MATH 4LT3/6LT3 Assignment #5
Due: Friday, 24 November, by 11:59pm

Unless otherwise stated, in your solutions you may use the Axiom of Choice, or any equivalent statement that has been discussed in the lectures

1. (a) Show that the Pairset Axiom can be deduced from the Replacement Axiom. Hint: first show that there is some two element set S, using some of the other axioms, and then show, if A and B are sets, that there is some definite unary function h such that $h[S] = \{A, B\}$.

   (b) Show that the Separation Axiom can be deduced from the Replacement Axiom.

2. Let A be a set and $\chi(A) = (h(A), \leq_{\chi(A)})$ be the well order given by Hartog's Theorem. Show that $\leq_{\chi(A)}$ is a best well ordering of the set $h(A)$.

3. Let $\mathcal{N}$ be the class

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \dots\}.$$

   So, $\mathcal{N}$ contains the emptyset and satisfies the property that if $x \in \mathcal{N}$ then $x \cup \{x\}$ is also in $\mathcal{N}$. Use the Replacement Axiom to show that $\mathcal{N}$ is a set.

   This question is related to question #8 from Assignment #1. The usual Axiom of Infinity essentially asserts the existence of the set $\mathcal{N}$. In order to show that $\mathcal{N}$ exists as a set using the axioms from the textbook requires the use of the Replacement Axiom.

4. Using the Axiom of Regularity (or the Principle of Foundation, or the Axiom of Foundation) show that sets with the following properties cannot exist:

   (a) A set A such that $A = \{A\}$.

   (b) For some $n > 0$, a sequence of sets A_i, $0 \leq i \leq n$ such that $A_{i+1} \in A_i$, for $0 \leq i < n$, and $A_1 = A_n$.

5. Use the Axiom of Regularity to show that the construction from question #1 (b) of Assignment #2 satisfies the ordered pair property (OP1). It also satisfies (OP2), but you don't need to show that.

Bonus Question: For κ a cardinal, the cofinality of κ, denoted $cf(\kappa)$, is given in Definition 9.23 of the textbook.

1. Show that $cf(\aleph_0) = \aleph_0$ and that $cf(\aleph_1) = \aleph_1$.
2. The gimel function on the class of cardinals is defined by: $\gimel(\kappa) = \kappa^{cf(\kappa)}$. Use König's Theorem to show that for any cardinal κ, $\kappa <_c \gimel(\kappa)$.