A SUBALGEBRA INTERSECTION PROPERTY FOR CONGRUENCE DISTRIBUTIVE VARIETIES

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ABSTRACT. We prove that if a finite algebra **A** generates a congruence distributive variety then the subalgebras of the powers of **A** satisfy a certain kind of intersection property that fails for finite idempotent algebras that locally exhibit affine or unary behaviour. We demonstrate a connection between this property and the constraint satisfaction problem.

1. INTRODUCTION

By an algebraic structure (or just algebra) **A** we mean a tuple of the form $\langle A, \mathcal{F}\rangle$, where A is a non-empty set and $\mathcal{F}$ is a set (possibly indexed) of finitary operations on A. A is called the universe of **A** and the functions in $\mathcal{F}$ are called the basic operations of **A**. This definition is broad enough to encompass most familiar algebras encountered in mathematics but not so broad that a systematic study of these structures cannot be undertaken. Since the start of this study in the 1930's it has been recognized that two important invariants of any algebra are its lattice of subuniverses and its lattice of congruences. In this paper we demonstrate, for certain finite algebras, a connection between the behaviour of their congruences and the subalgebras of their finite powers.

The link between universal algebra and the constraint satisfaction problem that has been developed over the past several years, starting with the ground breaking paper by Feder and Vardi ([8]) and continuing with the work of Jeavons, Bulatov, Krokhin and others, has brought to light a number of questions that are of interest to algebraists, independent of their connection with the constraint satisfaction problem. The properties of subalgebras of finite algebras that we study in this

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paper have a direct connection with the constraint satisfaction problem. They also tie in with a classic result of Baker and Pixley ([1]) and the more recent work on the local structure of finite algebras developed by Hobby and McKenzie ([10]). In particular, the main result of this paper, Theorem 2.9, suggests an alternate characterization of some familiar classes of locally finite varieties in terms of intersections of subalgebras.

To conclude this section, we recall some of the basic definitions that will be used throughout this paper. Standard references for this material are [6] and [16].

Definition 1.1. *Let* $\mathbf{A} = \langle A, \mathcal{F} \rangle$ *be an algebra.*

(1) *A subuniverse of* $\mathbf{A}$ *is a subset of* A *that is closed under the operations in* $\mathcal{F}$. *An algebra* $\mathbf{B}$ *is a subalgebra of* $\mathbf{A}$ *if its universe if a non-empty subuniverse* B *of* $\mathbf{A}$ *and its basic operations are the restrictions of the basic operations of* $\mathbf{A}$ *to* B. *For* $\mathcal{K}$ *a class of algebras,* $\mathsf{S}(\mathcal{K})$ *denotes the class of all subalgebras of members of* $\mathcal{K}$.
(2) *A congruence of* $\mathbf{A}$ *is an equivalence relation* θ *on* A *that is compatible with the operations in* $\mathcal{F}$. *The set of all congruences of* $\mathbf{A}$, *ordered by inclusion, is called the congruence lattice of* $\mathbf{A}$ *and is denoted* $\operatorname{Con} \mathbf{A}$.

If the basic operations of the members of a class $\mathcal{K}$ of algebras all have similar indices, then it is possible to define the notion of a homomorphism from one member of $\mathcal{K}$ to another and of a homomorphic image of a member of $\mathcal{K}$. Cartesian products of similar algebras can also be defined in a standard manner. $\mathsf{H}(\mathcal{K})$ will denote the class of all homomorphic images of $\mathcal{K}$ while $\mathsf{P}(\mathcal{K})$ will denote the class of cartesian products of members of $\mathcal{K}$.

A fundamental theorem of universal algebra, due to G. Birkhoff, states that a class of similar algebras is closed under the operations of H, S and P if and only if the class can be defined via a set of equations. Such a class of algebras is known as a variety. For $\mathcal{K}$ a class of similar algebras, $\mathsf{V}(\mathcal{K})$ denotes the smallest variety that contains $\mathcal{K}$. It follows from the proof of Birkhoff's theorem that this class coincides with the class $\mathsf{HSP}(\mathcal{K})$.

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2. An Intersection Property

Definition 2.1. *Let $n > 0$ and A_i be sets for $1 \leq i \leq n$. For $k > 0$ and $B, C \subseteq \prod_{1 \leq i \leq n} A_i$ we say that B and C are k-equal, and write $B =_k C$ if for every subset I of $\{1, 2, \dots, n\}$ of size at most k, the projection of B and C onto the coordinates I are equal.*

If $B =_k \prod_{1 \leq i \leq n} A_i$ then we say that B is k-complete with respect to $\prod_{1 \leq i \leq n} A_i$.

Note that being 1-complete with respect to $\prod_{1 \leq i \leq n} A_i$ is equivalent to being subdirect.

Definition 2.2. *Let $\mathbf{A}$ be an algebra and $k > 0$.*

1. *For $n > 0$ and $\mathbf{B}$ a subalgebra of $\mathbf{A}^n$, we denote the set of all subuniverses C of $\mathbf{A}^n$ with $C =_k B$ by $[\mathbf{B}]_k$.*
2. *We say that $\mathbf{A}$ has the **k-intersection property** if for every $n > 0$ and subalgebra $\mathbf{B}$ of $\mathbf{A}^n$, $\bigcap [\mathbf{B}]_k \neq \emptyset$.*
3. *We say that $\mathbf{A}$ has the **strong k-intersection property** if for every $n > 0$ and subalgebra $\mathbf{B}$ of $\mathbf{A}^n$, $\bigcap [\mathbf{B}]_k =_k B$.*
4. *We say that $\mathbf{A}$ has the **k-complete intersection property** if for every $n > 0$, $\bigcap [\mathbf{A}^n]_k \neq \emptyset$.*

The following proposition lists some elementary facts about the above properties. Note that if $\mathbf{A} = \langle A, \mathcal{F} \rangle$ is an algebra and $\mathcal{G}$ is a subset of the derived operations of $\mathbf{A}$, then the algebra $\mathbf{B} = \langle A, \mathcal{G} \rangle$ is known as a **reduct** of $\mathbf{A}$.

