INDEPENDENCE OF THE CONTINUUM HYPOTHESIS

CAPSTONE

MATT LUTHER

1. Introduction

This paper will summarize many of the ideas from logic and set theory that are needed in order to follow Paul Cohen's proof that the Continuum Hypothesis is not implied by the axioms of Zermelo-Fraenkel set theory with Choice. We will then trace the majority of his proof, in which we will see the technique of forcing and how it is used to build a model in which the Continuum Hypothesis fails.

2. History

Georg Cantor began development of set theory in the 1870s while investigating trigonometric series and the structure of the real numbers. Cantor used the existence of one-to-one correspondences between the elements of sets to compare their sizes, or cardinalities. His first major results regarding the comparison of cardinalities of sets came in 1874 when he published a paper containing a proof that the set of algebraic numbers could be put in one-to-one correspondence with the natural numbers and another proof that there was no such correspondence between the real numbers and the naturals. The latter of these proofs established for the first time that there are infinite sets of different sizes.

Over the next two decades, Cantor expanded his theory, and by the 1890s had developed the transfinite cardinal and ordinal numbers, which serve as representatives for the different sizes and order types of sets, respectively. Among his publications were proofs that the cardinality of the reals is the same as the cardinality of the power set of the naturals, $\mathcal{P}(\mathbb{N})$, and that for any set X, its power set $\mathcal{P}(X)$ has a larger cardinality. Cantor proved that by taking successive power sets, one could always find higher cardinality sets. However, it was unclear whether or not the cardinality of $\mathcal{P}(X)$ is the next cardinal, κ^+, whenever the cardinality of X is some infinite cardinal κ. That is, it was unclear whether the power set operation always gives the next infinite cardinal, or if there might be some cardinal between $|X|$ and $|\mathcal{P}(X)|$. He proposed that the cardinality of the reals $|\mathbb{R}|$, and equivalently $|\mathcal{P}(\mathbb{N})|$, was indeed the next smallest cardinal after the size of $\mathbb{N}$, but he was unable to prove this. This claim is known as the Continuum Hypothesis.

The transfinite cardinals begin at the cardinality of the naturals, which is denoted by the cardinal number $\aleph_0$. Sets with cardinality $\aleph_0$ are called countable. Assuming the Axiom of Choice, we have that the next smallest cardinal after $\aleph_0$ is the cardinality of the set of all ordinals with cardinality less than or equal to $\aleph_0$, that is, the set of all at-most countable ordinals. This next smallest cardinal is denoted $\aleph_1$. We also know that the size of $\mathcal{P}(X)$ for any set X is the cardinality of the set of functions from X to the set $\{0, 1\}$, which is $2^{|X|}$. Thus we can write the Continuum Hypothesis (CH) quite succinctly as

$$\aleph_1 = 2^{\aleph_0}$$

This can be read as stating that the next cardinality after $\aleph_0$ is the size of the power set of any set with cardinality $\aleph_0$. It can also be read as saying that the set of all ordinals with cardinality $\leq \aleph_0$ is the same size as the collection of all subsets of a set with cardinality $\aleph_0$. More tangibly, this claims that the size of the set of reals is the next smallest cardinality after the size of the naturals. Cantor made a general version of this claim, reasonably named the Generalized Continuum

Hypothesis (GCH), which is written as

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha} \text{ for all ordinals } \alpha$$

This says that the next cardinality after $|X|$ is always the size of the power set, $|\mathcal{P}(X)|$. In the absence of the Axiom of Choice, we do not know what the smallest cardinal after $\aleph_\alpha$ is, so the GCH would be stated as follows.

Given an infinite cardinal κ, there exists no cardinal λ such that $\kappa < \lambda < 2^\kappa$

In the years after Cantor's development of what would later became known as naive set theory, work was done to axiomatize the intuitive idea of sets. In 1908, Ernst Zermelo put forth his axiomatic set theory, which eventually became Zermelo-Fraenkel set theory with the Axiom of Choice. Zermelo-Fraenkel set theory is commonly abbreviated as ZF when treated without the Axiom of Choice, and abbreviated ZFC when the Axiom of Choice is included. We will abbreviate the Axiom of Choice as AC when thought of as a statement on its own. The major work regarding the Continuum Hypothesis following Cantor's investigations would later all take place within ZF and ZFC.

Little progress was made regarding the truth of the Continuum Hypothesis until 1938 when Kurt Gödel established that if the set ZF of statements that serve as the axioms of ZF set theory is consistent, then there is a model of ZF in which the Generalized Continuum Hypothesis and the Axiom of Choice are both true. As a result of Gödel's completeness theorem from logic, the existence of such a model implies that neither GCH nor CH can be disproven in either ZF or ZFC set theory, provided that ZF set theory is consistent. This however did not demonstrate that either form of the Continuum Hypothesis is true or provable, just that they cannot be disproven in these theories.

The question of the Continuum Hypothesis again went without much progress until 1963 when Paul Cohen showed that if the set of statements ZF which make up the axioms of ZF set theory is consistent, then there is a model of ZF in which the Generalized Continuum Hypothesis fails but the Axiom of Choice holds. For the same reason that Gödel's result shows that $\neg GCH$ cannot be proven, Cohen's result shows that CH cannot be proven in ZFC set theory either.

The end result of Gödel and Cohen's work is that regardless of whether we are working in ZF or ZFC set theory, both the Continuum Hypothesis and Generalized Continuum Hypothesis cannot be proven or disproven. Statements such as these which cannot be proven true or false in some theory are called independent of the theory.

3. Models, Consistency, and Independence

We will begin with several concepts from mathematical logic which will allow us to better understand exactly what it means for the Continuum Hypothesis to be independent and what the proofs by Gödel and Cohen actually say.

Definition 3.1. A **theory** is a set of sentences in a formal language.

For example, we may have a theory T_O intended to represent strictly ordered sets which has only the following two statements as elements.

- The Axiom of Asymmetry: there are no a, b such that $a < b$ and $b < a$.
- The Axiom of Transitivity: for all a, b, c, if $a < b$ and $b < c$, then $a < c$.

Definition 3.2. A **model** for a theory T is a structure that tells how to interpret the non-logical symbols of the language that T is in, so that every statement in the theory T is true.

Models provide a domain for variables to range over and provide definitions for the relations like the $<$ sign that appears in our above sentences. To continue with our example theory T_O, consider the structure M where we let our variables range over the set $\{0,1,2\}$ and we define $<$ as the usual less-than relation on integers. We can see that M is a model for T_O because

- The Axiom of Asymmetry holds: for any two numbers selected from $\{0,1,2\}$, at least one of them is not less than the other.
- The Axiom of Transitivity holds: for any a, b, c in $\{0,1,2\}$, if a is less than b and b is less than c, then a is less than c.

However, if we consider the structure N where the variables range over the same set $\{0,1,2\}$ but the relation $<$ is defined by $a < b$ if and only if $b = a + 1$, then we see that N is not a model for T_O because the Axiom of Transitivity is false; we have $0 < 1$ and $1 < 2$, yet $0 < 2$ is false since $2 \neq 0 + 1$.

Definition 3.3. We say that a statement S is **provable** in a theory T if there exists a proof of S which may contain the statements in T as assumptions. That is, S is provable in T if there exists a finite sequence of statements $(S_1, S_2, ..., S_n)$ such that each statement S_i is either a statement in T or follows from the previous statements by some rule of logic.

The following theorem relates the provable statements of a theory to the models of that theory.

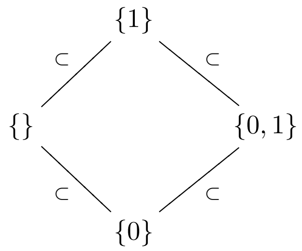
Theorem 3.4. ***(The Soundness Theorem):*** *If a statement S is provable in a theory T, then S is true in every model of T.*

Simply put, this says that if we can deduce a statement from a set of assumptions, then that statement is true in every structure satisfying those assumptions. Equivalently, this theorem tells us that if we have a theory T and a model for it in which some statement S is false, then there cannot be a proof of S in T.

Now, given a statement and a theory, we might wonder if we can prove that statement in that theory. Consider again our theory T_O of strictly ordered sets. If we were to propose an additional axiom for our theory, we would be interested in whether or not the axiom was already provable in T_O or if it conflicted with the axioms we already have in the theory. Suppose the following axiom is suggested.

- The Axiom of Trichotomy: for all a, b, we have $a < b$ or $a = b$ or $b < a$.

By the Soundness Theorem, if we could prove the Axiom of Trichotomy in T_O, then the axiom would have to be true in any model of T_O. But, consider the structure M_1 where variables range over the set $S = \{\{\}, \{0\}, \{1\}, \{0,1\}\}$ and our relation $<$ is taken to be the proper subset relation $\subset$ between sets, as indicated in the diagram below.



This structure is a model for T_O, because no two of the elements in S are proper subsets of each other, and the subset relation is transitive. However, the Axiom of Trichotomy fails in M_1, because we have $\{0\}$ and $\{1\}$ in S, but $\{0\} \not\subset \{1\}$, $\{0\} \neq \{1\}$, and $\{1\} \not\subset \{0\}$. Thus, because the Axiom of Trichotomy fails in a model of T_O, the Soundness Theorem tells us that the Axiom of Trichotomy is not provable in T_O.

