# Solutions to Test 1    Math. 3X03. Complex Analysis

**1** a) $\sin(z) = i$

$$\frac{e^{iz} - e^{-iz}}{2i} = i \Rightarrow e^{iz} - e^{-iz} = -2$$

$$\Rightarrow (e^{iz})^2 - 1 + 2e^{iz} = 0 \Rightarrow e^{iz} = \frac{-2 \pm \sqrt{4+4}}{2} = \boxed{-1 \pm \sqrt{2}}$$

$$z = -i \log(-1 \pm \sqrt{2})$$

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**1** b) $\bar{z}^3 = 2 \Rightarrow \overline{z^3} = 2 \Rightarrow$

$$z^3 = \bar{2} = 2 \Rightarrow z = \sqrt[3]{2},\ \sqrt[3]{2}\, e^{\frac{2\pi i}{3}},\ \sqrt[3]{2}\, e^{\frac{4\pi i}{3}}.$$

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2 a) Either of def. :

- $K$ is closed and bounded.
- Any seq. in $K$ has a convergent sub-sequence.

(- Any open cover has a finite sub-cover)

→ was not discussed in class but is in the text.

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2 (b) $C$ is not connected $\Longleftrightarrow$ we can find open sets $U, V$ such that

- $C \subset U \cup V$
- $(C \cap U) \cap (C \cap V) = \emptyset$.

For $C = [1,2] \cup [3,4] \cup [5,6]$ Take $U = (0, 2\frac{1}{2})$ &

$V = (2\frac{1}{2}, 6\frac{1}{2})$.

3.a)

$$\log(re^{i\theta}) = \log(r) + i\theta$$

So $u(x,y)$ is the real part of log.
and hence is harmonic on $\mathbb{C} \setminus$ neg. x-axis.

and $v(x,y) = \arg(z)$ is a harmonic conjugate for it.

To show $u(x,y)$ is harmonic on the whole $\mathbb{C} \setminus 0$
choose other branches of logarithm.

On the other hand we can show
directly that $u$ is harmonic:

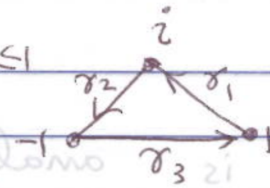
$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1\cdot(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

By symmetry of expression with resp. to y we have:

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{x^2+y^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

4. $\gamma_1(t) = \overbrace{1 + t(i-1)}^{\boxed{(1-t) + it}} \quad 0 \le t \le 1$




$\gamma_2(t) = \overbrace{i + t(-1-i)}^{\boxed{-t - it + i}} \quad 0 \le t \le 1$

$\gamma_3(t) = t \quad -1 \le t \le 1.$

$$\int_{\gamma_1} Re(z) = \int_0^1 (1-t)(i-1)\, dt = (i-1)\left(t - \frac{t^2}{2}\right)\Big|_0^1$$

$$= \frac{i-1}{2}$$

$$\int_{\gamma_2} Re\, z = \int_0^1 -t(-1-i)\, dt = -(1+i)\left(\frac{-t^2}{2}\right)\Big|_0^1$$

$$= \frac{1+i}{2}$$

$$\int_{\gamma_3} Re\, z = \int_{-1}^1 t\, dt = \frac{t^2}{2}\Big|_{-1}^1 = \frac{1}{2} - \frac{1}{2} = 0$$

$$\int_{\gamma} Re(z) = \frac{i-\cancel{1}}{2} + \frac{\cancel{1}+i}{2} = \boxed{i}$$

$\int_{\gamma} z^3 = 0$ because $z^3$ has anti-derivative $\frac{z^4}{4}$

which is defined everywhere. (Fundamental Theorem of Calculus).

5

$\log(z)$ is analytic on $\mathbb{C} \setminus \{x+0i \mid x \leq 0\}$.

$\sqrt{z}$ " " " " .

$\sqrt{\log(z)+1}$ to be analytic

$\log(z)+1$ should not lie in $\overset{y=0}{x \leq 0}$.

If it does, then

~~That means log~~ let $z = re^{i\theta}$

$\Rightarrow \log(z)+1 = \log(r) + i\theta + 1 \Rightarrow$

$\theta = 0$ and $\log(r)+1 \leq 0$.

$\Rightarrow \log(r) \leq -1 \Rightarrow \boxed{0 < r \leq e^{-1}}$

$\arg(z) = 0$.

So $z$ should not lie in $0 < |z| \leq \frac{1}{e}$.

Thus $\sqrt{\log(z)+1}$ is analytic on

$\mathbb{C} \setminus \{x+0i \mid 0 < x \leq \frac{1}{e}\}$.