol{k}_{\mathcal{E}})$ where $\boldsymbol{k}_{\mathcal{E}}$ is the difference subfield of $\boldsymbol{k}$ generated by r and the residue difference field of E. Then there is a good map $\mathcal{E} \to \mathcal{E}'$ that sends a to a', r to r' and γ to γ'. Now we can apply Theorem 6.2 to obtain the result. □

Corollary 7.3. *Each subset of $\Gamma^m \times \boldsymbol{k}^n$ that is definable in $\mathcal{K}$ is a finite union of rectangles $X \times Y$ with $X \subseteq \Gamma^m$ definable in the ordered difference group Γ and $Y \subseteq \boldsymbol{k}^n$ definable in the difference field $\boldsymbol{k}$.*

Proof. By standard arguments we can reduce to the following situation: $\mathcal{K}$ is $\aleph_1$-saturated, $\mathcal{E} = (E, \Gamma_E, \boldsymbol{k}_E; \dots) \preceq \mathcal{K}$ is countable, $\gamma, \gamma' \in \Gamma^m$ have the same type over Γ_E, and $r, r' \in \boldsymbol{k}^n$ have the same type over $\boldsymbol{k}_E$. It suffices to show that then (r, γ) and (r', γ') have the same type over $\mathcal{E}$ in $\mathcal{K}$. Our assumptions imply that

$$\mathrm{rqftp}^{\mathcal{K}}(\mathrm{s}(\gamma), \gamma, r|\mathcal{E}) = \mathrm{rqftp}^{\mathcal{K}}(\mathrm{s}(\gamma'), \gamma', r'|\mathcal{E}).$$

Hence we can apply Theorem 7.2. □

8. Kaplansky Fields and the ϕ-hensel Configuration

Throughout this section we fix a prime number p and all valued fields that get mentioned are assumed to have characteristic $p > 0$.

Definition 8.1. *A* Kaplansky field *is a valued field $(K, \Gamma, \boldsymbol{k}; v, \pi)$ such that*

- *Γ is p-divisible;*
- *For all n, and $\alpha_0, \dots, \alpha_n \in \boldsymbol{k}$ with $\alpha_i \neq 0$ for some i, the equation*

  $$1 + \alpha_0 x + \alpha_1 x^p + \cdots + \alpha_n x^{p^n} = 0$$

  has a solution in $\boldsymbol{k}$.

We shall prove the following theorem of Kaplansky [8] using a formalism that relies on the connection between valued difference fields and valued fields with positive characteristic.

Theorem 8.2. *If $\mathcal{K}$ is a Kaplansky field, then $\mathcal{K}$ has, up to isomorphism over $\mathcal{K}$, a unique algebraically maximal, immediate, algebraic extension.*

The proof will be given at the end of this section. Let $\mathcal{K}$ be a valued field. We consider the polynomial ring $K[\phi^i(x) : i \in \mathbb{N}]$ where

$$x = \phi^0(x),\ \phi^1(x),\ \phi^2(x),\ \dots$$

are distinct indeterminates. An element of this ring will be called a *ϕ-polynomial* over K. We shall interpret ϕ as the Frobenius endomorphism. To be precise: for any K-algebra R and $a \in R$, "evaluation at a" is the unique K-algebra morphism,

$$F \mapsto F(a) : \ K[\phi^i(x) : i \in \mathbb{N}] \to R$$

sending $\phi^i(x)$ to a^{p^i} for all $i \in \mathbb{N}$.

Now we fix n and work in $K[\phi^i(x) : 0 \leq i \leq n]$. For an $(n+1)$-tuple $\boldsymbol{i} = (i_0, i_1, \dots, i_n) \in \mathbb{N}^{n+1}$ we set

$$\boldsymbol{\phi}(x)^{\boldsymbol{i}} := x^{i_0} \cdot \phi(x)^{i_1} \cdots \phi^n(x)^{i_n}.$$

With $t(\boldsymbol{i}) := i_0 + i_1 p + \cdots + i_n p^n$ we have

$$v(\boldsymbol{\phi}(a)^{\boldsymbol{i}}) = t(\boldsymbol{i}) v(a) \qquad \text{for all } a \in K.$$

So $\boldsymbol{\phi}(x)^{\boldsymbol{i}}$ induces the endomorphism $\gamma \mapsto t(\boldsymbol{i})\gamma$ of Γ, to be denoted by $\boldsymbol{\phi}^{\boldsymbol{i}}$; we call $t(\boldsymbol{i})$ the *slope* of $\boldsymbol{\phi}^{\boldsymbol{i}}$. If $|\boldsymbol{i}| = 1$ with $i_m = 1$, then $\boldsymbol{\phi}^{\boldsymbol{i}}\gamma = p^m\gamma$ for all $\gamma \in \Gamma$.