Since the axiom is not provable in T_O, and we have seen that it fails in at least one model, we might wonder if we could disprove it. That is, we would be interested in whether we could deduce the negation of the Axiom of Trichotomy in our theory. However, if we consider again our earlier model M where variables range over $\{0, 1, 2\}$ and our relation is simply the usual $<$ from the integers, we see that the Axiom of Trichotomy holds. The Soundness Theorem now tells us that the negation of the axiom is not provable in T_O, because if the negation were provable, then the axiom would have to be false in every model, but here we have a counterexample.

Definition 3.5. We say that a theory T is **consistent** if there is no statement S such that both S and its negation, $\neg S$, are provable in T. That is, T is consistent if no contradiction can be deduced from it. We extend this notion and say that a statement S is **consistent with** a theory T if the theory $T \cup \{S\}$ is consistent.

Another theorem establishes the relationship between models and consistency.

Theorem 3.6. ***(The Completeness Theorem)****: A theory is consistent if and only if it has a model.*

Thus we can quickly see from our above examples that both the Axiom of Trichotomy and its negation are consistent with the theory of strictly ordered sets, because we have found a model for both $T_O \cup \{$the Axiom of Totality$\}$ and $T_O \cup \{$the negation of the Axiom of Totality$\}$. This leads us to our next definition.

Definition 3.7. We call a statement S **independent** of a theory T when both S is consistent with T and $\neg S$ is consistent with T.

Equivalently, in light of the above theorem, we can say that a statement S is independent of a theory T if and only if there exists a model M_S for $T \cup \{S\}$ and a model $M_{\neg S}$ for $T \cup \{\neg S\}$. Remembering that having a model in which a statement fails means that the statement cannot be proven from the theory, we can state this in yet one more equivalent way: a statement is independent of a theory if and only if it cannot be proven or disproven from the theory.

So, we can now see exactly what is meant by the independence of the Continuum Hypothesis. It is a statement CH such that both $ZFC \cup \{CH\}$ and $ZFC \cup \{\neg CH\}$ are consistent, and as such, neither $\neg CH$ nor CH is provable in ZFC.

Gödel's consistency proof builds a model of $ZF \cup \{GCH,\ AC\}$, that is, a model where all of the axioms of ZF set theory hold, the Generalized Continuum Hypothesis is true, and the Axiom of Choice is true. This establishes both the consistency of $ZF \cup \{GCH\}$ and $ZFC \cup \{GCH\}$, and thus that GCH and CH cannot be proven false in either ZF or ZFC set theory. Cohen's model establishes the consistency of $ZF \cup \{\neg CH,\ AC\}$, and thus that CH cannot be proven in ZF or ZFC set theory. Since GCH implies CH, this additionally means that GCH cannot be proven in these theories either.

Both Gödel and Cohen assume the existence of some model of ZF at the start of their proofs, and as such, both of their proofs are contingent on the consistency of ZF. However, note that if ZF is not consistent, then it contains a contradiction and thus any statement is provable in it anyway.

4. Some Concepts from Set Theory

Before we can begin finding the models we need, we require a few definitions and properties relating to ordinal numbers.

Definition. A relation $<$ **orders** a set S if

(1) For all x, y in S, exactly one of the following holds.

$$x = y,\ x < y,\ y < x$$

(2) If $x < y$ and $y < z$, then $x < z$.

Definition. A relation $<$ **well-orders** a set S if $<$ orders S and for any nonempty subset $B \subseteq S$, there exists an $x \in B$ such that for any $y \in B$, we have either $x = y$ or $x < y$.

A relation thus well-orders a set if every non-empty subset has a least element with respect to that relation. We say that a set S is **well-ordered** if S is a set with a relation $<$ such that $<$ well-orders S.

A familiar example of a well-ordered set is the set of natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$ under its usual ordering. Given a well-ordered set S we can see that any nonempty subset $X \subseteq S$ is also well-ordered, since if Y is a nonempty subset of X, then Y is a nonempty subset of S, and so Y has a least element since S is well-ordered. Thus, any nonempty subset of the natural numbers will also serve as an example.

Definition. A set x is **transitive** if $z \in y$ and $y \in x$ implies that $z \in x$.

A simple example of a transitive set is the set $x = \{\emptyset, \{\emptyset\}\}$. The only elements of this set are $y_0 = \emptyset$ and $y_1 = \{\emptyset\}$. To see that x is a transitive set, we need to check that every element of y_o is in x and that every element of y_1 is in x. We can see that y_0 has no elements, and we see that the only element $z \in y_1$ is the set $\emptyset$, which is an element of x.

Definition. An **ordinal** is a set α which is transitive and well-ordered by the set membership relation $\in$. We denote that a set α is an ordinal by $On(\alpha)$.

In set theory, we develop natural numbers by induction in the following way.

$$\begin{array}{ccc} 0 & = & \emptyset \\ 1 & = & \{\emptyset\} \\ 2 & = & \{\emptyset, \{\emptyset\}\} \\ 3 & = & \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\ & \text{and in general} & \\ n+1 & = & n \cup \{n\} \end{array}$$

Since each successive number contains all of the elements of the previous number, we see that each n is transitive. This development also ensures that each natural number is a set well-ordered by $\in$. Thus, each natural number is an ordinal. This construction causes $\in$ to act identically like the usual less-than relation between natural numbers, and so when the entire set $\mathbb{N} = \{0, 1, 2, ...\}$ is considered, we can see that it is also an ordinal.

In a similar way to how the cardinal numbers generalize the natural numbers and convey information about size, the ordinals generalize the natural numbers and convey information about order. Sets have the same cardinality when there is a one-to-one, onto function between them. Two sets with order relations, $(X, <_X)$ and $(Y, <_Y)$, have the same order type when there is a one-to-one, onto function f between them such that $x_1 <_X x_2$ if and only if $f(x_1) <_Y f(x_2)$. One example of the distinction between these ideas can be seen by comparing the cardinalities and order types of the natural and rational numbers. Both $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality, because there is a one-to-one correspondence between these sets. However, they have different order types when considered under their usual orderings, because any one-to-one, onto function f between $\mathbb{N}$ and $\mathbb{Q}$ which preserves order as required would have to assert the existence of some $n \in \mathbb{N}$ such that $1 < n < 2$, since $1 < 2$ implies $f(1) < f(2)$, but $f(1)$ and $f(2)$ are distinct rational numbers, and so there is some rational between them that is the output for some $n \in \mathbb{N}$. This rational $f(n)$ is such that $f(1) < f(n) < f(2)$, and so we must have $1 < n < 2$, which is impossible. This example demonstrates that there are distinct ordinals with the same cardinality.

The ordinals serve as representatives for the order types of sets, similarly to how cardinals serve as representatives for the sizes of sets. An important thing for us to know as we move forward is that we can perform transfinite induction on ordinals. This lets us extend the idea of strong induction on the integers to any well-ordered set, including infinite sets with cardinality larger than $\aleph_0$. The models we consider will involve inductively building up structures by starting at 0, and then worrying about what happens at some ordinal α when every earlier ordinal $\beta < \alpha$ has been taken care of.

5. Concepts from Gödel's Model

In order to carry out his consistency proof, Gödel builds a model featuring only certain kinds of sets that arise in a particular way. The ideas and theorems involved in Gödel's proof play a significant role in Cohen's work.

The Axiom of Replacement in ZF set theory allows very liberal use of properties to define new sets. These properties can do things such as range over collections of sets including even the set that is currently being defined. Gödel's idea was essentially to restrict the properties that are acceptable for this axiom so that they range only over the sets that were already defined. By starting out with the empty set, and iteratively building up new sets in this way, we end up with what are called the constructible sets.

Definition 5.1. Let A be a formula, and let X be a set. We denote by A_X the formula A with all of its variables which are bound by a quantifier restricted so that they range only over the set X. We say that A_X is the formula A **restricted to** X.

What the above definition says is that, starting with the formula A that contains some quantifiers such as "for all x" and "there exists y" in it, A_X is the formula one gets when A is changed so that each quantifier is instead read as "for all $x \in X$" and "there exists $y \in X$".

Definition 5.2. Let X be a set. We define X' as the union of X and the set of all sets Y for which there is some formula $A(z, t_1, ..., t_k)$ in ZF such that for some $t_1^*, ..., t_k^*$ in X, we have $Y = \{z \in X \mid A_X(z, t_1^*, ..., t_k^*)$.

This definition gives us that X' contains all of the elements of X, along with all of the sets Y that can be built by using a formula which has variables ranging only over X. This is the set of everything that can be built from X using a formula restricted to X.

Definition 5.3. Given an ordinal α, define M_α by $M_0 = \emptyset$ and $M_\alpha = (\bigcup_{\beta<\alpha} M_\beta)'$.

Note that in the above definition, the set M_α is of the form X'. We start with $M_0 = \emptyset$, then iteratively make these M_α which consist of all of the elements of the previous sets along with all of the new sets that can be created using formulas restricted to the previous sets. This definition ensures that each M_α is transitive, since it continues to include every element of M_α into all future sets, each of which is only adding subsets of the previous sets.

Definition 5.4. A set x is **constructible** if there is an ordinal α such that $x \in M_\alpha$. We say that x is **constructed at** α or at stage α when this is the case.

