Geometry and Categoricity

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Zilber's Thesis

Fundamental structures are canonical

Fundamental mathematical structures can be characterized in an appropriate logic.

Conversely, characterizable structures are 'fundamental'.

The relevant notion of 'characterize' is categoricity in power.

Specifying the thesis I

Find an axiomatization for $\mathrm{Th}(\mathcal{C}, +, \cdot, \exp)$.

Specifying the thesis II

Zilber Conjecture

Every strongly minimal first order theory is

1. disintegrated
2. group-like
3. field-like

Specifying the thesis III

Cherlin-Zilber Conjecture

Every simple ω-stable group is an algebraic group over an algebraically closed field.

This led to:
Is there an ω-stable field of finite Morley rank with a definable proper subgroup of the multiplicative group?

Challenge

Hrushovski Construction I

There is a strongly minimal set which is not locally modular and not field-like. (Hrushovski)

Hrushovski Construction II

There is an ω-stable field of finite Morley rank with a definable proper subgroup of the multiplicative group. (Baudisch, Hils, Martin-Pizarro, Wagner)

Response

Response I

Strengthen the hypotheses:
Extend first order to more powerful "Logics".

1. $L_{\omega_1,\omega}(Q)$
2. Zariski Structures

Response II

Weaken the conclusion:
Replace first order interpetable in $(\mathcal{C}, +, \cdot)$ by 'analytically' definable.

Response

COMPLEX EXPONENTIATION

Consider the structure $(C, +, \cdot, e^x, 0, 1)$.

It is Godelian:

The integers are defined as $\{a : e^a = 1\}$.
The first order theory is undecidable and 'wild'.

ZILBER'S INSIGHT

Maybe Z is the source of all the difficulty. Fix Z by adding the axiom:

$$(\forall x) e^x = 1 \rightarrow \bigvee_{n \in Z} x = 2n\pi .$$

GEOMETRIES

Definition

A closure system is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$

(G, cl) is pregeometry if in addition:
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If points are closed the structure is called a geometry.

STRONGLY MINIMAL I

M is *strongly minimal* if every first order definable subset of any elementary extension M' of M is finite or cofinite.

$a \in \mathrm{acl}(B)$ if for some $\mathbf{b} \in B$ and some $\phi(x, \mathbf{y})$: $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only **finitely many** solutions.

Exercise: If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from $\mathrm{acl}(X)$ to $\mathrm{acl}(Y)$.

STRONGLY MINIMAL II

A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T extends to an isomorphism of the models

QUASIMINIMALITY I

Trial Definition M is *'quasiminimal'* if every first order ($L_{\omega_1,\omega}$?) definable subset of M is countable or cocountable.

$a \in \mathrm{acl}'(X)$ if there is a first order formula with **countably many** solutions over X which is satisfied by a.

Zilber

If M is quasiminimal then (M, acl') is a closure system.

Pillay (Itai, Tsuboi, Wakai) characterizes when these closure systems are geometries in terms of symmetry properties of the generic type.

QUASIMINIMALITY II

Exercise ? If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from $\mathrm{acl}'(X)$ to $\mathrm{acl}'(Y)$.

QUASIMINIMALITY II

Exercise ? If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from $\mathrm{acl}'(X)$ to $\mathrm{acl}'(Y)$.

How is the geometry connected to the isomorphism type of M?

Geometric Homogeneity

An infinite dimensional pregeometry (M, cl) is
Geometrically Homogenous if
for each finite $B \subset M$, $\{a \in M : a \notin \mathrm{cl}(B)\}$ are the realizations in M of a unique complete type in $S(B)$.

QUASIMINIMAL EXCELLENCE I

Basic Conditions

Let **K** be a class of L-structures such that $M \in \mathbf{K}$ admits a closure relation cl_M mapping $X \subseteq M$ to $\mathrm{cl}_M(X) \subseteq M$ that satisfies the following properties.