Proposition 2.3. *Let $\mathbf{A}$ be an algebra and $k > 0$.*

1. *The strong k-intersection property implies the k-intersection property, which implies the k-complete intersection property.*
2. *If $\mathbf{A}$ fails one of these properties then any reduct of it does as well.*
3. *If $\mathbf{A}$ has a constant term then $\mathbf{A}$ satisfies the k-intersection property.*
4. *If $\mathbf{A}$ satisfies the (strong) k-intersection property then so does every algebra in $\mathsf{HSP}(\mathbf{A})$. If $\mathbf{A}$ satisfies the k-complete intersection property then so does every quotient and cartesian power of $\mathbf{A}$.*

Proof. The proofs of these claims are elementary and are left to the reader. □

Example 2.4 (E. Kiss). *Define $\mathbf{A}$ to be the algebra on $\{0, 1, 2\}$ with a single ternary basic operation $p(x, y, z)$ defined by: $p(x, y, y) = x \cdot y$*

and $p(x, y, z) = z$ otherwise, where $x \cdot y$ is the operation:

$\cdot$	0	1	2
0	0	1	0
1	0	1	1
2	2	2	2

It can be shown directly, or by using results from [14] that $\mathbf{A}$ satisfies the 2-complete intersection property. Since $\mathbf{A}$ has a two element subalgebra (with universe $\{0, 1\}$) that is term equivalent to a set, then this example demonstrates that the class of algebras that satisfy the k-complete intersection property for some $k \geq 2$ is not closed under taking subalgebras.

The following Proposition exhibits some relevant examples.

Definition 2.5. (1) *For $k > 2$, a k-ary function f on a set A is a near unanimity function if for all x, $y \in A$, the following equalities hold:*

$$f(x, x, \dots, x, y) = f(x, x, \dots, x, y, x) = \dots = f(y, x, \dots, x) = x.$$

A k-ary term t of an algebra $\mathbf{A}$ is a near unanimity term of $\mathbf{A}$ if the function $t^{\mathbf{A}}$ is a near unanimity function on A.

(2) *A function f on a set A is idempotent if for all $x \in A$,*

$$f(x, x, \dots, x) = x.$$

An algebra $\mathbf{A}$ is idempotent if all of its term operations are idempotent. The idempotent reduct of an algebra $\mathbf{A}$ is the algebra with universe A and with basic operations the set of all idempotent term operations of $\mathbf{A}$.

Note that if $\mathbf{A}$ is an idempotent algebra and α is a congruence of $\mathbf{A}$ then every α-class is a subuniverse of $\mathbf{A}$. In fact, this property characterizes the idempotent algebras (just apply it to the congruence 0_A).

Proposition 2.6. (1) *If $\mathbf{A}$ has a k-ary near unanimity term (for $k > 2$) then it satisfies the strong $(k-1)$-intersection property.*

(2) *If $\mathbf{A}$ is the idempotent reduct of a module or is term equivalent to a set then $\mathbf{A}$ fails the k-complete intersection property for every $k > 0$.*

(3) *If $\mathbf{A}$ is the 2 element meet semi-lattice then it fails the strong k-intersection property for all $k > 0$.*

Proof. The first claim is a direct consequence of a result of Baker and Pixley. From Theorem 2.1 of [1] it follows that if $\mathbf{A}$ has a k-ary near unanimity term and $\mathbf{B}$ is a subalgebra of $\mathbf{A}^n$ then the only subuniverse of $\mathbf{A}^n$ that is $(k-1)$-equal to B is B.

Suppose that $\mathbf{A}$ is the idempotent reduct of the module $\mathbf{M}$ over the ring $\mathbf{R}$ and let $k > 0$. Consider the submodule $\mathbf{E}$ of $\mathbf{M}^{k+1}$ with universe $\{(m_1, \dots m_{k+1}) : \sum_{1 \le i \le k+1} m_i = 0\}$. Let $c \in M$ with $c \neq 0$ and define O to be the set $\{(m_1, \dots m_{k+1}) : \sum_{1 \le i \le k+1} m_i = c\}$. Then O is a coset of E in $\mathbf{M}^{k+1}$ that is disjoint from E. It is not hard to see that as subsets of M^{k+1}, $E =_k O$ and that in fact they are both k-complete. It is also straightforward to show that E and O are subuniverses of $\mathbf{A}^{k+1}$ since $\mathbf{A}$ is the idempotent reduct of $\mathbf{M}$. From this it follows that $\mathbf{A}$ fails the k-complete intersection property. Since any reduct of $\mathbf{A}$ also fails this property then it follows that any algebra term equivalent to a set also fails this property.

Let $\mathbf{S}$ be the 2 element meet semi-lattice on $\{0, 1\}$ and $k > 1$. Let $S_1 = S^k \setminus \{\sigma : \sigma \text{ is a co-atom of } \mathbf{S}^k\}$ and $S_2 = S^k \setminus \{\langle 1 \rangle\}$ ($\langle 1 \rangle$ is the top element of $\mathbf{S}^k$). It is not hard to show that both S_1 and S_2 are subuniverses of $\mathbf{S}^k$ and that both are $(k-1)$-complete. The intersection of S_1 and S_2 is not $(k-1)$-complete since it contains no co-atom of $\mathbf{S}^k$ nor the element $\langle 1 \rangle$. □

In this paper we will correlate, for finite idempotent algebras, these intersection properties with some more familiar properties of finite algebras.

Definition 2.7. *An algebra* $\mathbf{A}$ *is said to be* ***congruence distributive*** *if its congruence lattice satisfies the distributive law. A class of algebras is congruence distributive if all of its members are.*

For $k > 0$, $\mathbf{A}$ *is in the class* $CD(k)$ *if it has a sequence of ternary terms* $p_i(x, y, x)$, $0 \le i \le k$ *that satisfies the identities:*

$$\begin{aligned} p_0(x,y,z) &= x \\ p_k(x,y,z) &= z \\ p_i(x,y,x) &= x \text{ for all } i \\ p_i(x,x,y) &= p_{i+1}(x,x,y) \text{ for all } i \text{ even} \\ p_i(x,y,y) &= p_{i+1}(x,y,y) \text{ for all } i \text{ odd} \end{aligned}$$

Note that an algebra is in $CD(1)$ if and only if it has size 1 and is in $CD(2)$ if and only if it has a ternary near unanimity term (a majority term). A sequence of terms of an algebra $\mathbf{A}$ that satisfies the above equations will be referred to as Jónsson terms of $\mathbf{A}$. The following celebrated theorem of Jónsson relates congruence distributivity to the existence of Jónsson terms.