Let $F \in K[\phi^i(x) : 0 \leq i \leq n]$. As for difference polynomials,

$$F = f(\phi^0(x), \dots, \phi^n(x))$$

for a unique polynomial $f \in K[x_0, \dots, x_n]$. Using the Taylor expansion of f, we obtain

$$F(x+y) = \sum_{\boldsymbol{i}} F_{(\boldsymbol{i})}(x) \cdot \boldsymbol{\phi}(y)^{\boldsymbol{i}}, \quad \text{all } F_{(\boldsymbol{i})}(x) \in K[\phi^i(x) : 0 \le i \le n],$$

with distinct indeterminates x, y over K. We also have

$$(F_{(\boldsymbol{i})})_{(\boldsymbol{j})} = \begin{pmatrix} \boldsymbol{i}+\boldsymbol{j} \\ \boldsymbol{i} \end{pmatrix} F_{(\boldsymbol{i}+\boldsymbol{j})}.$$

We say that a ϕ-polynomial $F = \sum_{\boldsymbol{i}} a_{\boldsymbol{i}} \cdot \boldsymbol{\phi}(x)^{\boldsymbol{i}}$ is *small* if each component of $\boldsymbol{i}$ is less than p for all $\boldsymbol{i}$ such that $a_{\boldsymbol{i}} \neq 0$. Note that the small ϕ-polynomials over K (in x) are the elements of a K-vector subspace $K[\phi^i(x) : i \in \mathbb{N}]_{\text{small}}$ of $K[\phi^i(x) : i \in \mathbb{N}]$.

Let $i \in \mathbb{N}$, take the least n such that $i < p^{n+1}$, and let $\boldsymbol{i} = (i_0, i_1, \dots i_n)$ be the unique $(n+1)$-tuple in $\{0, \dots, p-1\}^{n+1}$ such that

$$i = \sum_{m=0}^{n} i_m p^m.$$

We have a K-linear bijection $\Psi : K[x] \to K[\phi^i(x) : \ i \in \mathbb{N}]_{\text{small}}$ given by

$$\Psi(x^i) = \boldsymbol{\phi}(x)^{\boldsymbol{i}}$$

where $\boldsymbol{i}$ depends on i as indicated above. Note that if $f \in K[x]$ and $F = \Psi(f)$, then $F(a) = f(a)$ for all $a \in K$. We shall mostly work with small ϕ-polynomials instead of ordinary polynomials using the bijection Ψ.

Let F be a small ϕ-polynomial and $\boldsymbol{j} \neq 0$. It is easy to see that $F_{(\boldsymbol{j})}$ is also a small ϕ-polynomial and if $F = \Psi(f)$, $F_{(\boldsymbol{j})} = \Psi(f_1)$ then the degree of f_1 is less than the degree of f.

Lemma 8.3. *Let F be a small ϕ-polynomial and $a \in K$. Then $v(F_{(\boldsymbol{i})(\boldsymbol{j})}(a)) = v(F_{(\boldsymbol{i}+\boldsymbol{j})}(a))$ for all $(n+1)$-tuples $\boldsymbol{i}, \boldsymbol{j}$.*

Proof. If a component of $(\boldsymbol{i}+\boldsymbol{j})$ is greater than $p-1$ then $F_{(\boldsymbol{i})(\boldsymbol{j})}(x), F_{(\boldsymbol{i}+\boldsymbol{j})}(x) = 0$ since F is small. Otherwise the result follows from

$$(F_{(\boldsymbol{i})})_{(\boldsymbol{j})} = \begin{pmatrix} \boldsymbol{i}+\boldsymbol{j} \\ \boldsymbol{i} \end{pmatrix} F_{(\boldsymbol{i}+\boldsymbol{j})}.$$

□

In the rest of this section we let $f(x) \in K[x]$ be a nonzero polynomial. Suppose that the degree of f is i and n is such that $i < p^{n+1}$. Let $F(x) = \Psi(f)$, and let $\boldsymbol{i} = (i_0, \dots, i_n), \boldsymbol{j} = (j_0, \dots, j_n), \boldsymbol{l} = (l_0, \dots, l_n)$ range over $\mathbb{N}^{n+1}$.