So, the constructible sets are the sets gained in this method of iteratively building up new sets from the empty set using only formulas restricted to the sets which were already constructed at a previous stage.

Definition 5.5. Given an ordinal α, we define X_α as $\bigcup_{\beta<\alpha} M_\beta$. We say that a set x is **constructed before** stage α if we have $x \in X_\alpha$.

Note that we could now also write $M_\alpha = (X_\alpha)'$ for each ordinal α. The X_α are the sets of elements that exist by the time stage α of the construction is taking place, but X_α does not yet include the new elements that will be added at this stage. The X_α collect up all the existing elements by the time we get to α, the M_α collect up the elements, and additionally include the new sets that can be constructed. We define these terms separately so that we can more easily speak about when an element is constructed. Note, however, that elements which are constructed before α are also in a sense "reconstructed" at α, because we have $X_\alpha \subseteq M_\alpha$. These phrases do not necessarily refer to the minimal α for which x begins to appear.

Notation 5.6. We denote by L the class of all constructible sets, and we denote by V the class of all sets.

Gödel's proof that $ZF \cup \{GCH\}$ is consistent if ZF is consistent comes as the result of three theorems. The first theorem is the verification that L satisfies the axioms of ZF set theory and thus is a model of ZF. The next theorem verifies that L additionally satisfies the statement $V = L$; that is, when the relativized constructible sets are built inside of the actual class of constructible sets, the result is the same as when the constructible sets are built in ZF set theory as a whole. The final theorem establishes that $(V = L) \rightarrow (AC \ \& \ GCH)$ holds within ZF set theory. Taken together, these theorems establish that the Axiom of Choice and the Generalized Continuum Hypothesis hold within L, and thus they are both consistent with the axioms of ZF set theory.

Before we move on to developing Cohen's model, there are a couple more definitions we will need which relate to constructibility.

Definition 5.7. The **transitive closure** of a set X is the set $\bigcup_n X_n$ where $X_0 = X$ and $X_{n+1} = \bigcup X_n$.

That is, the transitive closure of X is the union over all successive unions starting at X, then $\bigcup X$, then $\bigcup(\bigcup X)$, and so on. This ensures that the transitive closure of a set is actually transitive by adding the necessary elements.

Definition 5.8. Given a set y, we define $M_\alpha(y)$ inductively by setting $M_0(y)$ as the transitive closure of y, and $M_\alpha(y) = (\bigcup_{\beta<\alpha} M_\beta(y))'$. Likewise, we define $X_\alpha(y)$ as $\bigcup_{\beta<\alpha} M_\beta(y)$. We say that an element $x \in M_\alpha(y)$ is **constructed from** y **at** α, and similarly we say that an element $x \in X_\alpha(y)$ is **constructed from** y **before** α.

The above definition just generalizes the earlier definitions about constructibility by allowing the process to begin at a set other than the empty set. We will however continue to simply say that x is constructible if it is constructed from the empty set at some ordinal. Note that a set which is constructible from some set is not necessarily constructible from the empty set, and so we may have that a set x is constructible from a set a, and yet x is not in L, the class of constructible sets. This distinction is important to keep in mind when Cohen's model is considered, because we will see that there are nonconstructible sets which are nevertheless constructed from some other set. This allows Cohen's model to fail $V = L$ while still maintaining a convenient structure.

6. The Minimal Model

Within Cohen's proof that there is a model for $ZF \cup \{\neg CH\}$, we will assume the following additional axiom.

Axiom 6.1. *The Standard Model Axiom (SM): There is a set M with a relation $R = \{(x, y) \mid x \in y \ \& \ x \in M \ \& \ y \in M\}$ such that M is a model for ZF under the relation R.*

This axiom gives us not just the existence of a model for ZF, but a model where the relation $\in$ in the axioms of ZF is interpreted simply as the usual set membership relation restricted to the elements of some set M. A model which interprets $\in$ in such a way is called **standard**. The above axiom allows us to infer the existence of a particular model which will be useful.

Theorem 6.2. *$ZF \cup \{SM\}$ implies that there exists a unique transitive countable model $\mathcal{M}$ such that if N is any standard model, then there is an isomorphism from*

$\mathcal{M}$ onto a subset of N which preserves the $\in$ relation. Furthermore, $\mathcal{M}$ satisfies $V = L$.

Notation. Throughout the rest of this paper, $\mathcal{M}$ will refer to the unique minimal model of ZF whose existence is given by the above theorem. We have that $\mathcal{M}$ is standard, transitive, and countable, and it satisfies $V = L$. Additionally, α_0 will denote $sup\{\alpha \mid \alpha \in \mathcal{M}\}$, the supremum of the ordinals in $\mathcal{M}$.

There is one more theorem concerning this model which will be useful later on.

Theorem 6.3. *For each element x in $\mathcal{M}$, there is a formula $A(y)$ in ZF such that x is the unique element of $\mathcal{M}$ that satisfies $A_{\mathcal{M}}(x)$, the formula A restricted to the elements of the model $\mathcal{M}$.*

This theorem asserts that there is a way to uniquely identify each of the x in $\mathcal{M}$ using formulas in ZF.

7. Motivation for Cohen's Model

Gödel's proof of the consistency of the Generalized Continuum Hypothesis involved establishing that $(V = L) \to (AC \ \& \ GCH)$ holds within ZF set theory. This tells us that if we are to find a model where GCH fails, that model must fail to satisfy $V = L$. So, we will need to build a model which contains sets that are not constructible relative to that model.

Moreover, the following theorem tells us that we cannot hope to find such a model by restricting the sets of ZF to those which satisfy some formula.

Theorem 7.1. *Let $A(x)$ be any formula in ZF set theory. It cannot be proven in ZF that the axioms of ZF and the statement $\neg(V = L)$ hold when restricted to the class of all sets x which satisfy $A(x)$.*

Proof. Assume we have a formula $A(x)$ for which it can be proven that the class of all x for which $A(x)$ is true satisfies $ZF \cup \{\neg(V = L)\}$. We will show that this leads to a contradiction.

Recall that $\mathcal{M}$ is the unique minimal model, and let $\mathcal{M}^* = \{x \mid x \in \mathcal{M} \ \& \ A_{\mathcal{M}}(x)\}$. Now, since $\mathcal{M}$ is a standard model of ZF, and using the assumption, we have that $\mathcal{M}^*$ is a standard model of ZF which also satisfies $\neg(V = L)$.

We have that $\mathcal{M}^* \subseteq \mathcal{M}$ by definition, and by Theorem 6.2 there is an isomorphism from $\mathcal{M}$ onto a subset of $\mathcal{M}^*$, so we must have that $\mathcal{M}$ and $\mathcal{M}^*$ are isomorphic. Thus, since $V = L$ holds in $\mathcal{M}$, it must also hold in $\mathcal{M}^*$. Hence $\mathcal{M}^*$ satisfies both $V = L$ and $\neg(V = L)$, which is impossible. □

The above two results make it clear that if we wish to find a model for $ZF \cup \{\neg CH\}$, we will need sets that appear non-constructible relative to the model, and we cannot find this model by restricting some collection of sets or by looking inside a model like $\mathcal{M}$; we will have to add new sets. Cohen's forcing technique gives us a way to introduce the new sets we need.

8. Forcing

Plainly, forcing will allow us to introduce a new set by using a sequence of finite sets of membership statements. The idea is similar to trying to specify the set of even natural numbers $E = \{0, 2, 4, ...\}$ by the following sequence.

$$\begin{aligned} P_0 = &\quad \{0 \in E\} \\ P_1 = &\quad \{0 \in E, 2 \in E\} \\ P_2 = &\quad \{0 \in E, 2 \in E, 4 \in E\} \\ &\quad \vdots \\ P_n = &\quad \{2 \cdot m \in E \mid m \leq n \;\&\; m \in \mathbb{N}\} \end{aligned}$$

Note that for no single n does P_n contain enough information to uniquely determine the set, and it is this vagueness in the determination of the set that we want to take advantage of. Forcing will allow us to add sets to the model $\mathcal{M}$ for which any question about membership will be eventually resolved, and yet a set determined in this way provides a minimal enough amount of information so that relative to the model, it will not appear constructible.

The following concept is necessary because as we are building our new model, we will need a way to refer to its members before the model has been explicitly defined. To facilitate this, we will essentially label formulas that will later be satisfied by the elements of the new model.

Definition 8.1. A **labeling** is a mapping defined in ZF, which assigns to each ordinal $0 < \alpha < \alpha_0$

- a set S_α called a **label space**,
- a function ϕ_α defined over S_α

We define $S = \bigcup_\alpha S_\alpha$, and we have a subset $G \subset S$ whose elements are called the **generic sets**. The sets S_α are mutually disjoint, and if $c \in S_\alpha$, then $\phi_\alpha(c)$ is a formula $A(x)$ which may contain elements from S_β with $\beta < \alpha$ as constants. We will eventually denote by $\bar{c}$ the set of elements that that satisfy the formula given by $\phi_\alpha(c)$.

In our usage of this idea, each $c \in S$ will act as a label for a unique formula defined over the variables and constants that came before c. Since each S_α is required to be disjoint from the others, we have that each $c \in S$ is in a unique S_α. The set G of generic sets is a collection of labels, some of which will refer to the sets that we will use forcing to identify and introduce to our model.

