MATH3XO3 (Complex Analysis) Spring 2010

Problem Set 3

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1.

(a) Show that the series $\sum_{n=0}^{\infty} 1/(n^2+z^2)$ converges on the set $\mathbb{C} \setminus \{z = ni \mid n \in \mathbb{Z}\}$.

(b) Show that the convergence is uniform and absolute on each closed disk contained in this region.

Solution: First notice that if $z = ni$ then $z^2 + n^2 = 0$ and hence the series is undefined at $z = ni$. Since every point in $\mathbb{C} \setminus \{ni \mid n \in \mathbb{Z}\}$ is contained in a closed disk in this domain, it is enough to show that the series converges uniformly and absolutely on any closed disk in this domain. Let $D_r = \{z \mid |z - z_0| \leq r\}$ be a closed disk centred at some point z_0 with $D_r \subset \mathbb{C} \setminus \{ni \mid n \in \mathbb{Z}\}$, i.e. D_r does not contain any point ni, $n \in \mathbb{Z}$. Now, since D_r is bounded there is a an integer $R > 0$ such that for any $z \in D_r$, $|z| < R$. For $n > R$ and $z \in D_r$ we have:

$$|n^2 + z^2| = |n^2 - (-z^2)| \geq n^2 - |z^2| \geq n^2 - R^2 \text{ (triangle inequality)},$$

and hence

$$\frac{1}{|n^2 + z^2|} \leq \frac{1}{n^2 - R^2}.$$

But for fix R the series of numbers $\sum_{n=R+1}^{\infty} 1/(n^2 - R^2)$ is convergent (e.g. compare with the series $\sum_{n=2R}^{\infty} 2/n^2$). Now by the Weierstrass M-test it follows that $\sum_{n=R+1}^{\infty} 1/(z^2 + n^2)$ converges uniformly and absolutely on D_r. Adding the finite number of terms from $n = 0$ to $n = R$ does not change the convergence and hence we conclude that $\sum_{n=0}^{\infty} 1/(n^2 + z^2)$ converges absolutely and uniformly on any closed disk D_r.

2. Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{z^n}{1+2^n}$.

Solution: Use the ratio test. Then

$$\lim_{n\to\infty} |\frac{z^n}{1+2^n}|^{1/n} = \lim_{n\to\infty} \frac{|z|}{(1+2^n)^{1/n}} = |z|/2.$$

Thus by the ratio test, if $|z| < 2$ the series converges and if $|z| > 2$ the series diverges. It follows that the radius of convergence is 2.

3. Compute the first four terms of the Taylor series of $1/(1 + e^z)$ around $z_0 = 0$. What is the radius of convergence?

Solution: Let $f(z) = 1/(1 + e^z)$. One computes that

$$f'(z) = \frac{-e^z}{(1+e^z)^2},$$

$$f''(z) = \frac{e^{2z} - e^z}{(1+e^z)^3},$$

$$f'''(z) = \frac{-e^{3z} + 4e^{2z} - e^z}{(1+e^z)^4},$$

and hence $f(0) = 1/2$, $f'(0) = -1/4$, $f''(0) = 0$ and $f'''(0) = 2/16 = 1/8$. Thus the first four terms of the Taylor series are:

$$1/2 - z/4 + 0 + z^3/48.$$

Alternatively, you can find the first few terms of the Taylor series by plugging in the Taylor series of $w = e^z$ into the Taylor series of $1/(1 + w)$.

By part of Taylor's theorem (Theorem 3.2.7) the circle of convergence is the largest circle (centred at $z_0 = 0$ inside which f is analytic. In our case, $1/(1 + e^z)$ is not analytic only if $1 + e^z = 0$, that is, $e^z = -1$. Solving for z we obtain $z = (\pi + 2k\pi)i$, for any $k \in \mathbb{Z}$ (use polar coordinates). The largest open disk centred at $z_0 = 0$ which does not contain any of these points is the disk of radius $R = \pi$.

4. Expand $\frac{1}{z(z-1)(z-2)}$ in a Laurent series in the following regions: (a) $0 < |z| < 1$, (b) $1 < |z| < 2$.

Solution: (a) By partial fractions we have $1/(z-1)(z-2) = 1/(z-2) - 1/(z-1)$. We use the geometric series $1/(1-w) = 1 + w + w^2 + \cdots$ to find the Laurent expansion:

$$\frac{1}{z-2} = \frac{(-1/2)}{1-(z/2)} = (-1/2)(1 + (z/2) + (z/2)^2 + \cdots .$$

and

$$\frac{1}{z-1} = \frac{-1}{1-z} = -1 - z - z^2 - \cdots .$$

Both series converge when $|z| < 1$. Thus

$$\frac{1}{z(z-1)(z-2)} = \sum_{n=-1}^{\infty} (1 - \frac{1}{2^{n+2}}) z^n.$$

(b) Again write $1/(z-1)(z-2) = 1/(z-2) - 1/(z-1)$. In the region $1 < |z| < 2$ we have $|1/z| < 1$ and $|z/2| < 1$. Write

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-(1/z)} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=-1}^{\infty} z^n.$$

Also

$$\frac{1}{z-2} = \frac{(-1/2)}{1-(z/2)} = (-1/2)(1 + (z/2) + (z/2)^2 + \cdots .$$

And both series converge in $1 < |z| < 2$. Putting this together we get:

$$\begin{aligned} \frac{1}{z(z-1)(z-2)} &= -\frac{1}{2z} \sum_{n=0}^{\infty} \frac{z^n}{2^n} - \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{1}{z^n} \\ &= -\sum_{n=0}^{\infty} \frac{z^{n-1}}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{z^{n+2}} = -\sum_{n=-1}^{\infty} \frac{z^n}{2^{n+2}} - \sum_{n=-2}^{\infty} z^n \ . \end{aligned}$$

Bonus question: Question 2 in Section 3.1, p. 201 in the text book.

Solution: (a) Let $r = |c| - 1$. By induction we show $|z_n| \geq |c| r^{n-1}$. First $|z_1| = |c| = |c| r^0$. Now suppose that $|z_n| \geq |c| r^{n-1}$. We know $|z_{n+1}| = |z_n^2 + c|$. We need to show $|z_n^2 + c| \geq |c| r^n$. But

$$|z_n^2 + c| = |z_n||z_n + c/z_n| \geq |c| r^{n-1} |z_n + c/z_n|,$$

by induction hypothesis. Thus it is enough to show

$$|z_n + c/z_n| \geq r = |c| - 1.$$

But

$$|z_n + c/z_n| \geq |z_n| - |c|/|z_n| \geq |c| - |c|/|z_n|. \text{ (triangle inequality)}$$

Again by induction hypothesis we have $|c|/|z_n| \leq 1/r^{n-1} \leq 1$. Thus

$$|z_n + c/z_n| \geq |c| - |c|/|z_n| \geq |c| - 1 = r,$$

as required. Now since $r > 1$, $\lim_{n\to\infty} |c|r^{n-1} = \infty$ and thus $\lim_{n\to\infty} z_n = \infty$.

(b) Suppose there is a value of k with $|z_k| > 2$. Using induction and repeating similar argument as in the part (a) (with c replaced with z_k) we can show that for any $p > 0$, $|z_{k+p}| \geq |z_k|r^p$ where $r = |z_k| - 1 > 1$. As $\lim_{p\to\infty} |z_k|r^p = \infty$, we conclude that $\lim_{n\to\infty} z_n = \lim_{p\to\infty} z_{k+p} = \infty$.