Lemma 8.4. *Let $\{a_\rho\}$ be a pc-sequence in K. Assume a is a pseudo-limit of $\{a_\rho\}$ in some extension and $\gamma_\rho = v(a - a_\rho)$. Then there is a nonzero $\boldsymbol{i}$ such that for all nonzero $\boldsymbol{j} \neq \boldsymbol{i}$*

$$v\big(F(a_\rho) - F(a)\big) = v(F_{(\boldsymbol{i})}(a)) + \boldsymbol{\phi}^{\boldsymbol{i}}\gamma_\rho < v(F_{\boldsymbol{j}}(a)) + \boldsymbol{\phi}^{\boldsymbol{j}}\gamma_\rho, \quad \textit{eventually},$$

and thus $F(a_\rho) \rightsquigarrow F(a)$.

Proof. For each ρ we have

$$F(a_\rho) - F(a) = \sum_{|\boldsymbol{i}|\geq 1} F_{(\boldsymbol{i})}(a)\cdot \boldsymbol{\phi}(a_\rho - a)^{\boldsymbol{i}}.$$

As F is non-constant not all $F_{(\boldsymbol{i})}(a) = 0$. For each $\boldsymbol{i}$ with $|\boldsymbol{i}| \neq 0$ and $F_{(\boldsymbol{i})}(a) \neq 0$, consider the function

$$\gamma \to v(F_{(\boldsymbol{i})}(a)) + \boldsymbol{\phi}^{\boldsymbol{i}}(\gamma) : \ \Gamma \to \Gamma.$$

By applying Lemma 2.2 from [1] to these functions we obtain nonzero $\boldsymbol{i}$ such that for all nonzero $\boldsymbol{j} \neq \boldsymbol{i}$

$$v(F_{(\boldsymbol{i})}(a)) + \boldsymbol{\phi}^{\boldsymbol{i}}(\gamma_\rho) < v(F_{(\boldsymbol{j})}(a)) + \boldsymbol{\phi}^{\boldsymbol{j}}(\gamma_\rho), \quad \text{eventually.}$$

Therefore

$$v\big(F(a_\rho) - F(a)\big) = v(F_{(\boldsymbol{i})}(a)) + \boldsymbol{\phi}^{\boldsymbol{i}}(\gamma_\rho), \quad \text{eventually.}$$

□

Definition 8.5. *Let $a \in K$. We that say (f, a) is in ϕ-*hensel configuration *if $f \notin K$ and there is $\boldsymbol{i}$ with $|\boldsymbol{i}| = 1$, $\gamma \in \Gamma$ such that*

(*i*) $v(F(a)) = v(F_{(\boldsymbol{i})}(a)) + \boldsymbol{\phi}^{\boldsymbol{i}}\gamma \leq v(F_{(\boldsymbol{j})}(a)) + \boldsymbol{\phi}^{\boldsymbol{j}}\gamma$ *whenever* $|\boldsymbol{j}| = 1$,

(*ii*) $v(F_{(\boldsymbol{j})}(a)) + \boldsymbol{\phi}^{\boldsymbol{j}}\gamma < v(F_{(\boldsymbol{j}+\boldsymbol{l})}(a)) + \boldsymbol{\phi}^{\boldsymbol{j}+\boldsymbol{l}}\gamma$ *whenever* $|\boldsymbol{j}|, |\boldsymbol{l}| \neq 0$ *and* $F_{(\boldsymbol{j})} \neq 0$.

If (f, a) is in ϕ-hensel configuration, then $F_{(\boldsymbol{j})}(a) \neq 0$ whenever $\boldsymbol{j} \neq 0$ and $F_{(\boldsymbol{j})} \neq 0$. Hence $F(a) = f(a) \neq 0$ and γ as above satisfies

$$v(F(a)) = \min_{|\boldsymbol{j}|=1} v(F_{(\boldsymbol{j})}(a)) + \boldsymbol{\phi}^{\boldsymbol{j}}\gamma,$$

so is unique and we set $\gamma(f, a) := \gamma$.