Theorem 2.8 (Jónsson). *An algebra* $\mathbf{A}$ *generates a congruence distributive variety if and only if* $\mathbf{A}$ *is in* $CD(k)$ *for some* $k > 0$.

The proof of this theorem can be found in any standard reference on universal algebra, for example in [6]. Some of the results contained in this paper deal with local invariants of finite algebras developed by Hobby and McKenzie that form part of Tame Congruence Theory. Details of this theory may be found in [10] or [7]. In this paper we will only introduce some of the basic terminology of the theory and will omit most details.

In Tame Congruence Theory, five local types of behaviour of finite algebras are identified and studied. The five types are, in order:

(1) the unary type,
(2) the affine or vector-space type,
(3) the 2 element Boolean type,
(4) the 2 element lattice type,
(5) the 2 element semi-lattice type.

For $1 \leq \mathbf{i} \leq 5$, we say that an algebra $\mathbf{A}$ omits type $\mathbf{i}$ if, locally, the corresponding type of behaviour does not occur in $\mathbf{A}$. A class of algebras $\mathcal{C}$ is said to omit type $\mathbf{i}$ if all finite members of $\mathcal{C}$ omit that type. $\mathrm{typ}(\mathbf{A})$ denotes the set of types that occur in $\mathbf{A}$ and $\mathrm{typ}(\mathcal{C})$ denotes the union of $\mathrm{typ}(\mathbf{A})$ over all finite $\mathbf{A} \in \mathcal{C}$.

If $\mathbf{A}$ is a finite simple algebra then one can speak of its type since (rather than its type set), since it is shown by Hobby and McKenzie that the local behaviour of a finite simple algebra is uniform. So, for $\mathbf{A}$ finite and simple, $\mathrm{typ}(\mathbf{A}) = \{\mathbf{i}\}$ for some $\mathbf{i}$ and we say that $\mathbf{A}$ has type $\mathbf{i}$.

The following theorem is the main result of this paper and will be proved over the next two sections.

Theorem 2.9. *Let* $\mathbf{A}$ *be a finite algebra.*

(1) *If* $\mathbf{A}$ *generates a congruence distributive variety then it satisfies the 2-complete intersection property.*
(2) *If* $\mathbf{A}$ *is idempotent and satisfies the* k*-intersection property for some* $k > 0$ *then* $\mathsf{HSP}(\mathbf{A})$ *omits types* $\mathbf{1}$ *and* $\mathbf{2}$.
(3) *If* $\mathbf{A}$ *is idempotent and satisfies the strong* k*-intersection property for some* $k > 0$ *then* $\mathsf{HSP}(\mathbf{A})$ *omits types* $\mathbf{1}$, $\mathbf{2}$ *and* $\mathbf{5}$.

3. Omitting Types

Consider the usual ordering on the set of possible types of a finite algebra, i.e., $\mathbf{1} < \mathbf{2} < \mathbf{3} > \mathbf{4} > \mathbf{5} > \mathbf{1}$. This order corresponds to the relative strength of the set of operations on the algebras associated with each of the five different local types. For example the 2 element Boolean algebra has a lattice reduct and a vector space reduct, hence $\mathbf{2}, \mathbf{4} < \mathbf{3}$.

The following proposition is a generalization of Proposition 4.14 from [4] in the type $\mathbf{1}$ case. It also generalizes the corresponding type $\mathbf{1}, \mathbf{2}$ result worked out with B. Larose. An algebra is strictly simple if it is simple and has no proper subuniverse with more than 1 element.

Proposition 3.1. *Let* $\mathbf{A}$ *be a finite idempotent algebra. If* $\mathbf{i}$ *is in* typ$(\mathsf{HSP}(\mathbf{A}))$ *then for some* $\mathbf{j} \leq \mathbf{i}$ *there is a finite strictly simple algebra of type* $\mathbf{j}$ *in* $\mathsf{HS}(\mathbf{A})$.

Proof. Suppose that $\mathbf{i} \in \mathrm{typ}(\mathsf{HSP}(\mathbf{A}))$. If $\mathbf{i} = \mathbf{3}$ then there is nothing to prove since $\mathsf{HS}(\mathbf{A})$ contains a strictly simple algebra. If $\mathbf{i} \neq \mathbf{3}$ we first show that the variety generated by $\mathbf{A}$ contains a strictly simple algebra of type $\mathbf{j}$ for some $\mathbf{j} \leq \mathbf{i}$. Since $\mathbf{i} \in \mathrm{typ}(\mathsf{HSP}(\mathbf{A}))$ then we can find a finite $\mathbf{B} \in \mathsf{HSP}(\mathbf{A})$ of minimal size whose type set contains $\mathbf{j}$ for some $\mathbf{j} \leq \mathbf{i}$. By the minimality of $|B|$, it follows that there is some congruence $0_B \prec \beta$ of $\mathbf{B}$ with $\mathrm{typ}(\langle 0_B, \beta\rangle) = \mathbf{j}$.

Let $C \subseteq B$ be a nontrivial β-class. Since $\mathbf{A}$, and hence $\mathbf{B}$, is idempotent then C is a subuniverse of $\mathbf{B}$. If $\mathbf{j} = \mathbf{2}$ (or $\mathbf{1}$) then β is an abelian (or strongly abelian) congruence and so the algebra $\mathbf{C}$ is abelian (or strongly abelian). Unless $C = B$ we obtain a contradiction to the minimality of $|B|$ and so we have that $\beta = 1_B$ and hence that $\mathbf{B}$ is a finite simple abelian (or strongly abelian) algebra. It is now elementary to show that $\mathsf{HS}(\mathbf{B})$ contains a strictly simple algebra of type $\mathbf{2}$ or $\mathbf{1}$, but it follows by the main result of [18] that $\mathbf{B}$ itself is strictly simple.

The remaining cases are when $\mathbf{j} = \mathbf{4}$ or $\mathbf{5}$. Let $N = \{0, 1\}$ be a $\langle 0_B, \beta\rangle$-trace of $\mathbf{B}$ contained in C, let ν be the congruence of $\mathbf{C}$ generated by N^2 and let μ be a congruence of $\mathbf{C}$ that is covered by ν. We claim that $\mathrm{typ}(\langle \mu, \nu\rangle) \leq \mathbf{j}$. Let M be a $\langle \mu, \nu\rangle$-trace. Then there is a unary polynomial $p(x)$ of $\mathbf{C}$ that maps N into M with $(p(0), p(1)) \notin \mu$. As p is the restriction to C of some unary polynomial $p'(x)$ of $\mathbf{B}$ then it follows that $N' = \{p(0), p(1)\}$ is a $\langle 0_B, \beta\rangle$-trace contained in M.