Notation. We will write $\forall_\alpha x$ or $\exists_\alpha x$ to indicate that a variable x bound by a quantifier is restricted to the set X_α.

Definition 8.2. A **limited statement** is a statement in ZF in which every quantifier is of the form $\forall_\alpha$ or $\exists_\alpha$ for some ordinal $\alpha < \alpha_0$ and in which elements of S may appear as constants.

Definition 8.3. An **unlimited statement** is a statement in ZF in which elements of S may appear as constants.

Definition 8.4. Given a generic set G, a **forcing condition** P is a finite set of limited statements each of the form $n \in a$ or $\neg(n \in a)$, where $n \in \mathbb{N}$ and $a \in G$, and where for any such n and a, at most one of $(n \in a)$ and $\neg(n \in a)$ is in P.

The last part of the above definition ensures that no forcing condition contains conflicting information as to whether something is an element of one of the sets a.

Definition 8.5. A **condition set** is a set U of forcing conditions defined in $\mathcal{M}$ and a relation $<$ on U, also defined in $\mathcal{M}$, such that $<$ is reflexive and transitive, along with a map ψ in $\mathcal{M}$ such that for all forcing conditions $P \in U$, $\psi(P)$ is some set of statements of the form $c_1 \in c_2$ with c_2 in G, and if $c_2 \in S_\alpha$, then $c_1 \in S_\beta$ for some $\beta < \alpha$. The relation $<$ is required to satisfy the property that whenever $P < Q$, we have $\psi(P) \subseteq \psi(Q)$.

Now we will define a hierarchy on the limited statements which will be used for induction in the definitions and proofs that follow.

Definition 8.6. Given a limited statement A, we define $rank(A) = (\alpha, i, r)$ where

- α is the least ordinal such that whenever $\forall_\beta$ or $\exists_\beta$ appear in A, we have $\beta \leq \alpha$, and whenever $c \in S_\beta$ appears in A, we have $\beta < \alpha$.
- $i = 0$ if $\alpha = \beta + 1$ for some ordinal β and neither $\forall_\alpha$ nor $\exists_\alpha$ appear in A, and no term of the form $c \in x$, $c = x$, or $x = c$ appears in A with $c \in S_\beta$. Otherwise, $i = 1$.
- r is the length of the statement A; i.e., it is the number of symbols in A.

The way α is chosen in the above definition is directly tied to the definition of X_α and ensures that if $rank(A) = (\alpha, i, r)$, then A only references variables and constants from X_α.

Now, with $rank(A)$ defined, we can define the forcing of statements.

Definition 8.7. Given a labeling with $S = \bigcup_\alpha S_\alpha$, a generic set $G \subset S$, and a condition set U, we define P **forces** A, where P is a forcing condition in U and A is a limited statement, by induction on $rank(A)$ as follows.

(1) P forces $\exists_\alpha x\ B(x)$ if for some $c \in S_\beta$ with $\beta < \alpha$, P forces $B(c)$.
(2) P forces $\forall_\alpha x\ B(x)$ if for all forcing conditions $Q \supseteq P$ and all $c \in S_\beta$ with $\beta < \alpha$, we have that Q does not force $\neg B(c)$.
(3) P forces $\neg B$ if for all $Q \supseteq P$ we have that Q does not force B.
(4) P forces $B \& C$ if P forces B and P forces C.
(5) P forces $B \vee C$ if P forces B or P forces C.
(6) P forces $A \to B$ if P forces $\neg A$ or P forces B.
(7) P forces $A \leftrightarrow B$ if P forces $A \to B$ and P forces $B \to A$.
(8) P forces $c_1 = c_2$, where $c_1 \in S_\alpha$, $c_2 \in S_\beta$, and $\gamma = max(\alpha, \beta)$, if either $\gamma = 0$ and $c_1 = c_2$ as elements in S_0, or if $\gamma > 0$ and P forces $\forall_\gamma x\ (x \in c_1 \leftrightarrow x \in c_2)$.
(9) P forces $c_1 \in c_2$, where $c_1 \in S_\alpha$, $c_2 \in S_\beta$ with $\alpha < \beta$, if
    (a) $\neg c_2 \in G$ and P forces $A(c_1)$ where $A(x) = \phi_\beta(c_2)$, the unique formula corresponding to c_2 in the labeling.
    (b) $c_2 \in G$ and for some $c_3 \in S_\gamma$ with $\gamma < \beta$, we have $\{c_3 \in c_2\} \in \psi(P)$ and P forces $c_1 = c_3$.
(10) P forces $c_1 \in c_2$ where $c_1 \in S_\alpha$ and $c_2 \in S_\beta$ with $\alpha \geq \beta$, if for some $c_3 \in S_\gamma$ with $\gamma < \beta$, we have P forces $\forall_\alpha x\ (x \in c_1 \leftrightarrow x \in c_3)\ \&\ (c_3 \in c_2)$.

We will soon see an analogous list of definitions for unlimited statements, but first we will consider two examples that explore the consequences of the definition of forcing.

Example 8.8. Suppose we have the following sequence that is intended to describe a set a.

$$\begin{aligned} P_0 &= \{1 \in a\} \\ P_1 &= \{1 \in a, 2 \in a\} \\ &\vdots \\ P_n &= \{m \in a \mid 0 < m \leq n\} \end{aligned}$$

The sequence of conditions first asserts $1 \in a$ and then at each step asserts that the next natural number is also in a, along with all of the assertions from the previous condition. We can see that for no n does P_n ever contain the statement $0 \in a$, and so for every n we have that P_n does not force $0 \in a$. Yet, we do not have that any P_n forces $\neg(0 \in a)$ either, because each P_n could be contained in a forcing condition Q_n which forces $0 \in a$. In particular, we can simply let $Q_n = P_n \cup \{0 \in a\}$. Interestingly, this means that every P_n in this sequence fails to force $(0 \in a) \vee \neg(0 \in a)$.

On the other hand, every P_n in the sequence forces $1 \in a$, and also forces $\neg\neg(1 \in a)$ because for any $Q \supseteq P_n$, we have $(1 \in a) \in Q$, and so there exists an $R \supseteq Q$ which forces $(1 \in a)$ since R contains this statement; Q itself can serve as this R. Thus Q does not force $\neg(1 \in a)$ for any $Q \supseteq P_n$, and we have that P_n forces $\neg\neg(1 \in a)$.

We will later only consider certain sequences which eventually, for any statement A, force either A or $\neg A$ at some step in the sequence.

Example 8.9. Consider the statement "a is infinite" written as $\forall_1 x\ \exists_1 y\ (y > x \,\&\, y \in a)$. Cohen notes that while $>$ is not an admissible symbol in our statements, we can informally see how every forcing condition P would force that "a is infinite". This is because for any natural number n, no P forces the statement "there is no element of a greater than n", $\neg\exists_1 y\ (y > n \,\&\, y \in a)$, since P can always be contained in a larger forcing condition Q which has a statement $y \in a$ where y is larger than n.

We now proceed with the definition of forcing for unlimited statements.

Definition 8.10. P **forces** A, where P is a forcing condition and A is an unlimited statement, is defined by induction on the length of A by the following.

(1) P forces $\exists x\ B(x)$ if for some $c \in S$, P forces $B(c)$.
(2) P forces $\forall x\ B(x)$ if for all $c \in S$, if $Q \supseteq P$, then Q does not force $\neg B(c)$.

The definitions for P forces $\neg B$, $B \& C$, $B \vee C$, $B \to C$, $B \leftrightarrow C$, $c_1 \in c_2$, and $c_1 = c_2$ are exactly as in the case for limited statements.

9. Properties of Forcing Conditions

In this section we will prove a few results regarding when a condition forces a statement, and we will prove the existence of a sequence which, given some statement A, will force either A or $\neg A$ after a finite number of steps. It is this sort of sequence with which we are primarily concerned.

Throughout this section, A denotes a statement which is either limited or unlimited, and P, Q denote forcing conditions.

Lemma 9.1. *For any P and any A, we do not have both P forces A and P forces $\neg A$.*

Proof. If P forces $\neg A$, then by Definition 8.7.3, since $P \supseteq P$, we have that P does not force A. □

Lemma 9.2. *If P forces A and $Q \supseteq P$, then Q forces A.*

Proof. We prove this for limited statements A by induction on $rank(A)$.

If P forces $\exists_\alpha x\ B(x)$, then P forces $B(c)$ for some $c \in S_\beta$ with $\beta < \alpha$. By induction, Q forces $B(c)$ and so also forces $\exists_\alpha x\ B(x)$.

If P forces $\forall_\alpha x\ B(x)$, then if $R \supseteq Q$, we have that $R \supseteq P$, so by the definition of forcing, R does not force $\neg B(c)$ for any $c \in S_\beta$ with $\beta < \alpha$. So, Q also forces $\forall_\alpha x\ B(x)$.

If P forces $\neg B$, then if $R \supseteq Q$, we have $R \supseteq P$ and so by the definition of forcing R does not force B. Thus, Q also forces $\neg B$.