If (f, a) is not in ϕ-hensel configuration, we set $\gamma(f, a) := \infty$.

Given (f, a) in ϕ-hensel configuration we aim to find $b \in K$ such that $v(f(b)) > v(f(a))$ and (f, b) is in p-hensel configuration. This requires the assumption that $\mathcal{K}$ is a Kaplansky field.

Lemma 8.6. *Suppose that $\mathcal{K}$ is a Kaplansky field and $a \in K$. Assume (f, a) is in ϕ-hensel configuration. Then there is $b \in K$ such that*

(1) $v(b - a) \geq \gamma(f, a)$, $v(f(b)) > v(f(a))$,

(2) *either $f(b) = 0$ or (f, b) is in ϕ-hensel configuration with $\gamma(f, b) > \gamma(f, a)$.*

For any such b we have $v(b - a) = \gamma(f, a)$ and $\gamma(f, b) > \gamma(f, a)$.

Proof. Let $\gamma = \gamma(f, a)$, pick $\epsilon \in K$ with $v(\epsilon) = \gamma$. Let $b = a + \epsilon u$ where $u \in K$ is to be determined later, we only impose $v(u) \geq 0$ for now. Consider

$$F(b) = F(a) + \sum_{|\boldsymbol{i}|\geq 1} F_{(\boldsymbol{i})}(a)\cdot \boldsymbol{\phi}(b - a)^{\boldsymbol{i}}.$$

Therefore $F(b) = F(a)\cdot(1 + \sum_{|\boldsymbol{i}|\geq 1} c_{\boldsymbol{i}} \cdot \boldsymbol{\phi}(u)^{\boldsymbol{i}})$, where

$$c_{\boldsymbol{i}} = \frac{F_{(\boldsymbol{i})}(a)\cdot \boldsymbol{\phi}(\epsilon)^{\boldsymbol{i}}}{F(a)}.$$

From $v(\epsilon) = \gamma$, we obtain $\min_{|\boldsymbol{i}|=1} v(c_{\boldsymbol{i}}) = 0$, and $v(c_{\boldsymbol{j}}) > 0$ if $|\boldsymbol{j}| > 1$. Then imposing $v(F(b)) > v(F(a))$ forces $\bar{u}$ to be a solution of the equation

$$1 + \sum_{|\boldsymbol{i}|=1} \bar{c}_{\boldsymbol{i}} \cdot \bar{\boldsymbol{\phi}}(x)^{\boldsymbol{i}} = 0.$$

Since $\mathcal{K}$ is a Kaplansky field we can take u with this property, and then $v(u) = 0$, so $v(b-a) = \gamma(f,a)$ and $v(f(b)) > v(f(a))$.

Assume $f(b) \neq 0$. It remains to show that then (f,b) is in ϕ-hensel configuration with $\gamma(f,b) > \gamma$. Let $\boldsymbol{i} \neq 0$, $F_{(\boldsymbol{i})} \neq 0$ and consider

$$F_{(\boldsymbol{i})}(b) = F_{(\boldsymbol{i})}(a) + \sum_{|\boldsymbol{j}|\geq 1} F_{(\boldsymbol{i})(\boldsymbol{j})}(a) \cdot \boldsymbol{\phi}(b-a)^{\boldsymbol{j}},$$

where $F_{(\boldsymbol{i})}(a) \neq 0$. As F is a small ϕ-polynomial, $v(F_{(\boldsymbol{i})(\boldsymbol{j})}(a)) = v(F_{(\boldsymbol{i}+\boldsymbol{j})}(a))$. Therefore

$$v\big(F_{(\boldsymbol{i})(\boldsymbol{j})}(a) \cdot \boldsymbol{\phi}(b-a)^{\boldsymbol{j}}\big) > v(F_{(\boldsymbol{i})}(a))$$