1. Each cl_M defines a pregeometry on M.
2. For each $X \subseteq M$, $\mathrm{cl}_M(X) \in \mathbf{K}$.
3. If f is a partial monomorphism from $H \in \mathbf{K}$ to $H' \in \mathbf{K}$ taking $X \cup \{y\}$ to $X' \cup \{y'\}$ then $y \in \mathrm{cl}_H(X)$ iff $y' \in \mathrm{cl}_{H'}(X')$.

Homogeneity over models

$\aleph_0$-homogeneity over models

Let $G \subseteq H, H' \in \mathbf{K}$ with G empty or a countable member of $\mathbf{K}$ that is closed in H, H'.

1. If f is a partial G-monomorphism from H to H' with finite domain X then for any $y \in \mathrm{cl}_H(X)$ there is $y' \in H'$ such that $f \cup \{\langle y, y' \rangle\}$ extends f to a partial G-monomorphism.
2. If f is a bijection between $X \subset H \in \mathbf{K}$ and $X' \subset H' \in \mathbf{K}$ which are *separately* cl-independent (over G) subsets of H and H' then f is a G-partial monomorphism.

QUASIMINIMAL EXCELLENCE II

A class $(\mathbf{K}, \mathrm{cl})$ is *quasiminimal excellent* if it admits a combinatorial geometry which satisfies on each $M \in \mathbf{K}$

1. Basic Conditions
2. $\aleph_0$-homogeneity over countable models.
3. countable closure property (ccp)
4. and the 'excellence condition' which follows.

Towards Categoricity and Existence

Beginning the induction

If $(\mathbf{K}, \mathrm{cl})$ satisfies the basic conditions and $\aleph_0$-homogeneity over countable models then

1. For any finite set $X \subset M$, if $a, b \in M - \mathrm{cl}_M(X)$, a, b realize the same $L_{\omega_1,\omega}$-type over X.
2. M is quasiminimal (for $L_{\omega_1,\omega}$) and (M, cl) is geometrically homogeneous.
3. There is one and only one model M in $\aleph_1$.
4. Whence (by Shelah) there is an $L_{\omega_1,\omega}(Q)$-extension in $\aleph_2$.

Further Assumptions

To guarantee existence and uniqueness in larger cardinals we use ccp and excellence.

Notation for Excellence

In the following definition it is essential that $\subset$ be understood as *proper* subset.

Definition

1. For any Y, $\mathrm{cl}^{-}(Y) = \bigcup_{X \subset Y} \mathrm{cl}(X)$.
2. We call C (the union of) *an n-dimensional* cl*-independent system* if $C = \mathrm{cl}^{-}(Z)$ and Z is an independent set of cardinality n.

Elements of Excellence

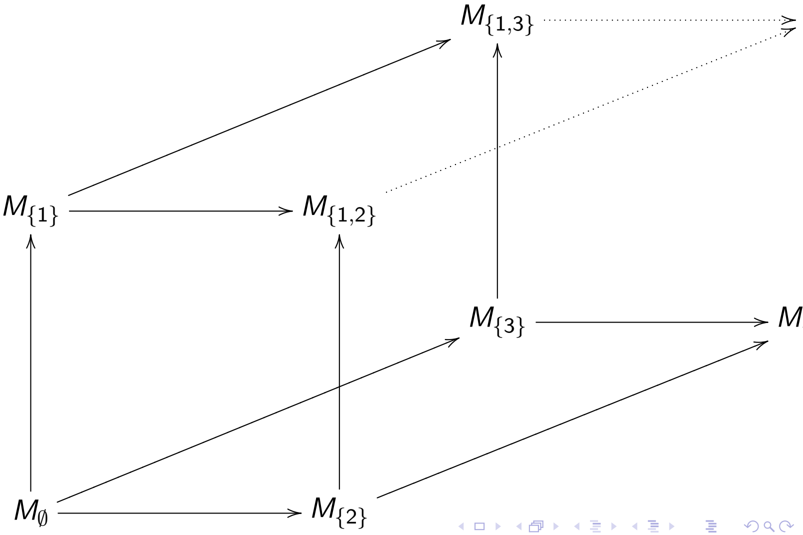
There is a primary model over any finite independent system.