The remaining cases of P forces A where A is of some other form reduce to P forces B where B has a lower rank than A, and so they hold by induction. The base cases, where the rank is lowest, are simple membership statements. The lemma holds here since if P forces $(c_1 \in c_2)$, then we have that $(c_1 \in c_2)$ is a statement in P, but $P \subseteq Q$, so $(c_1 \in c_2)$ is a statement in Q and so Q forces $(c_1 \in c_2)$. The proof for unlimited statements is similar, but inducts on the length of A. □

Lemma 9.3. *For all P and A, there is a $Q \supseteq P$ such that either Q forces A or Q forces $\neg A$.*

Proof. If P does not force $\neg A$, then by Definition 8.7.3, there is some $Q \supseteq P$ such that Q forces A. Thus, either $P \supseteq P$ forces $\neg A$, or some $Q \supseteq P$ forces A. □

Definition 9.4. A **complete sequence** is a sequence $\{P_n\}$ of forcing conditions such that for every n, we have $P_n \subseteq P_{n+1}$, and for each A, there is an n such that either P_n forces A or P_n forces $\neg A$.

Lemma 9.5. *A complete sequence exists.*

Proof. Since $\mathcal{M}$ is countable, we can enumerate all statements A_n defined in $\mathcal{M}$. We can construct a complete sequence inductively by first setting P_0 as a forcing condition that forces either A_0 or $\neg A_0$, and then selecting P_{n+1} as any forcing condition $Q \supseteq P_n$ such that Q forces A_{n+1} or Q forces $\neg A_{n+1}$.

Intuitively, we have created the sequence by picking each successive forcing condition so that it decides one way or the other on the next membership statement. Thus we have a sequence where each condition is a subset of the next condition, and for any statement A_n, we know that either A_n or $\neg A_n$ is forced by P_n. □

10. Cohen's Model $\mathcal{N}$

Recall that the cardinality of the continuum is the cardinality of the set of subsets of $\mathbb{N}$. So, if we are to build a model where the continuum hypothesis fails, then we need to ensure that in our model, there are more than $\aleph_1$ subsets of $\mathbb{N}$. We will accomplish this by selecting an arbitrary cardinal $\aleph_\tau$ with $\tau \geq 2$ and introducing $\aleph_\tau$ many subsets to $\mathcal{M}$ which are identified by a sequence of forcing conditions.

First, fix some cardinal $\aleph_\tau$ in $\mathcal{M}$ with $\tau \geq 2$. We now select our labeling and generic sets. Our goal with this labeling is to eventually be able to use a label

$W \in S$ such that $\bar{W}$ is the set of ordered pairs consisting of elements $\delta < \aleph_\tau$ paired with the new subsets a_δ that we will introduce to the model. Ordered pairs (δ, a_δ) in set theory are sets of the form $\{\{\delta\}, \{\delta, a_\delta\}\}$, so getting to the point where we can refer to the ordered pairs will require a few steps of building up.

For each ordinal $\alpha < \aleph_\tau$, we set $S_\alpha = \{c_\alpha\}$ where c_α will later be sent by ϕ to a formula A which defines the set $\bar{c_\alpha} = \alpha$. We include each c_α in our set G of generic sets. For $\alpha = \aleph_\tau$, we include $\aleph_\tau$ many elements in S_α and denote these elements by a_δ with $\delta < \aleph_\tau$. Each a_δ is included in G, and each is such that $\bar{a_\delta}$ will be a subset of the natural numbers. Still with $\alpha = \aleph_\tau$, we set $S_{\alpha+1}$ as the union of two sets, each containing $\aleph_\tau$ many elements. The first set contains elements denoted by $c_{\{\beta\}}$ with $\beta < \aleph_\tau$, which will be eventually be such that $\bar{c_{\{\beta\}}} = \{\beta\}$. The second contains elements denoted $c_{\{\delta,a_\delta\}}$ with $\delta < \aleph_\tau$, which will be such that $\bar{c_{\{\delta,a_\delta\}}} = \{\delta, a_\delta\}$. All of these elements are also included in G. The label space $S_{\alpha+2}$ is where the labels for the ordered pairs appear; it has $\aleph_\tau$ many elements $c_{(\delta,a_\delta)}$ with $\delta < \aleph_\tau$, each such that $\bar{c_{(\delta,a_\delta)}} = (\delta, a_\delta)$. These are again included in G. The next space, $S_{\alpha+3}$ consists of a single element denoted by W, which is such that $\bar{W} = \{(\delta, a_\delta) \mid \delta < \aleph_\tau\}$, and once more, we include this element W in G. For $\alpha > \aleph_\tau + 3$, we have that S_α contains no elements of G and that S_α is in one-to-one correspondence with the set of formulas that range over $\bigcup(S_\gamma \mid \gamma < \alpha)$.

We will use only use forcing conditions which contain statements of the form $n \in a_\delta$ or $\neg n \in a_\delta$ where n is a natural number and $\delta < \aleph_\tau$. So, the forcing conditions will be finite sets of statements saying that a natural number is or is not a member of one of our a_δ, which are each a generic set in G. As a reminder, there are $\aleph_\tau$ many of these elements a_δ, and we are introducing them to the model $\mathcal{M}$ as new subsets of the natural numbers. This is what will cause the Continuum Hypothesis to fail: there will be at least $\aleph_\tau$ many subsets of the natural numbers in our model, and so the cardinality of the continuum, which is the cardinality of $\mathcal{P}(\mathbb{N})$, will be at least $\aleph_\tau$, which was selected to be greater than $\aleph_1$.

Our condition set will consist of the above forcing conditions, and our ordering $<$ will be defined by $P < Q$ whenever $P \subseteq Q$. We define our function ψ by setting $\psi(P)$ equal to the union of the following sets

$$\{c_n \in a_\delta \mid (n \in a_\delta) \in P\}$$
$$\{c_\alpha \in c_\beta \mid \alpha < \beta < \aleph_\tau\}$$
$$\{c_\alpha \in c_{\{\alpha\}} \mid \alpha < \aleph_\tau\}$$
$$\{c_\delta \in c_{\{\delta,a_\delta\}} \mid \delta < \aleph_\tau\}$$
$$\{a_\delta \in c_{\{\delta,a_\delta\}} \mid \delta < \aleph_\tau\}$$
$$\{c_{\{\delta\}} \in c_{(\delta,a_\delta)} \mid \delta < \aleph_\tau\}$$
$$\{c_{\{\delta,a_\delta\}} \in c_{(\delta,a_\delta)} \mid \delta < \aleph_\tau\}$$
$$\{c_{(\delta,a_\delta)} \in W \mid \delta < \aleph_\tau\}$$

Let $\{P_n\}$ be any complete sequence of forcing conditions. We will now define all of the $\bar{c}$ corresponding to the elements of our label spaces. We proceed by induction on the ordinal index α of the label spaces S_α as follows. For $c \in S_0$, let $\bar{c} = \emptyset$. For $c \in S_\alpha$ when all previous $c \in S_\beta$ with $\beta < \alpha$ have had their $\bar{c}$ defined, there are two cases.

(1) If c is not also in G, then $\bar{c}$ is the set of $\bar{x}$ such that $\bar{x}$ satifies the formula $A = \phi_\alpha(c)$ and such that $x \in S_\beta$ for some $\beta < \alpha$. That is, $\bar{c}$ is the set of already defined elements which satisfy the formula that corresponds to c.
(2) If c is in both S_α and G, then $\bar{c} = \{\bar{x} \mid \exists n \in \mathbb{N}\ s.t.\ (x \in c) \in \psi(P_n)\}$. Essentially, this means that if c is a generic set, then $\bar{c}$ contains exactly the elements that the forcing conditions in our complete sequence tell us belong to c.

Note that since each a_δ is one of the elements of $S_\alpha \cap G$, they fall into the second case above, and the definition of ψ ensures that each $\bar{a_\delta}$ will be exactly the subset of the natural numbers which our forcing sequence says belong to $\bar{a_\delta}$.

We can now finally build our new model, which we call $\mathcal{N}$. We will use the usual set membership relation in our structure, since our intention is to extend the model $\mathcal{M}$, which is a standard model. Let $G^* = \{\bar{a_\delta} \mid \delta < \aleph_\tau\}$ and define the domain of the new structure $\mathcal{N}$ to be

$$\bigcup\{M_\alpha(G^*) \mid \alpha < \alpha_0\}$$
or equivalently
$$\bigcup\{X_{\alpha_0}(G^*)\}$$

Recall that α_0 is the supremum of the ordinals in the unique minimal model $\mathcal{M}$. The above set is thus the set of elements that can be constructed before stage α_0 from our new sets, each of which is defined by our complete sequence of forcing conditions.

Lemma 10.1. *A statement A is true in the model $\mathcal{N}$ if and only if for some n, P_n forces A.*

Proof. We prove the lemma for limited statements A by induction on $rank(A)$.

If A is of the form $\exists_\alpha x\ B(x)$ and P_n forces A, then we have that P_n forces $B(c)$ where $c \in S_\beta$ with $\beta < \alpha$. By induction, since $\beta < \alpha$, we have $B(\bar{c})$ is true in $\mathcal{N}$ and so A is true in $\mathcal{N}$. Conversely, if A is true in $\mathcal{N}$, then we have that for some $c \in S_\beta$ with $\beta < \alpha$, $B(\bar{c})$ is true. By induction, there is a P_n which forces $B(c)$ and thus also forces A.