whenever $|\boldsymbol{j}| \geq 1$ and hence $v(F_{(\boldsymbol{i})}(b)) = v(F_{(\boldsymbol{i})}(a))$ whenever $|\boldsymbol{i}| \neq 0$ and $F_{(\boldsymbol{i})} \neq 0$. As Γ is p-divisible, we can pick $\gamma_1 \in \Gamma$ such that $F(b) = \min_{|\boldsymbol{i}|=1} v(F_{(\boldsymbol{i})}(b)) + \boldsymbol{\phi}^{\boldsymbol{i}}\gamma_1$. Note that $\gamma_1 > \gamma$ because $v(F(b)) > v(F(a))$ and $v(F_{(\boldsymbol{i})}(b)) = v(F_{(\boldsymbol{i})}(a))$ for $\boldsymbol{i}$ with $|\boldsymbol{i}| \neq 0$. Also if $\boldsymbol{i}, \boldsymbol{j} \neq 0$ and $\theta \in \Gamma$ with $\theta > 0$ we have $\boldsymbol{\phi}^{\boldsymbol{i}}\theta < \boldsymbol{\phi}^{\boldsymbol{i}+\boldsymbol{j}}\theta$. Now the inequality

$$v(F_{(\boldsymbol{i})}(a)) + \boldsymbol{\phi}^{\boldsymbol{i}}\gamma < v(F_{(\boldsymbol{i}+\boldsymbol{j})}(a)) + \boldsymbol{\phi}^{\boldsymbol{i}+\boldsymbol{j}}\gamma$$

together with $\gamma_1 > \gamma$ leads to

$$v(F_{(\boldsymbol{i})}(a)) + \boldsymbol{\phi}^{\boldsymbol{i}}\gamma_1 < v(F_{(\boldsymbol{i}+\boldsymbol{j})}(a)) + \boldsymbol{\phi}^{\boldsymbol{i}+\boldsymbol{j}}\gamma_1.$$

Hence (f,b) is in ϕ-hensel configuration with $\gamma_1 = \gamma(f,b)$. □

Definition 8.7. *A Kaplansky field $\mathcal{K}$ is* ϕ-henselian *if for all (f,a) in ϕ-hensel configuration there is $b \in K$ such that $v(b-a) = \gamma(f,a)$ and $f(b) = 0$.*

Lemma 8.8. *Let $\{a_\rho\}$ from K be a pc-sequence of algebraic type over K with minimal polynomial $f(x)$ over K; and with pseudolimit a in some extension which has p-divisible value group. Then, with $\gamma_\rho := v(a - a_\rho)$:*

1. *(f, a_ρ) is in ϕ-hensel configuration with $\gamma(f, a_\rho) = \gamma_\rho$, eventually;*
2. *If $f(a) \neq 0$, then (f,a) is in ϕ-hensel configuration and $\gamma(f,a) > \gamma_\rho$, eventually.*

Proof. By Lemma 8.4 there is a nonzero $\boldsymbol{i}$ such that for all nonzero $\boldsymbol{j} \neq \boldsymbol{i}$,

$$v\big(F(a_\rho) - F(a)\big) = v\big(F_{(\boldsymbol{i})}(a)\big) + \boldsymbol{\phi}^{\boldsymbol{i}}\gamma_\rho < v\big(F_{(\boldsymbol{j})}(a)\big) + \boldsymbol{\phi}^{\boldsymbol{j}}\gamma_\rho, \quad \text{eventually.}$$

Now $F(a_\rho) \rightsquigarrow 0$ and $F(a_\rho) \rightsquigarrow F(a)$, so $v\big(F(a)\big) > v\big(F(a_\rho)\big)$ eventually, and hence for all nonzero $\boldsymbol{j} \neq \boldsymbol{i}$:

$$(*) \quad v\big(F(a_\rho)\big) = v\big(F_{(\boldsymbol{i})}(a)\big) + \boldsymbol{\phi}^{\boldsymbol{i}}\gamma_\rho < v\big(F_{(\boldsymbol{j})}(a)\big) + \boldsymbol{\phi}^{\boldsymbol{j}}\gamma_\rho, \quad \text{eventually.}$$

Since f is a minimal polynomial of $\{a_\rho\}$, for each nonzero $\boldsymbol{j}$ we have $v(F_{(\boldsymbol{j})}(a_\rho)) = v(F_{(\boldsymbol{j})}(a))$, eventually, hence for all nonzero $\boldsymbol{j} \neq \boldsymbol{i}$,

$$(**) \quad v\big(F(a_\rho)\big) = v\big(F_{(\boldsymbol{i})}(a_\rho)\big) + \boldsymbol{\phi}^{\boldsymbol{i}}\gamma_\rho < v\big(F_{(\boldsymbol{j})}(a_\rho)\big) + \boldsymbol{\phi}^{\boldsymbol{j}}\gamma_\rho, \quad \text{eventually.}$$

