

MATH3XO3 (Complex Analysis) Spring 2010

Test 2 Solutions

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1.

- (a) [5 marks] Give the definition of a simply connected region.

Solution: A set A is simply connected if it is connected and if every closed curve in A is homotopic (as a closed curve) to a point (i.e. a constant curve).

- (b) [5 marks] State the (homotopy form of) Cauchy's theorem.

Solution: Let f be a complex function which is analytic on a region A . Let γ be a closed curve in A which is homotopic to a point. Then $\int_{\gamma} f = 0$.

2. [12 marks] Show that the upper-half of the unit circle, connecting 1 to -1 , is homotopic (with fixed end-points) to the lower-half of the unit circle (in \mathbb{C}). Show that these two curves are **not** homotopic in the region $\mathbb{C} \setminus \{0\}$. (Cite theorem(s) you use.)

Solution: Let γ_1 (respectively γ_2) be the upper-half (respectively lower-half) of the unit circle. The curves γ_1, γ_2 can be parameterized as

$$\gamma_1(t) = (\cos(\pi t), \sin(\pi t)), \quad 0 \leq t \leq 1.$$

$$\gamma_2(t) = (\cos(\pi t), -\sin(\pi t)), \quad 0 \leq t \leq 1.$$

Consider the function $1 - 2s$. When s goes from 0 to 1, $1 - 2s$ goes from 1 to -1 . Then a homotopy between the curve γ_1 and γ_2 can be written as $H : [0, 1] \times [0, 1] \rightarrow \mathbb{C}$:

$$H(s, t) = (\cos(\pi t), (1 - 2s) \sin(\pi t)).$$

Let $A = \mathbb{C} \setminus \{0\}$. The function $1/z$ is analytic on A . If γ_1 is homotopic to γ_2 in A then by the deformation theorem $\int_{\gamma_1} 1/z = \int_{\gamma_2} 1/z$. Thus $\int_{\gamma_1 - \gamma_2} 1/z = 0$. But $\gamma_1 - \gamma_2$ is the unit circle γ (traversed counter clockwise) and $\int_{\gamma} 1/z = 2\pi i$ which is not zero. The contradiction proves that γ_1 and γ_2 are not homotopic in A .

3. [10 marks] Compute the integral

$$\int_{\gamma} \frac{z+1}{z^2+z+1},$$

where γ is the circle of radius 2 centered at the origin.

Solution: The roots of the denominator are $\alpha = (-1/2) + (\sqrt{3}/2)i$ and $\beta = (-1/2) - (\sqrt{3}/2)i$, and we have $z^2+z+1 = (z-\alpha)(z-\beta)$. Consider two small circles γ_1, γ_2 of radius $1/2$ centered at the points α and β respectively. As $(z+1)/(z^2+z+1)$ is analytic in the region inside the circle γ and outside the circles γ_1, γ_2 , by the deformation theorem, $\int_{\gamma} (z+1)/(z^2+z+1)$ is equal to $\int_{\gamma_1} (z+1)/(z^2+z+1) + \int_{\gamma_2} (z+1)/(z^2+z+1)$.

The function $(z+1)/(z-\beta)$ is analytic on and inside the circle γ_1 . Similarly the function $(z+1)/(z-\alpha)$ is analytic on and inside the circle γ_2 . Now, by Cauchy's integral formula we have:

$$(0+1)/(0-\beta) = (1/2\pi i) \int_{\gamma_1} \frac{(z+1)/(z-\beta)}{(z-\alpha)},$$

$$(0+1)/(0-\alpha) = (1/2\pi i) \int_{\gamma_2} \frac{(z+1)/(z-\alpha)}{(z-\beta)}.$$

Thus we get

$$\int_{\gamma} (z+1)/(z^2+z+1) = -2\pi i \left(\frac{1}{\alpha} + \frac{1}{\beta} \right) = -2\pi i \left(\frac{\alpha+\beta}{\alpha\beta} \right) = 2\pi i.$$

4. [10 marks] Compute

$$\int_{\gamma} \frac{\sin(z)}{z^2},$$

where γ is the triangle with vertices $2, -2+i$ and $-2-i$, traversed counter-clockwise. (Cite the theorems you use.)

Solution: Since the curve γ goes once around the point 0 we have $I(\gamma, 0) = 1$. By Cauchy's integral formula for derivative (first derivative in this case) applied to $\sin(z)$ and $z_0 = 0$ we get:

$$\cos(0) = \sin'(0) = \frac{1!}{2\pi i} \int_{\gamma} \sin(z)/z^2.$$

Thus the integral in question is equal to $2\pi i \cos(0) = 2\pi i$.

5. [10 marks] Suppose that f is an entire function such that $|f'|$ is bounded. Show that f is either constant or a linear polynomial, i.e. there exist $a, b \in \mathbb{C}$ with $f(z) = az + b$. (Cite theorem(s) you use.)

Solution: By part of Cauchy's integral formula all the derivatives of f exists and in particular f' is also entire. Now by Liouville's theorem applied to f' we conclude that f' should be constant. Thus $f' = a$ for some constant $a \in \mathbb{C}$. Then the derivative of the function $f(z) - az$ is 0 which implies that $f - az$ is equal to a constant say b . Hence $f(z) = az + b$ for all z .