If A is of the form $\forall_\alpha x\ B(x)$ and P_n forces A, then for some $c \in S_\beta$ with $\beta < \alpha$, some P_m with $m > n$ forces $B(c)$, because for $\forall_\alpha x\ B(x)$ to be forced by P_n, no P_m can force $\neg B(c)$, but our conditions are in a complete sequence. Induction gives that $B(\bar{c})$ is true in $\mathcal{N}$ and so A is also true in $\mathcal{N}$. Conversely, if A is true in $\mathcal{N}$, then no P_n can force $\exists_\alpha x\ \neg B(x)$ since that would imply that for some $c \in S_\beta$ with $\beta < \alpha$, P_n forces $\neg B(c)$. Then $\neg B(\bar{c})$ would hold in $\mathcal{N}$ and A would be false in $\mathcal{N}$. So, since no P_n can force $\exists_\alpha x\ \neg B(x)$, but our sequence is complete, we have that some P_n forces $\neg\exists_\alpha x\ \neg B(x)$. By the definition of forcing, this gives that for every $Q \supseteq P_n$, Q does not force $\exists_\alpha x\ \neg B(x)$. This means that every $Q \supseteq P_n$ does not force $\neg B(c)$ for any $c \in S_\beta$ with $\beta < \alpha$, which again by the definition of forcing means that P_n forces $\forall_\alpha x\ B(x)$.

If A is of the form $\neg B$ and P_n forces A, then if $\neg B$ were false in $\mathcal{N}$, B is true in $\mathcal{N}$ and so by induction, some P_m forces B. However, by Lemma 9.2, we would have that $P_{max(n,m)}$ forces B and forces $\neg B$, which Lemma 9.1 tells us is impossible. Thus, $\neg B$ must be true in $\mathcal{N}$. Conversely, if $\neg B$ is true in $\mathcal{N}$, then supposing $\neg B$ is not forced by any P_n, we have that B is forced by some P_n since our sequence is complete. But B is a lower rank statement than $\neg B$, so by induction this means that B is true in $\mathcal{N}$, which is impossible. So, we must have that some P_n forces $\neg B$.

The other cases just reduce the rank of A and are covered by induction, and the base cases are membership statements, for which the lemma holds because our $\bar{c}$ and $\bar{a_\delta}$ have been defined so that the membership statements that are true about them are exactly the statements contained in some $\psi(P_n)$. The proof for unlimited statements is similar but inducts on the length of A. □

We can now see that the sets we are introducing to the model will be distinct within the model.

Theorem 10.2. *Given a_{δ_1}, α_{δ_2} with $\delta_1 \neq \delta_2$, we have that $\bar{a_{\delta_1}}$ and $\bar{a_{\delta_2}}$ will give distinct sets in $\mathcal{N}$.*

Proof. Any forcing condition P is finite and can be contained in a forcing condition Q which has $(m \in a_{\delta_1})$ and $\neg(m \in a_{\delta_2})$ for some $m \in \mathbb{N}$, and thus no P forces the elements of the two sets to be the same. The definition of forcing tells us that this means each of the P_n in our complete sequence actually forces $\neg(a_{\delta_1} = a_{\delta_2})$, and so by the above lemma, the sets are not equal in $\mathcal{N}$. □

Theorem 10.3. *$\mathcal{N}$ is a model of ZF.*

Proof. We have to verify that each of the axioms of ZF set theory is true in $\mathcal{N}$. We do so by checking that the axioms hold when they are restricted to $\mathcal{N}$. That is, for each axiom A in ZF, we show that $A_{\mathcal{N}}$ holds. Throughout the proof, we are permitted to use the usual axioms of ZF set theory.

(1) The Axiom of the Null Set

We want to show that $\emptyset \in \mathcal{N}$, or in other words that $\emptyset$ is constructible from G^* at some stage before α_0. We can easily obtain $\emptyset$ as the set of x in $M_0(G^*)$ satisfying the property $x \neq x$. This property ranges only over $M_0(G^*)$ and so $\emptyset$ appears in $M_1(G^*)$. Hence, $\emptyset$ is in $\mathcal{N}$.

(2) The Axiom of Extensionality

We must show that if x, y are two sets in $\mathcal{N}$ which share the same elements in $\mathcal{N}$, then $x = y$. If x and y are in $\mathcal{N}$, then $x \in M_{\alpha_1}(G^*)$ and $y \in M_{\alpha_2}(G^*)$ for some $\alpha_1, a_2 < \alpha_0$. Let α be the larger of α_1, α_2 and note that by definition $M_\alpha(G^*) = (\bigcup_{\beta<\alpha} M_\beta(G^*))'$ and thus $M_\alpha(G^*)$ contains both x and y. Since every $M_\beta(G^*)$ is transitive, all elements of x and all elements of y are also inside $M_\alpha(G^*)$. Thus, if x and y share the same elements inside $\mathcal{N}$, then they truly share all of their elements, and so by the usual Extensionality of ZF, we have $x = y$.

(3) The Axiom of Unordered Pairs

Given x, y in $\mathcal{N}$, we have that $x \in M_{\alpha_1}(G^*)$ and $y \in M_{\alpha_2}(G^*)$ for some $\alpha_1, a_2 < \alpha_0$. Both x and y are thus in $M_\alpha(G^*)$ where α is the larger of α_1, α_2. So, we can obtain the set $\{x, y\}$ as the set

$$\{z \mid z \in M_\alpha(G^*) \ \& \ (z = x \text{ or } z = y)\}$$

This is a set defined by a property restricted to $M_\alpha(G^*)$, and so $\{x, y\}$ will be an element of $M_{\alpha+1}(G^*)$. Thus, $\{x, y\}$ is in $\mathcal{N}$.

(4) The Axiom of Union

Given x in $\mathcal{N}$, we have that $x \in M_\alpha(G^*)$ with $\alpha < \alpha_0$. Since $M_\alpha(G^*)$ is transitive, we have that all elements $y \in x$ are also in $x \in M_\alpha(G^*)$. We can obtain the union over x as the set

$$\{z \mid z \in M_\alpha(G^*) \ \&\ \exists y(y \in M_\alpha(G^*) \ \&\ z \in y \ \&\ y \in x\}$$

which is defined by a property containing variables restricted to $M_\alpha(G^*)$ and thus is an element of $M_{\alpha+1}(G^*)$.

(5) The Axiom of Infinity

We will show this by showing the stronger condition that every ordinal α is in the set $M_{\alpha+1}(G*)$, and hence the least infinite ordinal ω is in $\mathcal{N}$.
If we have a set x inside a transitive set X, then x has the same elements when membership is restricted to X as it does when membership is unrestricted. The definition of an ordinal refers only to the elements of the set in question, and so, x will be an ordinal when the definition is restricted to X if and only if it is actually an ordinal. Written with the restricted formula notation, this means that for any transitive set X and any element $x \in X$, we have $On(x)$ if and only if $On_X(x)$.
By our argument for (1), we know that $0 = \emptyset$ is in $M_1(G^*)$. Assume α is the least ordinal for which $\alpha \notin M_{\alpha+1}(G^*)$. We will find a contradiction. Since α is assumed to be least, we have that $\beta < \alpha$ implies $\beta \in M_{\beta+1}(G^*)$ where $\beta + 1 \leq \alpha$. If we let $X = \bigcup_{\gamma \leq \alpha} M_\gamma(G^*)$ then this means that for $\beta < \alpha$, we have $\beta \in X$. We now consider the set $\gamma = \{x \in X \mid On_X(x)\}$. We can see that X is transitive, and so by the above paragraph, we have that γ is a set of ordinals and is actually itself an ordinal itself. The ordinal γ is larger than any ordinal it contains, and it contains all $\beta < \alpha$, since each $\beta < \alpha$ is an ordinal in X. So we have that $\gamma \geq \alpha$ and hence either $\gamma = \alpha$ or γ contains α. Now we observe that the set X is $X_{\alpha+1}$ and thus X' is transitive and is a subset of $M_{\alpha+1}(G^*)$. Since γ is defined by a property ranging over $X_{\alpha+1}$, we have that γ is in $M_{\alpha+1}(G^*)$. So, if $\gamma = \alpha$, we are done, and if γ contains α, then we have that α is in $M_{\alpha+1}(G^*)$ by transitivity.
We have established that $\alpha \in M_{\alpha+1}(G^*)$ for every ordinal α. We now simply note that the least infinite ordinal ω is an element of $M_{\omega+1}(G*)$ and hence is an element in $\mathcal{N}$, so the Axiom of Infinity holds.

(6) The Axiom of Regularity

We are required to show that any x in $\mathcal{N}$ has a member y, also in $\mathcal{N}$, which contains no elements from x. So, suppose we have some $x \in M_\alpha(G^*)$. Since we are able to use the usual Axiom of Regularity from ZF, we know that x has an element y containing no elements from x, and we need only show that this y is in $\mathcal{N}$. But, we know that $M_\alpha(G*)$ is transitive, and so $y \in x$ and $x \in M_\alpha(G^*)$ implies that $y \in M_\alpha(G^*)$. Thus, the y that we want is in $\mathcal{N}$.