Suppose also that for all $\boldsymbol{j}, \boldsymbol{l} \neq 0$ with $F_{(\boldsymbol{j})} \neq 0$ we have

$$(***) \qquad v(F_{(\boldsymbol{j})}(a_\rho)) < v\big(F_{(\boldsymbol{j}+\boldsymbol{l})}(a_\rho)\big) + \boldsymbol{\phi}^{\boldsymbol{l}}\gamma_\rho, \quad \text{eventually.}$$

Then $|\boldsymbol{i}| = 1$: otherwise $\boldsymbol{i} = \boldsymbol{j} + \boldsymbol{l}$ with $|\boldsymbol{j}| = 1$ and $\boldsymbol{l} \neq 0$, so $F_{(\boldsymbol{i})} \neq 0$ yields $F_{(\boldsymbol{j})} \neq 0$, and so (**) and (***) lead to a contradiction. From $|\boldsymbol{i}| = 1$, (**) and (***) we obtain (1).

Let $\boldsymbol{j}, \boldsymbol{l} \neq 0$ with $F_{(\boldsymbol{j})} \neq 0$. It remains to prove that then (***) holds. If $F_{(\boldsymbol{j})} \in K$, then clearly (***) holds, so assume $F_{(\boldsymbol{j})} \notin K$. Lemma 8.4 with $F_{(\boldsymbol{j})}$ in the role of F shows that

$$v\big(F_{(\boldsymbol{j})}(a_\rho) - F_{(\boldsymbol{j})}(a)\big) \leq v\big(F_{(\boldsymbol{j})(\boldsymbol{l})}(a)\big) + \boldsymbol{\phi}^{\boldsymbol{l}}\gamma_\rho, \quad \text{eventually.}$$

Since F is a small ϕ-polynomial, $v(F_{(\boldsymbol{j})(\boldsymbol{l})}(a)) = v(F_{\boldsymbol{j}+\boldsymbol{l}}(a))$. Hence we get

$$v\big(F_{(\boldsymbol{j})}(a_\rho) - F_{(\boldsymbol{j})}(a)\big) \leq v\big(F_{(\boldsymbol{j}+\boldsymbol{l})}(a)\big) + \boldsymbol{\phi}^{\boldsymbol{l}}\gamma_\rho, \quad \text{eventually.}$$

As f is a minimal polynomial of $\{a_\rho\}$, $v(F_{(\boldsymbol{j})}(a_\rho)) = v(F_{(\boldsymbol{j})}(a))$ and $v(F_{(\boldsymbol{j}+\boldsymbol{l})}(a_\rho)) = v(F_{(\boldsymbol{j}+\boldsymbol{l})}(a))$, eventually. Therefore

$$v(F_{(\boldsymbol{j})}(a_\rho)) \leq v\big(F_{(\boldsymbol{j}+\boldsymbol{l})}(a_\rho)\big) + \boldsymbol{\phi}^{\boldsymbol{l}}\gamma_\rho \quad \text{eventually.}$$

Note that $F_{(\boldsymbol{j})}(a_\rho) \neq 0$, eventually. Since γ_ρ is strictly increasing eventually we have,

$$v(F_{(\boldsymbol{j})}(a_\rho)) < v\big(F_{(\boldsymbol{j}+\boldsymbol{l})}(a_\rho)\big) + \boldsymbol{\phi}^{\boldsymbol{l}}\gamma_\rho, \quad \text{eventually.}$$

Thus we have established (***), and hence (1).