As C is a subuniverse of B then the polynomial clone of $\mathbf{C}|_{N'}$ is contained in the polynomial clone of $\mathbf{B}|_{N'}$. We can rule out $\mathrm{typ}(\langle \mu, \nu\rangle) = \mathbf{2}$ or $\mathbf{3}$ since in either case, M supports a polynomial that maps $p(0)$ to $p(1)$ and $p(1)$ to $p(0)$. In the type $\mathbf{3}$ case, the trace M consists of exactly two elements (and so is equal to N') and has a unary polynomial that acts as boolean complementation. If $\mathrm{typ}(\langle \mu, \nu\rangle) = \mathbf{2}$ then $\mathbf{C}|_M$ has a Mal'cev polynomial $d(x, y, z)$ and the unary polynomial $d(p(0), x, p(1))$ has the desired property. In either case, the restriction of this polynomial to N' belongs to the polynomial clone of $\mathbf{C}|_{N'}$ but cannot be contained in the polynomial clone of $\mathbf{B}|_{N'}$ since $\mathrm{typ}(\langle 0_B, \beta\rangle) = \mathbf{4}$ or $\mathbf{5}$. So, if $\mathbf{j} = \mathbf{4}$ then $\mathrm{typ}(\langle \mu, \nu\rangle) \leq \mathbf{j}$.

Finally, if $\mathbf{j} = \mathbf{5}$ then $\mathrm{typ}(\langle \mu, \nu\rangle)$ cannot equal $\mathbf{4}$ since then both traces M and N' consist of exactly two elements (and so are equal) and hence $\mathbf{B}|_{N'}$ would support both lattice meet and join operations, contrary to $\mathbf{j} = \mathbf{5}$. Note that the essential fact used in this part of the argument is that C is a subuniverse of $\mathbf{B}$ that contains a $\langle 0_B, \beta\rangle$-trace.

We have established that $\mathrm{typ}(\langle \mu, \nu\rangle) \leq \mathbf{j}$ and so by the minimality of B we conclude that $C = B$, implying that $\mathbf{B}$ is a simple algebra of type $\mathbf{j}$.

By the previous argument and the minimality of B we can conclude that no proper subuniverse of $\mathbf{B}$ contains a $\mathbf{B}$-minimal set and so the subuniverse generated by any $\mathbf{B}$-minimal set is B. From this it follows that every two element subset of B is a $\mathbf{B}$-minimal set. To see this, let $a \neq b$ in B. Since $\mathbf{B}$ is simple of type $\mathbf{4}$ or $\mathbf{5}$ then there is a $\mathbf{B}$-minimal set $\{0, a\}$ for some element $0 \in B$. We have just concluded that $\{0, a\}$ generates all of B and so there is a term $t(x, y)$ of $\mathbf{B}$ with $t(0, a) = b$. Since $\mathbf{B}$ is idempotent, we have that $t(a, a) = a$ and so the polynomial $t(x, a)$ maps the minimal set $\{0, a\}$ to the set $\{a, b\}$, establishing that $\{a, b\}$ is a $\mathbf{B}$-minimal set. Thus, $\mathbf{B}$ is strictly simple.

So, we have established that for some $\mathbf{j} \leq \mathbf{i}$, $\mathsf{HSP}(\mathbf{A})$ contains a strictly simple algebra of type $\mathbf{j}$. If $\mathbf{j} = \mathbf{3}$ or $\mathbf{4}$ then we can use Lemma 14.4 of [10] to conclude that $\mathsf{HS}(\mathbf{A})$ contains a strictly simple algebra of type $\mathbf{j}$. The following, elementary argument, handles all cases including these two but makes use of the idempotency of $\mathbf{A}$. It is essentially the argument found in the proof of Proposition 4.14 of [4]. To show that $\mathsf{HS}(\mathbf{A})$ contains a strictly simple algebra of type $\mathbf{j}$ for some $\mathbf{j} \leq \mathbf{i}$, let $n > 0$ be minimal with the property that $\mathsf{HS}(\mathbf{A}^n)$ contains such an algebra. We wish to show that $n = 1$. Assume that $n > 1$ and let $B \subseteq A^n$ be a subuniverse of $\mathbf{A}^n$ and θ a congruence of $\mathbf{B}$ with $\mathbf{S} = \mathbf{B}/\theta$ a strictly simple algebra of type $\mathbf{j} \leq \mathbf{i}$.

By the minimality of n it follows that the projection of $\mathbf{B}$ onto its first coordinate is a subalgebra $\mathbf{A}'$ of $\mathbf{A}$ that contains more than one element. For each $a \in A'$, let B_a be the subset of B consisting of all elements whose first coordinate is equal to a. Since $\mathbf{B}$ is idempotent, it follows that B_a is a subuniverse of B. If for some $a \in A'$, B_a is not contained in a θ-block, then modulo the restriction of θ to B_a, we obtain a nontrivial subuniverse of $\mathbf{S}$. Since $\mathbf{S}$ is strictly simple, we have that $\mathbf{S}$ must be a quotient of $\mathbf{B}_a$, contradicting the minimality of n.

So, each B_a is contained in some θ-block. Thus, the kernel of the projection of B onto A' is a congruence contained in θ. From this it follows that $\mathbf{A}'$ has a quotient isomorphic to $\mathbf{S}$. This final contradiction conclude the proof of this proposition. $\square$

Theorem 3.2 of [2] provides examples that show the necessity of idempotency in the previous proposition. Another example due to McKenzie can be found described in [13]. In [13] E. Kiss observes that if $\mathbf{A}$ is a finite algebra that generates a congruence modular variety then the type set of the variety coincides with the set of types that appear in the subalgebras of $\mathbf{A}$.