Let $C \subseteq H \in \mathbf{K}$ and let X be a finite subset of H. We say $\mathrm{tp}_{\mathrm{qf}}(X/C)$ is *defined over* the finite C_0 contained in C if it is determined by its restriction to C_0.

Density of 'isolated' types

Let $G \subseteq H, H' \in \mathbf{K}$ with G empty or in $\mathbf{K}$. Suppose $Z \subset H - G$ is an n-dimensional independent system, $C = \mathrm{cl}^{-}(Z)$, and X is a finite subset of $\mathrm{cl}(Z)$. Then there is a finite C_0 contained in C such that $\mathrm{tp}_{\mathrm{qf}}(X/C)$ is defined over C_0.

n-AMALGAMATION



EXCELLENCE IMPLIES CATEGORICITY

Excellence implies by a direct limit argument:

Lemma

An isomorphism between independent X and Y extends to an isomorphism of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$.

This gives categoricity in all uncountable powers if the closure of each finite set is countable.

EXCELLENCE IMPLIES CATEGORICITY: Sketch

Lemma

Suppose $H, H' \in \mathbf{K}$ satisfy the countable closure property. Let $\mathcal{A}, \mathcal{A}'$ be cl-independent subsets of H, H' with $\mathrm{cl}_H(\mathcal{A}) = H$, $\mathrm{cl}_{H'}(\mathcal{A}') = H'$, respectively, and ψ a bijection between $\mathcal{A}$ and $\mathcal{A}'$. Then ψ extends to an isomorphism of H and H'.

Outline

We have the obvious directed union $\{\mathrm{cl}(X) : X \subseteq \mathcal{A}; |X| < \aleph_0\}$ with respect to the partial order of finite subsets of X by inclusion.

And $H = \bigcup_{X \subset \mathcal{A}; |X| < \aleph_0} \mathrm{cl}(X)$. So the theorem follows immediately if

for each finite $X \subset \mathcal{A}$ we can choose $\psi_X : \mathrm{cl}_H(X) \to H'$ so that $X \subset Y$ implies $\psi_X \subset \psi_Y$.

We prove this by induction on $|X|$. If $|X| = 1$, the condition is immediate from $\aleph_0$-homogeneity and the countable closure property. Suppose $|Y| = n+1$ and we have appropriate ψ_X for $|X| < n+1$. We will prove two statements.

1. $\psi_Y^- : \mathrm{cl}^-(Y) \to H'$ defined by $\psi_Y^- = \bigcup_{X \subset Y} \psi_X$ is a monomorphism.
2. ψ_Y^- extends to ψ_Y defined on $\mathrm{cl}(Y)$.

1) is immediate from excellence; 2 requires an argument.

CATEGORICITY

Theorem Suppose the quasiminimal excellent (I-IV) class **K** is axiomatized by a sentence Σ of $L_{\omega_1,\omega}$, and the relations $y \in \mathrm{cl}(x_1, \ldots x_n)$ are $L_{\omega_1,\omega}$-definable.

Then, for any infinite κ there is a unique structure in **K** of cardinality κ which satisfies the countable closure property.

Note on Proof

The proof of existence of large models is inductive using categoricity in κ to obtain a model in κ^+.

Excellence for $L_{\omega_1,\omega}$

Any κ-categorical sentence of $L_{\omega_1,\omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory. ($EC(T, Atomic)$-class)

Shelah defined a notion of excellence; Zilber's is the 'rank one' case for $L_{\omega_1,\omega}$.
Zilber shows sufficiency for certain $L_{\omega_1,\omega}(Q)$-sentences.

ω-stabilty

$(\mathbf{K}, \prec_{\mathbf{K}})$ is the class of atomic models of a first order theory under elementary submodel.

Definitions

$p \in S_{at}(A)$ if $a \models p$ implies Aa is atomic.