For (2), assume $f(a) \neq 0$. Note that $F_{(\boldsymbol{i})}(a) \neq 0$. Take an extension of $\mathcal{K}$ which contains a and has p-divisible value group. Let Δ be the value group of this extension. For each $\boldsymbol{j}$ with $|\boldsymbol{j}| = 1$ and $F_{(\boldsymbol{j})}(a) \neq 0$, the map

$$\gamma \mapsto v(F_{(\boldsymbol{j})}(a)) + \boldsymbol{\phi}^{\boldsymbol{j}}\gamma : \ \Delta \to \Delta$$

is an order preserving bijection, since Δ is p-divisible. Also

$$v(F(a)) > v(F_{(\boldsymbol{i})}(a)) + \boldsymbol{\phi}^{\boldsymbol{i}}\gamma_\rho \leq v(F_{(\boldsymbol{j})}(a)) + \boldsymbol{\phi}^{\boldsymbol{j}}\gamma_\rho, \quad \text{eventually.}$$

It follows that we can take the unique $\gamma \in \Delta$ such that

$$v(F(a)) = \min_{|\boldsymbol{j}|=1} v(F_{(\boldsymbol{j})}(a)) + \boldsymbol{\phi}^{\boldsymbol{j}}\gamma.$$

Then $\gamma > \gamma_\rho$, eventually. Therefore, f being a minimal polynomial of $\{a_\rho\}$, if $\boldsymbol{j} \neq 0$ and $F_{(\boldsymbol{j})} \neq 0$, then $v(F_{(\boldsymbol{j})}(a_\rho)) = v(F_{(\boldsymbol{j})}(a))$, eventually, and thus for each $\boldsymbol{l} \neq 0$,

$$v(F_{(\boldsymbol{j})}(a)) + \boldsymbol{\phi}^{\boldsymbol{j}}\gamma < v(F_{(\boldsymbol{j}+\boldsymbol{l})}(a)) + \boldsymbol{\phi}^{\boldsymbol{j}+\boldsymbol{l}}\gamma.$$

This gives (2). □

Algebraically Maximal Kaplansky Fields.

We now fix a Kaplansky field $\mathcal{K}$. As before let $f(x) \in K[x]$ be a nonzero polynomial. Suppose that the degree of f is i and n is such that $i < p^{n+1}$. Let $F(x) = \Psi(f)$, and let $\boldsymbol{i} = (i_0, \dots, i_n), \boldsymbol{j} = (j_0, \dots, j_n)$ range over $\mathbb{N}^{n+1}$.

Lemma 8.9. *Let $a \in K$ and assume (f, a) is in ϕ-hensel configuration with $f(a) \neq 0$. Suppose also that there is no $b \in K$ with $f(b) = 0$ and $v(a - b) \geq \gamma(G, a)$. Then there is a pc-sequence $\{a_\rho\}$ from K without pseudolimit in K such that $f(a_\rho) \rightsquigarrow 0$.*

Proof. Let $\{a_\rho\}_{\rho<\lambda}$ be a sequence in K with λ an ordinal > 0, $a_0 = a$, and

(*i*) (f, a_ρ) is in ϕ-hensel configuration for all $\rho < \lambda$ and $f(a_\rho) \neq 0$,
(*ii*) $v(a_{\rho'} - a_\rho) = \gamma(f, a_\rho)$ whenever $\rho < \rho' < \lambda$,
(*iii*) $v(f(a_{\rho'})) > v(f(a_\rho))$ and $\gamma(f, a_{\rho'}) > \gamma(f, a_\rho)$ whenever $\rho < \rho' < \lambda$.

Suppose $\lambda = \mu + 1$ is a successor ordinal. Then Lemma 8.6 yields $a_\lambda \in K$ such that the extended sequence $\{a_\rho\}_{\rho<\lambda+1}$ has the above properties with $\lambda + 1$ instead of λ.

Suppose λ is a limit ordinal. Then $\{a_\rho\}$ is a pc-sequence and $f(a_\rho) \rightsquigarrow 0$. If $\{a_\rho\}$ has no pseudolimit in K we are done. Assume otherwise, and take a pseudolimit $a_\lambda \in K$ of $\{a_\rho\}$. The extended sequence $\{a_\rho\}_{\rho<\lambda+1}$ clearly satisfies (ii) with $\lambda + 1$ instead of λ. Since $f(a_\rho) \rightsquigarrow 0$ and $f(a_\rho) \rightsquigarrow f(a_\lambda)$,

$$v(f(a_\lambda)) > v(f(a_\rho))$$

for all $\rho < \lambda$. Let $\boldsymbol{i}$ be such that $|\boldsymbol{i}| \neq 0$ and $F_{(\boldsymbol{i})} \neq 0$.

