

MATH 3GR3 Assignment #2
Due: Friday, October 6, by 11:59pm.

Upload your solutions to the Avenue to Learn course website.

1. Produce the Cayley table for the group $U(16)$, the group of units of \mathbb{Z}_{16} . Is this group cyclic?
2. Let G be a group and S a nonempty subset of G . Define the following relation on G :

$$a \sim b \text{ if and only if } s_1 a s_2 = b \text{ for some } s_1, s_2 \in S.$$

- (a) Show that if S is a subgroup of G then \sim is an equivalence relation on G .
 - (b) Compute the equivalence classes of \sim for the group of symmetries of the equilateral triangle, using the subgroup $S = \{id, \mu_1\}$.
 - (c) Show, by example, that if S is not a subgroup, then \sim need not be an equivalence relation.
3. Let $G = \mathbb{Z} \times \mathbb{Z}$. Define a binary operation \diamond on G as follows:

$$(a, b) \diamond (c, d) = (a + c, (-1)^c b + d).$$

- (a) Show that G with the operation \diamond is a group.
 - (b) Is this group cyclic? Justify your answer.
4. Let H and K be subgroups of the group G . Show that $H \cap K$ is a subgroup of G . Provide an example that shows that $H \cup K$ is not necessarily a subgroup of G .
 5. Let a and b be integers and define $K = \{na + mb \mid n, m \in \mathbb{Z}\}$. Show that K is a subgroup of \mathbb{Z} . Since every subgroup of \mathbb{Z} is cyclic, then K also has this property. Find a generator for K , and justify your answer.
 6. What is the order of the element 9 in the group \mathbb{Z}_{24} ? Does \mathbb{Z}_{24} contain an element of order 5?

7. (a) Let G be a finite **cyclic** group that has at least 2 elements. Prove that there is some $g \in G$ such that $|g|$ is a prime number.
- (b) Let G be a finite group that has at least 2 elements. Prove that there is some $g \in G$ such that $|g|$ is a prime number.
8. Suppose that G is a group and let $T = \{g \in G \mid \text{the order of } g \text{ is finite}\}$. Show that if G is abelian, then T is a subgroup of G . Find an example of a non-abelian group G for which T is not a subgroup.
9. Read over the SageMath tutorials at the ends of Chapters 3 and 4 and perform the following calculations. To submit your calculations, either take a screenshot (or maybe a picture) of the webpage that contains them or include a copy of the link (i.e., URL) that is produced by the SageCell “share” button, or just copy and paste your commands and the results into the document that you upload to Avenue to Learn.
 - (a) Produce the cyclic group \mathbb{Z}_{1000} using the “AdditiveAbelianGroup” command, and then determine the order of the element 30 in this group using the “order” command.
 - (b) Create the multiplicative cyclic group G that has order 20 and that is generated by the element a (So $G = \{1, a^2, a^3, \dots, a^{19}\}$). Use the command “ $G.<a>= \text{AbelianGroup}([20])$ ” to do this. To create the subgroup of G generated by the element a^k , use the command $H = G.\text{subgroup}([a^k])$. Produce the subgroup of G generated by the element a^6 and print out the order of this subgroup.

Supplementary problems from the textbook
(not to be handed in)

- From Chapter 3, questions 34, 35, 49, 50
- From Chapter 4, questions 1, 2, 3, 10, 23, 24, 25, 30, 39