Corollary 3.2. *Let* $\mathbf{A}$ *be a finite idempotent algebra and let* T *be some set of types closed downwards with respect the ordering on types. Then* $\mathsf{HSP}(\mathbf{A})$ *omits the types in* T *if and only if* $\mathsf{HS}(\mathbf{A})$ *does. In particular,* $\mathsf{HSP}(\mathbf{A})$ *omits types* $\mathbf{1}$ *and* $\mathbf{2}$ *if and only if* $\mathsf{HS}(\mathbf{A})$ *does.*

If $\mathsf{HSP}(\mathbf{A})$ *fails to omit the types in* T *then for some* $\mathbf{j} \in T$ *there is a strictly simple algebra in* $\mathsf{HS}(\mathbf{A})$ *of type* $\mathbf{j}$.

Corollary 3.3. *Let* T *be a set of types closed downwards with respect to the ordering on types. The problem of determining which finite idempotent algebras generate varieties that omit the types in* T *can be solved in polynomial time as a function of the size of the algebra.*

Proof. Given T and a finite idempotent algebra $\mathbf{A}$, to determine if $\mathsf{HSP}(\mathbf{A})$ omits the types in T it suffices to determine whether $\mathsf{HS}(\mathbf{A})$ contains a strictly simple algebra of type $\mathbf{j}$ for some $\mathbf{j} \in T$. If this occurs then there is some 2-generated subalgebra of $\mathbf{A}$ whose type set includes $\mathbf{j}$ since every strictly simple algebra is 2-generated. The paper [2] provides a polynomial time algorithm to determine the type set of a given finite algebra and so to test whether $\mathsf{HSP}(\mathbf{A})$ omits the types in T we need only apply this algorithm to all 2-generated subalgebras of $\mathbf{A}$. $\square$

This result is used in a forthcoming paper with Ralph Freese [9] that, among other things, establishes that there is a polynomial time algorithm to determine if a finite idempotent algebra generates a congruence modular, distributive, or permutable variety.

Theorem 3.4. *Let* $\mathbf{A}$ *be a finite idempotent algebra. If* $\mathbf{1}$ *or* $\mathbf{2} \in \mathbf{typ}\{\mathsf{HSP}(\mathbf{A})\}$ *then for every* $k > 0$, $\mathbf{A}$ *fails the* k*-intersection property. In fact, some subalgebra of* $\mathbf{A}$ *fails the* k*-complete intersection property for all* $k > 0$.

Proof. If $\mathbf{1}$ or $\mathbf{2}$ appear in the type set of the variety generated by $\mathbf{A}$ then by Corollary 3.2, there is a finite strictly simple abelian algebra $\mathbf{S}$ in $\mathsf{HS}(\mathbf{A})$. By a result of A. Szendrei ([17]), if $\mathbf{S}$ is of type $\mathbf{1}$ then

S is term equivalent to a set. On the other hand, she shows that if **S** is of type **2** then **S** is term equivalent to the idempotent reduct of a module **M**, over some finite ring **R**. Part 2) of Proposition 2.6 can be used to conclude that in either case, **S** fails the k-complete intersection property for all $k > 0$.

Since **S** is in $\mathsf{HS}(\mathbf{A})$ then **S** is isomorphic to a quotient of some subalgebra **B** of **A**. From Proposition 2.3 it follows that for all $k > 0$, **B** fails the k-complete intersection property and **A** fails the k-intersection property. ☐

Theorem 3.5. *If* **A** *is a finite idempotent algebra with* **1***,* **2** *or* **5** $\in$ *typ*$\{\mathbf{HSP}(\mathbf{A})\}$ *then for all* $k > 0$*,* **A** *fails the strong* k*-intersection property. In fact, there is some subalgebra of* **B** *of* **A** *such that for all* $k > 0$*, the intersection of all* k*-complete subuniverses of* $\mathbf{B}^{k+1}$ *fails to be* k*-complete.*

Proof. The previous theorem handles the case when **1** or **2** appear in the type set of the variety generated by **A**. If $\mathsf{HSP}(\mathbf{A})$ omits types **1** and **2** but **5** appears then by Proposition 3.1 $\mathsf{HS}(\mathbf{A})$ contains a strictly simple idempotent algebra **S** of type **5**. According to A. Szendrei's characterization of idempotent strictly simple algebras found in [17] it follows that **S** is term equivalent to the 2 element meet-semilattice $\langle \{0, 1\}, \wedge \rangle$. Using Proposition 2.3 (4), the result then follows from part (3) of Proposition 2.6. ☐

4. Congruence Distributive Varieties

In order to prove part 1) of Theorem 2.9 it suffices to consider finite algebras whose basic operations consist of a sequence of Jónsson terms. This follows from part 2) of Proposition 2.3. So, for this section, let **A** be a finite algebra whose only basic operations consist of: $p_i(x, y, z)$ for $0 \le i \le n$ and which satisfy the Jónsson identities:

$$\begin{aligned}
p_0(x, y, z) &= x \\
p_n(x, y, z) &= z \\
p_i(x, y, x) &= x \text{ for all } i \\
p_i(x, x, y) &= p_{i+1}(x, x, y) \text{ for all } i \text{ even} \\
p_i(x, y, y) &= p_{i+1}(x, y, y) \text{ for all } i \text{ odd}
\end{aligned}$$

Definition 4.1. *For* $1 \le j \le n$*, and* $X \subseteq A$*, define* $J_j(X)$ *to be the smallest subuniverse* Y *of* **A** *containing* X *and satisfying:*

for all $u \in A$ *and* $c \in Y$*,* $p_j(u, u, c) \in Y$*, if* j *is odd and* $p_j(u, c, c) \in Y$ *if* j *is even.*

Call $J_j(X)$ the jth Jónsson ideal of $\mathbf{A}$ *generated by* X.

A subset of the form $J_j(X)$ is called a j-Jónsson ideal. Call the algebra $\mathbf{A}$ j-minimal if it contains no proper, non-empty j-Jónsson ideal.

Proposition 4.2. *For each $a \in A$, $J_1(\{a\}) = A$ and hence $\mathbf{A}$ is 1-minimal. For every $X \subseteq A$, $J_n(X)$ is the subuniverse of $\mathbf{A}$ generated by X. In particular, for $a \in A$, $J_n(\{a\}) = \{a\}$.*

We prove something more general than is needed to establish part 1) of Theorem 2.9. Let $\mathcal{V}$ be the variety generated by $\mathbf{A}$.