$$v\big(F_{(\boldsymbol{i})}(a_\lambda)\big) = v\big(F_{(\boldsymbol{i})}(a_\rho) + \sum_{|\boldsymbol{j}|\neq 0} F_{(\boldsymbol{i})(\boldsymbol{j})}(a_\rho) \cdot \boldsymbol{\phi}(a_\lambda - a_\rho)^{\boldsymbol{j}}\big)$$

for all $\rho < \lambda$. Since F is a small ϕ-polynomial, $v(F_{(\boldsymbol{i})(\boldsymbol{j})}(a_\rho)) = v(F_{(\boldsymbol{i}+\boldsymbol{j})}(a_\rho))$ for all $\boldsymbol{j}$. By Lemma 8.8, (f, a_ρ) is in ϕ-hensel configuration with $\gamma(f, a_\rho) = v(a_\lambda - a_\rho)$, so we have

$$v\big(F_{(\boldsymbol{i})(\boldsymbol{j})}(a_\rho) \cdot \boldsymbol{\phi}(a_\lambda - a_\rho)^{\boldsymbol{j}}\big) > v(F_{(\boldsymbol{i})}(a_\rho))$$

whenever $|\boldsymbol{j}| \neq 0$. Therefore $v(F_{(\boldsymbol{i})}(a_\lambda)) = v(F_{(\boldsymbol{i})}(a_\rho))$ for all $\rho < \lambda$. This proves the extended sequence $\{a_\rho\}_{\rho<\lambda+1}$ satisfies conditions (i) and (iii) as well.

This building process must come to an end. □

Corollary 8.10. *If $\mathcal{K}$ is algebraically maximal, then $\mathcal{K}$ is ϕ-henselian.*

Lemma 8.11. *Suppose $\mathcal{K}'$ is a Kaplansky field and an algebraically maximal extension of $\mathcal{K}$. Let $\{a_\rho\}$ from K be pc-sequence of algebraic type over K, with no pseudolimit in K, and with minimal polynomial $f(x)$ over K. Then there exists $b \in K'$ such that $a_\rho \rightsquigarrow b$ and $F(b) = 0$.*

Proof. The previous corollary provides a pseudolimit $a \in K'$ of $\{a_\rho\}$. Since $\mathcal{K}'$ is algebraically maximal Kaplansky field, it is ϕ-henselian. Therefore, if $f(a) \neq 0$, there is $b \in K'$ such that $v'(a - b) = \gamma(f, a)$ and $f(b) = 0$. Note that $a_\rho \rightsquigarrow b$ since $\gamma(f, a) > v(a - a_\rho) = \gamma_\rho$ eventually. □

This yields Theorem 8.2 stated in the beginning of this section.

Theorem 8.12. *If $\mathcal{K}$ is a Kaplansky field, then $\mathcal{K}$ is algebraically maximal if and only if it is ϕ-henselian.*

Proof. Suppose that $\mathcal{K}$ is a ϕ-henselian Kaplansky field. Assume towards a contradiction that $\mathcal{K}$ is not algebraically maximal. Then there is a pc-sequence of algebraic type $\{a_\rho\}$ in K, which has no pseudo-limit in K. Let $f(x) \in K[x]$ be a minimal polynomial of $\{a_\rho\}$ and $\gamma_\rho = v(a_{\rho+1} - a_\rho)$. By Lemma 8.8, eventually (f, a_ρ) is in ϕ-hensel configuration with $\gamma(f, a_\rho) = \gamma_\rho$. Since $\mathcal{K}$ is p-henselian, eventually for each ρ there is $b_\rho \in K$ such that $f(b_\rho) = 0$ and $v(b_\rho - a_\rho) = \gamma_\rho$. As $f(x) \in K[x]$ has finitely many zeroes in K, there is $b \in K$ such that $f(b) = 0$ and $v(b - a_\rho) = \gamma_\rho$ eventually, contradicting the assumption that $\{a_\rho\}$ has no pseudo-limit in K. Hence $\mathcal{K}$ is algebraically maximal.

Together with Corollary 8.10 we obtain the result. □

Corollary 8.13. *If $\mathcal{K}$ is a p-henselian Kaplansky field, then $\mathcal{K}$ is henselian.*

Proof. Follows from the previous theorem together with the fact that algebraically maximal valued fields are henselian. □

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