$\mathbf{K}$ is ω-stable if for every countable model M, $S_{at}(M)$ is countable.

Theorem

[Keisler/Shelah]
$(2^{\aleph_0} < 2^{\aleph_0})$ If $\mathbf{K}$ has $< 2^{\aleph_1}$ models of cardinality $\aleph_1$, then $\mathbf{K}$ is ω-stable.

Goodness

Definition

A set A is *good* if the isolated types are dense in $S_{at}(A)$.

For countable A, this is the same as $|S_{at}(A)| = \aleph_0$.

Are there prime models over good sets?

Goodness

Definition

A set A is *good* if the isolated types are dense in $S_{at}(A)$.

For countable A, this is the same as $|S_{at}(A)| = \aleph_0$.

Are there prime models over good sets?
YES, In $\aleph_0$ and $\aleph_1$
but not generally above $\aleph_1$ (J. Knight, Kueker, Laskowski-Shelah).

Yes, if excellent.

Excellence

Independence is now defined in terms of splitting.

Definition

1. **K** is (λ, n)*-good* if for any independent n-system $\mathcal{S}$ (of models of size λ), the union of the nodes is good.

   That is, there is a prime model over any countable independent n-system.

2. **K** is *excellent* if it is $(\aleph_0, n)$-good for every $n < \omega$.

Essence of Excellence

Let **K** be the class of models of a sentence of $L_{\omega_1,\omega}$.

K is excellent

K is ω-stable and any of the following equivalent conditions hold.

For any finite independent system of countable models with union C:

1. $S_{at}(C)$ is countable.
2. There is a unique primary model over C.
3. The isolated types are dense in $S_{at}(C)$.

Excellence implies large models

Shelah proved:

Theorem

Let λ be infinite and $n < \omega$. Suppose $\mathbf{K}$ has $(< \lambda, \leq n+1)$-existence and is $(\aleph_0, n)$-good. Then $\mathbf{K}$ has (λ, n)-existence.

Excellence implies large models

Shelah proved:

Theorem

Let λ be infinite and $n < \omega$. Suppose **K** has $(<\lambda, \leq n+1)$-existence and is $(\aleph_0, n)$-good. Then **K** has (λ, n)-existence.

This yields:

Theorem (ZFC)

If an atomic class **K** is excellent and has an uncountable model then

1. it has models of arbitrarily large cardinality;
2. if it is categorical in one uncountable power it is categorical in all uncountable powers.

Enough categoricity implies Excellence

VWGCH: $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$.

Enough categoricity implies Excellence

VWGCH: $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$.

VWGCH: Shelah 1983

An atomic class **K** that has at least one uncountable model and at most one model in $\aleph_n$ for each $n < \omega$ is excellent.

Show by induction:

Very few models in $\aleph_n$ implies $(\aleph_0, n-2)$-goodness.

Transition

I have discussed the general theory of categoricity for infinitary classes.

We now consider some examples.

Categoricity up to $\aleph_\omega$ is essential.

Theorem. [Hart-Shelah / Baldwin-Kolesnikov] For each $3 \leq k < \omega$ there is an $L_{\omega_1,\omega}$ sentence ϕ_k such that:

1. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
2. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$.

Categoricity up to $\aleph_\omega$ is essential.

Theorem. [Hart-Shelah / Baldwin-Kolesnikov] For each $3 \leq k < \omega$ there is an $L_{\omega_1,\omega}$ sentence ϕ_k such that:

1. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
2. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$.
3. ϕ_k has the disjoint amalgamation property;
4. Syntactic types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
5. But there are syntactic types over models of size $\aleph_{k-3}$ that split into $2^{\aleph_{k-3}}$-Galois types.
6. ϕ_k is not $\aleph_{k-2}$-Galois stable;
7. But for $m \leq k-3$, ϕ_k is $\aleph_m$-Galois stable;

$\mathbb{Z}$-Covers of Algebraic Groups

Definition A $\mathbb{Z}$-cover of a commutative algebraic group $\mathbb{A}(\mathcal{C})$ is a short exact sequence

$$0 \to Z^N \to V \stackrel{\exp}{\to} \mathbb{A}(\mathcal{C}) \to 1. \quad (1)$$

where V is a $\mathbb{Q}$ vector space and $\mathbb{A}$ is an algebraic group, defined over k_0 with the full structure imposed by $(\mathcal{C}, +, \cdot)$ and so interdefinable with the field.