Lemma 4.3. *Let $m > 0$ and $\mathbf{A}_i$ be finite members of $\mathcal{V}$ for $1 \le i \le m$. Let $1 \le j < n$ and assume that for each i, $\mathbf{A}_i$ is j-minimal. Let $\mathbf{B}$ be a 2-complete subalgebra of $\prod_{1 \le i \le m} \mathbf{A}_i$. If for each i, J_i is a $(j+1)$-Jónsson ideal of $\mathbf{A}_i$, then $B \cap \prod_{1 \le i \le m} J_i$ is a 2-complete subuniverse of $\prod_{1 \le i \le m} \mathbf{J}_i$. In fact, for every $1 \le u < v \le m$ and $a \in A_u$, $b \in A_v$ there is $\sigma \in B$ with $\sigma(u) = a$, $\sigma(v) = b$ and $\sigma(i) \in J_i$ for all $i \notin \{u, v\}$.*

Proof. We prove this by induction on m. By 2-completeness, the property holds for $m = 2$. Assume the property holds for m and let $\mathbf{B}$ be a 2-complete subalgebra of $\prod_{1 \le i \le m+1} \mathbf{A}_i$. By symmetry it suffices to show that if $a \in A_1$ and $b \in A_2$ then there is some $\sigma \in B$ with $\sigma(1) = a$, $\sigma(2) = b$ and $\sigma(i) \in J_i$ for all $i > 2$.

Define B_a to be the set of $\sigma \in B$ with $\sigma(1) = a$, and $\sigma(i) \in J_i$ for $2 < i \le m+1$. Note that B_a is a subuniverse of B and is nonempty since, if $c \in J_{m+1}$ then by induction there is some $\mu \in B$ with $\mu(1) = a$, $\mu(m+1) = c$ and $\mu(i) \in J_i$ for $2 < i \le m$.

Let I be the projection of B_a onto the coordinate 2. We claim that I is a j-Jónsson ideal of $\mathbf{A}_2$. Since I is nonempty and $\mathbf{A}_2$ is assumed to be j-minimal it follows that $I = A_2$. Our result follows from this. To prove the claim we need to show that if $c \in I$ and $u \in A_2$ then $p_j(u, u, c) \in I$ if j is odd and $p_j(u, c, c) \in I$ if j is even.

Let $\sigma \in B_a$ with $\sigma(2) = c$. By induction there is an element $\mu \in B$ with $\mu(1) = a$, $\mu(2) = u$ and $\mu(i) \in J_i$ for all $2 < i < m+1$. Let $\sigma(m+1) = v \in J_{m+1}$ and $\mu(m+1) = z \in A_{m+1}$. Let $\nu \in B$ be any element with $\nu(i) \in J_i$ for $2 < i < m+1$ and with $\nu(2) = u$ and $\nu(m+1) = v$ if j is odd and $\nu(2) = c$ and $\nu(m+1) = z$ if j is even. By induction, such an element exists.

We claim that the element $\tau = p_j(\mu, \nu, \sigma) \in B_a$. This will complete the proof, since then $\tau(2) \in I$ and by design $\tau(2) = p_j(u, u, c)$ if j is odd and $\tau(2) = p_j(u, c, c)$ if j is even. Using the identity $p_j(x, y, x) = x$ it follows that $\tau(1) = a$. For $2 < i < m+1$, $\tau(i) = p_j(x, w, y)$

for some elements x, w, and $y \in J_i$ and so belongs to J_i. $\tau(m+1) = p_j(z, v, v) = p_{j+1}(z, v, v) \in J_{m+1}$ if j is odd and $\tau(m+1) = p_j(z, z, v) = p_{j+1}(z, z, v) \in J_{m+1}$ if j is even. So, in either case we have established that $\tau \in B_a$. □

Theorem 4.4. *Let $m > 0$ and $\mathbf{A}_i$ be finite members of $\mathcal{V}$. Then the intersection of all 2-complete subuniverses of $\prod_{1 \le i \le m} \mathbf{A}_i$ is non-empty.*

Proof. We prove this by induction on the sum s of the cardinalities of the A_i's . For $s = m$ the result holds trivially since then each A_i has size 1. Assume that $s > m$ and that we have established the result for all smaller sums. As noted earlier, each $\mathbf{A}_i$ is 1-minimal and so we can choose $j \le n$ maximal with the property that each $\mathbf{A}_i$ is j-minimal. Since all n-minimal algebras in $\mathcal{V}$ have cardinality 1 we have that $j < n$.

For each $1 \le i \le m$, let J_i be a $(j+1)$-Jónsson ideal of $\mathbf{A}_i$ so that at least one of the J_i is proper, say $|J_1| < |A_1|$. By induction, the intersection, C, of all 2-complete subuniverses of $\prod_{1 \le i \le m} \mathbf{J}_i$ is non-empty. If $\mathbf{D}$ is a 2-complete subalgebra of $\prod_{1 \le i \le m} \mathbf{A}_i$ then by the previous lemma $D \cap \prod_{1 \le i \le m} J_i$ is a 2-complete subuniverse of $\prod_{1 \le i \le m} \mathbf{J}_i$ since each $\mathbf{A}_i$ is j-minimal. Then $C \subseteq D \cap \prod_{1 \le i \le m} J_i \subseteq D$. This completes the proof. □

Corollary 4.5. *If $\mathbf{A}$ is a finite algebra that generates a congruence distributive variety then $\mathbf{A}$ satisfies the 2-complete intersection property.*

Corollary 4.6. *Let $m > 0$ and $\mathbf{A}_i$ be $(n-1)$-minimal finite members of $\mathcal{V}$ for $1 \le i \le m$. Then $\prod_{1 \le i \le m} A_i$ is the only 2-complete subuniverse of $\prod_{1 \le i \le m} \mathbf{A}_i$.*

Proof. For each $1 \le i \le m$, choose an element $a_i \in A_i$. Since each $\mathbf{A}_i$ is $(n-1)$-minimal and each $\{a_i\}$ is a n-Jónsson ideal then by the Lemma we conclude that every 2-complete subuniverse of $\prod_{1 \le i \le m} \mathbf{A}_i$ contains the element $(a_1, \ldots, a_m)$. Since the a_i were chosen arbitrarily, the result follows. □

5. Connections with the Constraint Satisfaction Problem

The class of constraint satisfaction problems provides a framework in which a wide number of familiar complexity classes can be specified. There are a number of excellent surveys of this class, in particular [4]. Starting with the paper by Feder and Vardi ([8]) and continuing with the work of Jeavons, Bulatov, Krokhin and others ([5] for example) an interesting connection between the constraint satisfaction problem (CSP) and universal algebra has been developed. In this section we will give a brief overview of the constraint satisfaction problem and then tie in the results from the previous sections with the CSP.

