MATH 3GR3 Midterm Test #2 Sample Questions

1. (a) State Lagrange's Theorem.

   (b) Let G be a group and suppose that $a \in G$ has order 2. Show that $\{e, a\}$ is a subgroup of G.

   (c) Show that if G is a finite group that contains an element a of order 2, then $|G|$ is an even number.

2. Determine which of the following pairs of groups are isomorphic. Justify your answers to receive credit.

   (a) $\mathbb{Z}$ and $\mathbb{R}$.

   (b) $\mathbb{Z}$ and $3\mathbb{Z}$.

   (c) $\mathbb{Z}_6$ and S_3.

3. Consider the group $GL_3(\mathbb{R})$ of all 3×3 invertible matrices over $\mathbb{R}$, with group operation the usual matrix multiplication, and let

$$H = \left\{ \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{pmatrix} : r \neq 0 \right\} \text{ and } K = \{A \in GL_3(\mathbb{R}) : \det(A) = 1\}.$$

   (a) Show that H and K are subgroups of $GL_3(\mathbb{R})$.

   (b) Show that $GL_3(\mathbb{R})$ is isomorphic to $H \times K$. You may present an explicit isomorphism (with proof) between these two groups to establish the isomorphism, or prove that $GL_3(\mathbb{R})$ is the internal direct product of H and K.

4. Let G be a group and let H and N be subgroups of G. Show that if N is a normal subgroup of G then $HN = \{hn : h \in H \text{ and } n \in N\}$ is a subgroup of G.

5. Let G be a group and H a subgroup of G of order n.

   (a) If $g \in G$, show that the set gHg^{-1} is also a subgroup of G and that this subgroup has order n.

   (b) Suppose that H is the only subgroup of G that has order n. Show that H is a normal subgroup of G.