Axiomatizing $\mathbb{Z}$-Covers: first order

Let $\mathbb{A}$ be a commutative algebraic group over an algebraically closed field F.

Let T_A be the first order theory asserting:

1. $(V, +, f_q)_{q \in \mathbb{Q}}$ is a $\mathbb{Q}$-vector space.
2. The complete first order theory of $\mathbb{A}(F)$ in a language with a symbol for each k_0-definable variety (where k_0 is the field of definition of $\mathbb{A}$).
3. exp is a group homomorphism from $(V, +)$ to $(\mathbb{A}(F), \cdot)$.

Axiomatizing Covers: $L_{\omega_1,\omega}$

Add to T_A
$\Lambda = \mathbb{Z}^N$ asserting the kernel of exp is standard.

$$(\exists \mathbf{x} \in (\exp^{-1}(1))^N)(\forall y)[\exp(y) = 1 \to \bigvee_{\mathbf{m} \in \mathbb{Z}^N} \Sigma_{i<N} m_i x_i = y]$$

Categoricity Problem

Is $T_A + \Lambda = \mathbb{Z}^N$ categorical in uncountable powers?

paraphrasing Zilber:

> *Categoricity would mean the short exact sequence is a reasonable 'algebraic' substitute for the classical complex universal cover.*

What is known?

If $\mathbb{A}$ is

$\mathbb{Z}$-covers

1. $(\mathcal{C}, \cdot)$ then quasiminimal excellent (Zilber)
2. $(\tilde{F}_p, \cdot)$ then not small.
   Each completion is quasiminimal excellent. (Bays-Zilber)
3. elliptic curve w/o cm then ω-stable (Gavrilovich/Bays)
4. elliptic curve w cm then not ω-stable.

Geometry from Model Theory

Zilber has shown *equivalence* between certain 'arithmetic' statements about Abelian varieties (algebraic translations of excellence) and categoricity below $\aleph_\omega$ of the associated $L_{\omega_1,\omega}$-sentence.

The equivalence depends on weak extensions of set theory and Shelah's categoricity transfer theorem.

Hrushovski Construction

The dimension function

$$d : \{X : X \subseteq_{fin} G\} \rightarrow \mathbf{N}$$

satisfies the axioms:

D1. $d(XY) + d(X \cap Y) \leq d(X) + d(Y)$

D2. $X \subseteq Y \Rightarrow d(X) \leq d(Y)$.

THE GEOMETRY

Definition

For A, b contained M, $b \in \mathrm{cl}(A)$ if $d_M(bA) = d_M(A)$.

Naturally we can extend to closures of infinite sets by imposing finite character. If d satsfies:

D3 $d(X) \leq |X|$.

we get a full combinatorial (pre)-geometry with exchange.

ZILBER'S PROGRAM FOR $(\mathcal{C}, +, \cdot, \exp)$

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}$-sentence discovered by the Hrushovski construction.

Objective A

Expand $(\mathcal{C}, +, \cdot)$ by a unary function f which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1,\omega}$-sentence Σ satisfied by $(\mathcal{C}, +, \cdot, f)$ is categorical and has quantifier elimination.

Objective B

Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

PSEUDO-EXPONENTIAL

E is a pseudo-exponential if for any n linearly independent elements over $\mathbb{Q}$, $\{z_1, \ldots z_n\}$

$$d_f(z_1, \ldots z_n, E(z_1), \ldots E(z_n)) \geq n.$$

Schanuel conjectured that true exponentiation satisfies this equation.