Definition 5.1. *An instance of the constraint satisfaction problem is a triple $P = (V, A, \mathcal{C})$ with*

- V *a nonempty, finite set of variables,*
- A *a nonempty, finite domain,*
- $\mathcal{C}$ *a set of constraints $\{C_1, \dots, C_q\}$ where each C_i is a pair $(\vec{s}_i, R_i)$ with*
  - $\vec{s}_i$ *a tuple of variables of length m_i, called the scope of C_i, and*
  - R_i *a subset of A^{m_i}, called the constraint relation of C_i.*

Given an instance P of the CSP we wish to answer the following question:

> Is there a solution to P, i.e., does there exist a function $f : V \to A$ such that for each $i \leq q$, the m_i-tuple $f(\vec{s}_i) \in R_i$?

In general, the class of CSPs is NP-complete, but by restricting the nature of the constraint relations that are allowed to appear in an instance of the CSP, it is possible to find natural subclasses of the CSP that are tractable.

Definition 5.2. *Let A be a domain and Γ a set of finitary relations over A. CSP(Γ) is the collection of all instances of CSP with domain A and with constraint relations coming from Γ. Γ is called the constraint language of the class CSP(Γ).*

Definition 5.3. *Call a constraint language Γ globally tractable if the class of problems CSP(Γ) is tractable, i.e., there is a polynomial time algorithm that solves all instances of CSP(Γ). If each finite subset Γ' of Γ is globally tractable then we say that Γ is tractable.*

Γ is said to be NP-complete if the class of problems CSP(Γ) is NP-complete.

A key problem in this area is to classify the (globally) tractable constraint languages. One approach to this problem is to consider constraint languages that arise from finite algebras in the following manner:

Definition 5.4. *For $\mathbf{A}$ a finite algebra, define $\Gamma_{\mathbf{A}}$ to be the constraint language over the domain A consisting of all subuniverses of finite cartesian powers of $\mathbf{A}$.*

We call an algebra $\mathbf{A}$ (globally) tractable if the language $\Gamma_{\mathbf{A}}$ is.

The work of Jeavons and others ([11]) provides a reduction of the tractability problem for constraint languages to the problem of determining those finite idempotent algebras $\mathbf{A}$ for which $\Gamma_{\mathbf{A}}$ is (globally) tractable and much work has been done on the CSP in this algebraic setting.

One particular method for establishing the tractability of a constraint language is via local consistency properties. While a number of useful notions of local consistency have been studied, in this paper we will deal with one proposed by Bulatov and Jeavons ([4]) known as finite relational width.

Definition 5.5. *For $k > 0$, an instance $P = (V, A, \mathcal{C})$ of the CSP is k-minimal if:*

- *Every k-element subset of variables is within the scope of some constraint in $\mathcal{C}$,*
- *For every set I of at most k variables and every pair of constraints $C_i = (s_i, R_i)$ and $C_j = (s_j, R_j)$ from $\mathcal{C}$ whose scopes contain I, the projections of the constraint relations R_i and R_j onto I are the same.*

While the following definition and theorem apply to a wider class of constraint languages, to avoid some technical matters we will only present them in the algebraic setting.

Definition 5.6. *An algebra $\mathbf{A}$ has relational width k if whenever P is a k-minimal instance of CSP($\Gamma_{\mathbf{A}}$) whose constraint relations are all non-empty then P has a solution. $\mathbf{A}$ has bounded relational width if it has relational width k for some k.*

We note that if $\mathbf{A}$ is a finite algebra of relational width k then this property is preserved by taking cartesian powers, subalgebras and homomorphic images and so every finite member of $\mathsf{HSP}(\mathbf{A})$ has relational width k.

We say that two instances of the CSP are equivalent if they have the same set of solutions.

Theorem 5.7 (see [4])**.** *Let $\mathbf{A}$ be a finite algebra.*

(1) *For a fixed k, any instance of CSP($\Gamma_{\mathbf{A}}$) can be converted into an equivalent, k-minimal instance of CSP($\Gamma_{\mathbf{A}}$) in polynomial time.*
(2) *If $\mathbf{A}$ is of bounded relational width then it is globally tractable.*

A problem closely related to the problem of classifying the globally tractable constraint languages or idempotent algebras is that of classifying the constraint languages or idempotent algebras of finite relational width.

Using a different notion of width, Larose and Zádori ([15]) show that if a finite idempotent algebra has finite width then the variety that it generates omits types **1** and **2**. For this notion of width they conjecture that the converse is true. In [3], Bulatov establishes something similar for constraint languages having finite relational width and also conjectures that the converse is true. He does not make explicit mention of the tame congruence theoretic types, but rather makes use of a related local analysis of finite algebras.

We can use Theorem 2.9 to prove something similar to (but implied by) the Larose-Zádori result.

Lemma 5.8. *Let $\mathbf{A}$ be a finite idempotent algebra. If $\mathbf{A}$ has relational width k for some $k > 0$ then $\mathbf{A}$ satisfies the k-intersection property.*

Proof. Let $n > 0$ and $\mathbf{B}$ a subalgebra of $\mathbf{A}^n$. Let $P_\mathbf{B}$ be the instance of $\mathrm{CSP}(\Gamma_\mathbf{A})$ with variables x_i, $1 \leq i \leq n$, domain A and, for each subuniverse S of $\mathbf{A}^n$ with $B =_k S$, the constraint C_S having scope $(x_1, x_2, \ldots, x_n)$ and constraint relation S. Since each constraint relation of $P_\mathbf{B}$ is a non-empty subalgebra of $\mathbf{A}^n$ then it is in $\mathrm{CSP}(\Gamma_\mathbf{A})$. It is not hard to check that $P_\mathbf{B}$ is also k-minimal and so has a solution, since $\mathbf{A}$ is assumed to have relational width k.