(7) The Power Set Axiom

The proof for this axiom requires using some properties of $\mathcal{M}$, but we have intentionally made our new sets very vague from the perspective of $\mathcal{M}$, so we have to try to infer enough information from the forcing conditions to find the power set of our set.
Let $\bar{x}$ be a set in $\mathcal{N}$. We have that $\bar{x} \in S_\alpha$ with $\alpha < \alpha_0$. We want to show that the power set of $\bar{x}$ is also in $\mathcal{N}$. Note that all of the labels from our label spaces and all of the forcing conditions are statements in ZF and so they are statements in $\mathcal{M}$ since it is a model of ZF. Our first goal will be

to find some bound for the label spaces that could contain elements that refer to the elements of our power set. For each $c \in S$, define the following sets in $\mathcal{M}$.

$$R(c) = \{P \mid P \text{ forces } c \subseteq x\}$$
$$T(c) = \{(P, c^*) \mid P \text{ forces } (c^* \in c) \text{ where } c^* \in S_\beta \text{ with } \beta < \alpha\}$$
$$U(c) = \{(R(c), T(c))\}$$

Each $R(c)$ is a set of forcing conditions, so it is an element of the power set of the set of all forcing conditions relative to $\mathcal{M}$. Each $T(c)$ is similarly an element of the power set of the set of pairs of elements from the set of all forcing conditions and the set $\bigcup_{\beta<\alpha} S_\alpha$. These sets of forcing conditions and labels are all elements of $\mathcal{M}$, so we have that each $U(c)$ will be an element in $\mathcal{M}$. So, we can define the set $U^* = \{U(c) \mid c \in S\}$ in $\mathcal{M}$. We define a function f such that for every $u \in U^*$, $f(u)$ is the least ordinal β such that there exists a $c \in S_\beta$ for which $U(c) = u$, or $f(u) = 0$ is there is no such ordinal. We then let $\beta_0 = sup\{f(u) \mid u \in U^*\}$. We have that β_0 is in $\mathcal{M}$.

Now, if $\bar{c} \subseteq \bar{x}$, then some P_n forces $c \subseteq x$. The function f will send $U(c)$ to the least β for which some $c_1 \in \beta$ is such that $U(c_1) = U(c)$. Such a β will exist, and so by the way we have defined β_0, we have a $c_1 \in \beta$ with $\beta < \beta_0$ for which $U(c) = U(c_1)$. If $\bar{c} \neq \bar{c_1}$ were true, then either some element of $\bar{c}$ is not in $\bar{c_1}$ or some element of $\bar{c_1}$ is not in $\bar{c}$. We will assume the first without loss of generality, so there is some $c_2 \in S_\beta$ with $\beta < \alpha$ such that $\bar{c_2} \in \bar{c}$ and not $\bar{c_2} \in \bar{c_1}$. Then, by Lemma 10.1, some P_n forces $c_2 \in c$. But $U(c) = U(c_1)$ implies $T(c) = T(c_1)$, so P_n must also force $c_2 \in c_1$. Again by Lemma 10.1, this means that $\bar{c_2} \in \bar{c_1}$,which contradicts our assumption. Thus, we have that $\bar{c} = \bar{c_1}$. So, we now know that any subset of $\bar{x}$ will be $\bar{c}$ for some $c \in S_\beta$ with $\beta < \beta_0$.

Any subset of $\bar{x}$ will thus be an element of $X_{\beta_0}(G^*)$ in $\mathcal{N}$. So, we can obtain the powerset of $\bar{x}$ as the set $\{y \mid y \in X_{\beta_0} \ \& \ y \subseteq \bar{x}\}$ which is defined over X_{β_0} and thus will be an element in $(X_{\beta_0}(G^*))' = M_{\beta_0}(G^*)$, and as such is an element in $\mathcal{N}$.

(8) The Axiom of Replacement

We need to show that given some function f defined in $\mathcal{N}$, the image of f on any set is also a set in $\mathcal{N}$. So, suppose we have a formula $A(x, y)$ which defines a function f in $\mathcal{N}$ that gives y as a function of x. Let $c_0 \in S_\alpha$ be fixed. In a way similar to the above proof, we will find a bound for which label spaces S_β the range of f will hit.

Define in $\mathcal{M}$ the function g which sends any pair of a forcing condition and an element $c \in S$ to the least β for which there is some $c' \in \beta$ such that P forces $f(c) = c'$, or if no such β exists, then $g(P, c) = 0$. Let β_0 be $sup\{g(P, c) \mid P \text{ is a forcing condition } \& \ c \in S_\beta, \beta < \alpha\}$. So, in a similar way to the above proof, we know that the range of f on c_0 will be contained in $M_{\beta_0}(G^*)$. This tells us where the elements of the range are, but we still need to know that we can obtain the range of f on c_0 as a set in $\mathcal{N}$.

What follows will allow us to conclude that the formula A can be restricted to some set $\bar{c}$ and still define the range of f in $\mathcal{N}$. The formula A will be of the form

$$Q_1y_1 \cdots Q_my_m\ B(x_1,...,x_m,y_1,...,y_m)$$

where the Q_i are quantifiers, and the formula B contains no quantifiers. For every r with $1 \le r \le m$, there is a function $f_r(P, c_1, ..., c_n, c'_1, ..., c'_{r-1})$, defined on c_i, c'_j in $\bigcup_{\delta<\alpha} S_\alpha$, such that

(a) if Q_r is $\exists$, then f_r maps each tuple to the least γ such that P forces

$$Q_{r+1}y_{r+1} \cdots Q_my_m\ B(c_1,...,c_n,c'_1,...,c'_{r-1},y_{r+1},...,y_m)$$

for some $c_\gamma \in S$, or to 0 if there is no such γ.

(b) if Q_r is $\forall$, then f_r maps each tuple to the least γ such that P forces

$$\neg Q_{r+1}y_{r+1} \cdots Q_my_m\ B(c_1,...,c_n,c'_1,...,c'_{r-1},y_{r+1},...,y_m)$$

for some $c_\gamma \in S$, or to 0 if there is no such γ.

Now, for each r, let g_r be the function such that $g_r(c_1, ..., c_n, c'_1, ..., c'_n)$ is the supremum of $\{f_r(P, c_1, ..., c_n, c'_1, ..., c'_{r-1}) \mid P \text{ is a forcing condition}\}$. Define another function, h, such that $h(\gamma)$ is the supremum of the range of g_r across all r with $1 \le r \le m$ and all c_i, c'_j in $\bigcup_{\delta<\gamma} S_\gamma$. The function h is still definable in $\mathcal{M}$.

Let $\beta_1 = \beta$ and $\beta_{n+1} = h(\beta_n)$, and take $\beta' = sup\{B_n\}$. If we take $\bar{c_1}$ as the set defined by the formula that says $\bar{c_1}$ is $\bigcup_{\gamma<\beta'} M_\gamma(G^*)$, then $\bar{c_1}$ will contain the range of f on $\bar{c_0}$. So, we can obtain the range of f on $\bar{c_0}$ as the set $\{\bar{c} \mid \exists \bar{x} \in \bar{c_0}\ A_{\bar{c_1}}(x, \bar{c})\}$.

□

The following lemma is useful because constructibility provides a natural well-ordering on the elements that are constructed. We needed to design a model in which $V = L$ failed, but in order to have the Axiom of Choice in $\mathcal{N}$, we need to have that all sets can be well-ordered. The Axiom of Choice is desirable because it makes some of the remaining theorems easier to prove, and because satisfying AC will mean that our model demonstrates the consistency of $\neg CH$ with ZFC, which is a stronger condition than consistency with ZF.

Lemma 10.4. *Every element in $\mathcal{N}$ is constructible from $\bar{W}$.*

Proof. We want to verify that every x in $\mathcal{N}$ is a member of $M_\alpha(\bar{W})$ for some ordinal α. In other words, we want to show the underlying set of $\mathcal{N}$ is a subset of $\bigcup\{M_\alpha(\bar{W}) \mid \alpha \in \mathcal{M}\}$.

Remember that we defined $\mathcal{N}$ as the set of elements constructible from the collection of our new subsets $\bar{a_\delta}$ before stage α_0, and recall that $\bar{W}$ is the set of ordered pairs of the form $(\bar{\delta}, \bar{a_\delta}) = \{\{\bar{\delta}\}, \{\bar{\delta}, \bar{a_\delta}\}\}$. It is clear that all of the $\bar{a_\delta}$ are in the transitive closure of $\bar{W}$, and so the set of elements that are constructible from $\bar{W}$ contains the set of elements that are constructible from the set of $\bar{a_\delta}$, which was the set of elements in $\mathcal{N}$. □

We can see also see now why the $\bar{a_\delta}$ are not constructible in $\mathcal{N}$. Fix one of the $\bar{a_\delta}$ and some ordinal α. Let $c_\alpha \in S$ be the label such that $\bar{c_\alpha} = \alpha$. Let x be the constructible set constructed at step α in some well-ordering of the construction process (for example, ordered by M_β, then by formulas). The construction process will be the same in $\mathcal{N}$ as it is in ZF set theory, so within $\mathcal{N}$, x is the α-th element constructed. Any forcing condition P is finite and so can be contained in a condition

Q such that for some $n \in \mathbb{N}$, either $n \in x$ is true and $\neg(n \in a_\delta)$ is in Q or $\neg(n \in x)$ is true and $(n \in a_\delta)$ is in Q. That is, P can be contained in a Q that makes $\bar{a_\delta}$ different from x. If $\bar{a_\delta}$ were the α-th element constructed in $\mathcal{N}$, then some P_n would have to force this to be true. Any of the Q containing P_n which cause $\bar{a_\delta}$ to differ from x would then also have to force that statement, but if we consider a complete sequence $\{Q_n\}$ starting with $Q_0 = Q$, we see that in a model defined by it in the same way that we have made our $\mathcal{N}$, we would have that both x and $\bar{a_\delta}$ are the α-th element constructed, of which there can only be one, and yet $x \neq \bar{a_\delta}$.