THE AXIOMS

$L = \{+, \cdot, E, 0, 1\}$

$(K, +, \cdot, E) \models \Sigma$ if

1. K is an algebraically closed field of characteristic 0.
2. E is a homomorphism from $(K, +)$ onto $(K^{x}, \cdot)$ and there is $\nu \in K$ transcendental over $\mathbb{Q}$ with $\ker E = \nu Z$.
3. E is a pseudo-exponential
4. K is strongly exponentially algebraically closed.

CONSISTENCY AND CATEGORICITY

For a finite subset X of an algebraically closed field k with a partial exponential function. Let

$$\delta(X) = d_f(X \cup E(X)) - ld(X).$$

Apply the Hrushovski construction to the collection of such (k, f) with
$\delta(X) \geq 0$ for all finite X
and with standard kernel.

The result is a quasiminimal excellent class.

ALGEBRA FOR OBJECTIVE A:

Definition

A *multiplicatively closed divisible subgroup* associated with $a \in \mathcal{C}^*$, is a **choice** of a multiplicative subgroup isomorphic to $\mathbb{Q}$ containing a .

ALGEBRA FOR OBJECTIVE A:

Definition

$b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$, *determine the isomorphism type* of $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ over F if given subgroups of the form $c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ and ϕ_m such that

$$\phi_m : F(b_1^{\frac{1}{m}} \ldots b_\ell^{\frac{1}{m}}) \to F(c_1^{\frac{1}{m}} \ldots c_\ell^{\frac{1}{m}})$$

is a field isomorphism it extends to

$$\phi_\infty : F(b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}}) \to F(c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}}).$$

Thumbtack Lemma

Theorem (thumbtack lemma)

For any $b_1, \ldots b_\ell \subset \mathcal{C}^*$*, there exists an* m *such that* $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$, determine the isomorphism type *of* $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ *over* F.

TOWARDS EXISTENTIAL CLOSURE

The Thumbtack Lemma (over finite independent systems of fields) implies
the consistency of the strong exponential existential closure axioms with the rest of Σ.
These existential closure axioms imply the models of Σ satisfy:

1. the homogeneity conditions and
2. excellence.

GENUINE EXPONENTIATION?

Schanuel's conjecture: If $x_1, \ldots x_n$ are $\mathbb{Q}$-linearly independent complex numbers then $x_1, \ldots x_n, e^{x_1}, \ldots e^{x_n}$ has transcendence degree at least n over $\mathbb{Q}$.

Theorem (Zilber)

$(\mathcal{C}, +, \cdot, \exp) \in \mathcal{E}C^*_{st}$. $(\mathcal{C}, +, \cdot, \exp)$ has the countable closure property.

assuming Schanuel

Marker extended by Günaydin and Martin-Pizarro have verified existential closure axioms for $(\mathcal{C}, , \cdot, \exp)$ for irreducible polynomials $p(X, Y) \in \mathcal{C}[X, Y]$.

Axiomatizability/Characterizability

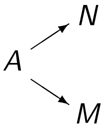
Quasiminimal Excellence is defined semantically.

But

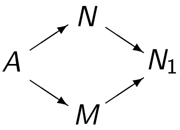
1. (Kirby) The class of models of a quasiminimal excellent class is axiomatizable in $L_{\omega_1,\omega}(Q)$.
2. Note the universal covers are axiomatized in $L_{\omega_1,\omega}$.

AMALGAMATION PROPERTY

The class **K** satisfies the *amalgamation property* for models if for any situation with $A, M, N \in \mathbf{K}$:

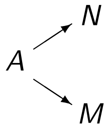


there exists an $N_1 \in \mathbf{K}$ such that



SET AMALGAMATION PROPERTY

The class **K** satisfies the *set amalgamation property* if for any situation with $M, N \in \mathbf{K}$ and $A \subset M, A \subset N$:



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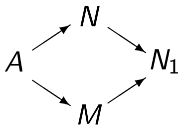

Is there a difference?

For a complete first order theory, Morley taught us:

There is no difference.

Is there a difference?

For a complete first order theory, Morley taught us:

There is no difference.