A solution of $P_\mathbf{B}$ is an n-tuple in A^n that lies in each subuniverse S of $\mathbf{A}^n$ that is k-equal to B and so is in $\bigcap[\mathbf{B}]_k$. This establishes that $\mathbf{A}$ satisfies the k-intersection property. $\square$

Theorem 5.9. *Let $\mathbf{A}$ be a finite idempotent algebra. If $\mathbf{A}$ is of finite relational width then $\mathsf{HSP}(\mathbf{A})$ omits types* **1** *and* **2**.

Using results from [10] we present an alternate proof of part 2) of Theorem 2.9 and of Theorem 5.9 that avoids using the material from Section 3.

Let $\mathbf{A}$ be a finite idempotent algebra that has relational width k for some $k > 0$. Then by Lemma 5.8 $\mathbf{A}$ satisfies the k-intersection property. We will use Lemma 9.2 of [10] to show that $\mathcal{V} = \mathsf{HSP}(\mathbf{A})$ omits types **1** and **2**. Suppose not, and assume that **2** appears in the typeset of $\mathcal{V}$. Then we can find a finite algebra $\mathbf{C}$ in $\mathcal{V}$ that has a congruence β with $0_C \prec \beta$ and $\mathrm{typ}(\langle 0_C, \beta\rangle) = \mathbf{2}$. If we select some $\langle 0_C, \beta\rangle$-trace S, then it follows that the algebra $\mathbf{C}|_S$ is polynomially equivalent to a one-dimensional vector space over some finite field $\mathbf{F}$.

We now construct a special Mal'cev condition (see Definition 9.1 of [10]) $\mathcal{W}_\mathbf{F}$ that is interpretable in $\mathcal{V}$ but not in the variety of all $\mathbf{F}$-vector spaces and hence not interpretable in $\mathsf{HSP}(\mathbf{CI}_S)$, the variety generated by $\mathbf{C}|_S$ with normal indexing (see Definition 6.12 of [10]). This will contradict Lemma 9.2 of [10].

Let $\mathbf{W} = \mathbf{F}^{k+1}$ be the $k+1$ dimension vector space over $\mathbf{F}$ and let E be the subspace of W consisting of all tuples whose entries sum to 0. Let D be some disjoint coset of E in W. Note that E and D are both k-equal to F^{k+1}.

Let $\mathbf{B}$ be the free algebra in $\mathcal{V}$ generated by F (so we think of the elements of F as free generators in B). Let $\mathbf{E}'$ be the subpower of $\mathbf{B}^{k+1}$ generated by the set E and let $\mathbf{D}'$ be the subpower of $\mathbf{B}^{k+1}$ generated by D. The subuniverses E' and D' are k-equal since their generators have this property.

Since $\mathbf{A}$ satisfies the k-intersection property then $\mathbf{B}$ does and so the intersection of E' and D' is non-empty. Since $\mathbf{E}'$ and $\mathbf{D}'$ are generated by E and D respectively then there are two terms s and t of $\mathcal{V}$ that witness this, i.e., when s is applied to E and t is applied to D we obtain the same element in B^{k+1}. Since the components of the tuples in E and D are free generators of $\mathbf{B}$ this equality of tuples in B^{k+1} translates as a set of $k+1$ equations involving the terms s and t that hold in $\mathcal{V}$. This system can be viewed as a special Mal'cev condition $\mathcal{W}_{\mathbf{F}}$ that is interpretable in $\mathcal{V}$. $\mathcal{W}_{\mathbf{F}}$ cannot be interpreted into the variety of $\mathbf{F}$-vector spaces since by applying the interpretation of s and t as idempotent $\mathbf{F}$-vector space terms to the elements of E and D in $\mathbf{W}$ would lead to an element in the intersection of the disjoint sets E and D.

That $\mathcal{V}$ omits type $\mathbf{1}$ follows from Lemma 9.4 of [10] and the fact that for any finite field $\mathbf{F}$, the special Mal'cev condition $\mathcal{W}_{\mathbf{F}}$ is interpretable in $\mathcal{V}$ but not in $\mathcal{S}ets$, the variety of all sets. What this argument actually establishes is that $\mathcal{V}$ satisfies the condition:

> for every finite field $\mathbf{F}$ there is a special Mal'cev condition $\mathcal{W}_{\mathbf{F}}$ that is interpretable into $\mathcal{V}$ but not into the variety of all $\mathbf{F}$-vector spaces.

and that this condition implies that $\mathcal{V}$ omits types $\mathbf{1}$ and $\mathbf{2}$. Note that this condition is implied by (2) in Theorem 9.10 of [10] and hence is equivalent to it.

6. Conclusion

Part 1) of Theorem 2.9 provides a partial converse to part 2) of the theorem. In light of the conjectures of Larose-Zádori and Bulatov and the connection between the k-intersection property and the constraint satisfaction problem we propose the following two conjectures:

Conjecture 1: Let $\mathbf{A}$ be a finite idempotent algebra such that $\mathsf{HSP}(\mathbf{A})$ omits types **1** and **2**. Then for some $k > 0$, $\mathbf{A}$ satisfies the k-intersection property.

Conjecture 2: Let $\mathbf{A}$ be a finite idempotent algebra such that $\mathsf{HSP}(\mathbf{A})$ omits types **1**, **2**, and **5**. Then for some $k > 0$, $\mathbf{A}$ satisfies the strong k-intersection property.

Note that Theorems 3.4 and 3.5 provide converses to these conjectures.

Question 3: Assuming that $\mathbf{A}$ is finite, idempotent and generates a congruence distributive variety, is there any relationship between the least k for which $\mathbf{A}$ is in $CD(k)$ and the least m for which $\mathbf{A}$ has relational width m, assuming that such an m exists?

In connection with this question, we note that E. Kiss and the author have shown that if a finite idempotent algebra is in $CD(3)$ (and so has a sequence of four Jónsson terms) then $\mathbf{A}$ satisfies the strong 2-intersection property. We also show (in [14]) that such an algebra has relational width $|A|^2$ and hence is globally tractable.

It is not the case that every finite algebra with Jónsson terms satisfies the strong 2-intersection property. It is possible to construct a four element algebra having a 5-ary near unanimity term for which the strong 2-intersection property fails. Nevertheless, by the Baker-Pixley Theorem it follows that the algebra satisfies the strong 4-intersection property and in fact, by the main result of [12] we know that the algebra has relational width 4.

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