Theorem 10.5. *The Axiom of Choice is true in $\mathcal{N}$. (So $\mathcal{N}$ is also a model of ZFC.)*

Informally, the above theorem holds since every element being constructible from $\bar{W}$ provides us with a well-ordering on the elements of $\mathcal{N}$. The elements can be ordered based on the stage at which they are constructed, an ordering on the formulas used to define them, and the well-ordering of $\bar{W}$ itself. This in turn makes every set well-ordered in our model, which is a condition equivalent to the Axiom of Choice.

Since we know that we have a model of ZFC in which we have at least $\aleph_\tau$ many distinct subsets of $\mathbb{N}$, our only concern is whether by introducing new sets to $\mathcal{M}$ we have made some bijective functions available which were not already present, and so perhaps may have made it so that some ordinals that appeared to have different cardinalities in $\mathcal{M}$ due to the lack of a one-to-one correspondence now have the same cardinality in $\mathcal{N}$. This would be a problem because it could potentially result in the cardinalities between $\aleph_1$ and $\aleph_\tau$ collapsing together. We need to know that $\aleph_\tau > \aleph_1$ still actually holds in $\mathcal{N}$. We will see that, in fact, every pair of ordinals with different cardinalities in $\mathcal{M}$ still have different cardinalities in $\mathcal{N}$, but first we need two more lemmas.

Definition 10.6. Let P and Q be forcing conditions. We call P and Q **incompatible** if there is no forcing condition R such that both $P < R$ and $Q < R$.

In our case, since we have chosen $\subseteq$ as the relation for our condition set, we have that P and Q are incompatible when they cannot both be a subset of another forcing condition. This happens simply when P and Q contain contrasting membership statements, for example if $1 \in a$ were in P and $\neg(1 \in a)$ were in Q.

Lemma 10.7. *If B is a set in $\mathcal{M}$ of mutually incompatible P, then B is countable in $\mathcal{M}$.*

Proof. Assume B is uncountable. Define B_n to be

$$\{P \in B \mid P \text{ contains fewer than } n \text{ statements}\}$$

for each $n \in \mathbb{N}$. Since $\mathcal{M}$ satisfies $V = L$, it also satisfies the Axiom of Choice, and so we know that the union of countably many countable sets is a countable set in $\mathcal{M}$. We have that $B = \bigcup_{n \in \mathbb{N}} B_n$, and so, since B is uncountable, but $\bigcup_{n \in \mathbb{N}} B_n$ is the union of countably many sets, we must have that some B_n is uncountable. Now, let k be the largest integer such that for some P_0, not in B, containing exactly k statements, there are uncountably many $P \in B_n$ where $P_0 < P$. Let B_{P_0} be the set of those $P \in B$ such that $P_0 < P$. Take any condition $P_1 \in B$ and let $A_1, A_2, ..., A_m$ be the statements in $P_1 - P_0$, of which there are finitely many since all forcing conditions are finite, and of which there is at least one since $P_0 \neq P_1$

and $P_0 < P_1$. Since B_{P_0} is still a set of mutually incompatible conditions, P_1 is incompatible with every other $P \in B_{P_0}$. Also, every $P \in B_{P_0}$ contains P_0 by definition of B_{P_0}. So, we must have that one of the A_i is such that uncountably many $P \in B_{P_0}$ contain $\neg A_i$, because if there were no such A_i, then only countably many sets would be incompatible with P_1. However, $\neg A_i$ cannot be in P_0 since we have $P_0 < P_1$, and so $P_0 \cup \{\neg A_i\}$ is a condition with $k + 1$ elements which is contained in uncountably many $P \in B_n$. This contradicts our selection of k, and thus we have that B must be countable. □

The next lemma will allow us to conclude that if a function defined in $\mathcal{N}$ has a domain with cardinality κ in $\mathcal{N}$, then the cardinality of its range cannot exceed $\aleph_0 \cdot \kappa$ in $\mathcal{M}$. This is what prevents any new functions in $\mathcal{N}$ from bridging a gap between cardinals.

Lemma 10.8. *Let f be a function defined in $\mathcal{N}$. There is a function g, defined in $\mathcal{M}$, which sends each $c \in S$ to a countable subset $g(c) \subseteq S$ and such that for all c, $f(\bar{c}) = \bar{c'}$ for some $c' \in g(c)$.*

Proof. In $\mathcal{N}$, set c' as the earliest element of S for which $f(\bar{c}) = \bar{c'}$. The definition of each S_α can be carried out in $\mathcal{N}$, and a well-ordering on S can be found in $\mathcal{N}$, so there is an unlimited statement T which says that c' is the first element of S for which $f(c) = c'$. Since this statement is true in $\mathcal{N}$ due to our selection of c', and since f is a function in $\mathcal{N}$, both the statement T and that f is a function must be forced by some P_n according to Lemma 10.1.

In $\mathcal{M}$, for every c, c' in S, define $A(c, c')$ to be the set of all forcing conditions P such that P forces f to be a function and such that c' is the earliest element of S under some well-ordering so that for all $Q > P$, we have that Q forces $f(c) = c'$. We can see that if c', c'' are distinct elements of S, then each of the elements in $A(c, c')$ must be incompatible with each of the elements in $A(c, c'')$, since if $P \in A(c, c')$ and $Q \in A(c, c'')$, then any R such that $P < R$ and $Q < R$ would have to force $f(c) = c' = c''$. Since the Axiom of Choice holds in $\mathcal{M}$, and by Lemma 10.7, there can be at most countably many c' such that $A(c, c') \neq \phi$, the function from our labeling. Let $g(c)$ be the countable set of these c'. We have that $A(c, c')$ is nonempty because of the condition P_n we found above, and we have that the c' such that $f(\bar{c}) = \bar{c'}$ holds is in $g(c)$. □

Theorem 10.9. *Given infinite ordinals α, β such that in $\mathcal{M}$, the cardinality of α, denoted $|\alpha|$, is less than the cardinality of β, denoted by $|\beta|$, we have that $|\alpha| < |\beta|$ in $\mathcal{N}$ as well.*

Proof. Suppose that f is a function in $\mathcal{N}$ from $|\alpha|$ onto $|\beta|$. Define

$$Z_\gamma = \bigcup \{S_\delta \mid \delta < \gamma\}$$

for every ordinal γ, and note that $\gamma \in Z_{\gamma+3}$ for all γ. In particular, we have that $\alpha \in Z_{\alpha+3}$. Extend the function f to a function f' on $X_{\alpha+3}$ such that $f'(x) = 0$ for all $x \notin \alpha$ and $f'(x) = f(x)$ for all $x \in \alpha$. Lemma 10.8 tells us that the range of f' is contained in $\bar{T}$ where T is a subset of $Z_\beta = \bigcup \{S_\delta \mid \delta < \beta\}$ and the cardinality of T is in $\mathcal{M}$ is at most $\aleph_0 \cdot |\alpha|$ and so is at most $|\alpha|$ since α is infinite. Now, because $|T| \leq |\alpha| < |\beta|$ in $\mathcal{M}$, there is an ordinal $\gamma < \beta$ such that $T \subseteq \{S_\delta \mid \delta < \gamma\}$. However, this means that $\bar{T}$ is constructed earlier than β, and so cannot contain

β, and thus f cannot be onto. So, there are no onto functions from α to β in the model $\mathcal{N}$, which means that $|\alpha| < |\beta|$ in $\mathcal{N}$. □

So, we have introduced $\aleph_\tau$ many subsets of $\mathbb{N}$ into our model, which views each of these subsets as distinct sets, and we know that the cardinals are still distinct, so $\aleph_\tau \geq \aleph_2$ in $\mathcal{N}$ as we intended. Thus, the cardinality of the continuum in $\mathcal{N}$ is at least $\aleph_\tau$, which is strictly greater than $\aleph_1$, and so the Continuum Hypothesis is false. This establishes the consistency of $ZF \cup \{\neg CH \ \& \ AC\}$. Together with Gödel's work, this demonstrates that both the Continuum Hypothesis and Generalized Continuum Hypothesis are independent of the axioms of ZF and ZFC set theory.

References

[1] Paul J. Cohen. Set Theory and the Continuum Hypothesis. Dover Publications: Mineola, New York, 2008.
[2] Timothy Gowers, ed. The Princeton Companion to Mathematics. Princeton University Press: Princeton, New Jersey, 2008.
[3] Richard Hodel. An Introduction to Mathematical Logic. PWS Publishing: Boston, Massachusetts, 1995.
[4] Karel Hrbacek and Thomas Jech. Introduction to Set Theory, 3rd ed. CRC Press: Boca Raton, Florida, 1999.