Tweak the language and we obtain set amalgamation.

(Tweak: put predicates for every definable set in the language)

There is a difference!

Zilber's examples of quasiminimal excellent classes have amalgamation over models but the interesting examples do not have set amalgamation.

Model amalgamation is the natural notion for study in infinitary logic (AEC).

Challenge and Response

We return to the challenge of the Hrushovski construction.

Challenge and Response

Response: Strengthen the Hypotheses

Zariski Structures: Motive

Generalize the notion of Zariski geometry.
Specialize the notion of strongly minimal set.

Zariski Structures: Technique

Define axioms on a set X and the powers X^n specifying a topology and relations between the powers to characterize the notion of 'smooth algebraic variety'.

Positive result: Gloss

Hrushovski-Zilber

Every non-linear (ample) Noetherian Zariski structure is interpretable in an algebraically closed field.

Positive result: Hrushovski-Zilber (detail)

If M is non-linear Noetherian Zariski structure then there is an algebraically closed field K, a quasiprojective algebraic curve $C_M = C_M(K)$ and a surjective map

$$p : M \mapsto C_M$$

of finite degree such that for every closed $S \subseteq M^n$, the image $p(S)$ is Zariski closed in C_M^n (in the sense of algebraic geometry);

if $\hat{S} \subseteq C_M^n$ is Zariski closed, then $p^{-1}(\hat{S})$ is a closed subset of M^n (in the sense of the Zariski structure M).

Whoops!

But if the ‘non-linear’ hypothesis is dropped there are finite covers of $P^1(\mathbf{K})$ that can not be interpreted in an Algebraically closed field.

Such counterexamples arise from finite covers of the affine line and of elliptic curves.

These structures induce an nth root quantum torus.

Response II: Weaken the conclusion

Instead of interpreting the models in algebraically closed fields by first order formulas, find an analytic model.

This is impossible for the finite rank case but has interesting consequences for infinite rank.

Green Fields

The naughty or green field (Poizat)

Expand an algebraically closed fields by a unary predicate for a proper subgroup of the multiplicative group.

$$\delta(X) = 2d_f(X) - \mathrm{ld}(X \cap G).$$

This yields an ω-stable theory of rank $\omega \times 2$.

Green Field

Concrete Models (Zilber)

Assume Schanuel's conjecture; let $\epsilon = 1 + i$.

1. The naughty field.

$$(\mathcal{C}, +, \cdot, \mathbb{G})$$

where $G = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Q}\}$ and exp is complex exponentiation.

Green Field

Concrete Models (Zilber)

Assume Schanuel's conjecture; let $\epsilon = 1 + i$.

1. The naughty field.

$$(\mathcal{C}, +, \cdot, \mathbb{G})$$

where $G = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Q}\}$ and exp is complex exponentiation.

2. A superstable version (chartreuse field)

$$(\mathcal{C}, +, \cdot, \mathbb{G}')$$

where $G' = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Z}\}$ and exp is complex exponentiation.

Non-Commutative Geometry

Three related objects

1. A structure
2. A finitely generated non-commutative C^*-algebra
3. A foliation

Assume Schanuel's conjecture; let $\epsilon = 1 + i$.

Quantum Tori

The concrete superstable (chartreuse field) version $\mathbb{R}^2/\mathbb{G}'$ where

$$\mathbb{G}' = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Z}\}$$

and exp is complex exponentiation

is the leaf space of the Kronecker foliation, the quantum torus.

a near Quantum Tori

The concrete naughty (green) field.
$\mathbb{R}^2/\mathbb{G}$ where

$$\mathbb{G} = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Q}\}$$

and exp is complex exponentiation. is apparently a 'new' structure to topologists, a quotient of the quantum torus by $\mathbb{Q}$. (Baldwin-Gendron)

Consequences of Zilber's Thesis

1. Deeper understanding of complex exponentiation
2. Specific conjectures in many areas of mathematics
3. Significance of Infinitary Logic
4. Broad Connections across mathematics and physics?!