MATH3XO3 (Complex Analysis) Spring 2010

Problem Set 2 Solutions

Instructor: Kiumars Kaveh

1. Compute each of the following integrals (e.g. directly using definition, Fundamental Theorem of Calculus or Cauchy's theorem).

(a) $\int_\gamma (\sin(z) + 1/z)dz$, where γ is the line segment joining 2 to i.

(b) $\int_\gamma \frac{dz}{z^2-1}$, where γ is the circle of radius 1 centered at 1. Hint: use partial fractions.

Solution: a) $\sin(z)$ is entire and its anti-derivative is $\cos(z)$. Also anti-derivative of $1/z$ is $\log(z)$ which (using principal branch) is analytic on $\mathbb{C} \setminus \{x + i0 \mid x \leq 0\}$. This region contains the line segment joining 2 and i. Thus by Fundamental Theorem of Calculus we have

$$\int_\gamma \sin(z) + (1/z)dz = (-\cos(i) + \cos(2)) + (\log(i) - \log(2)).$$

We can slightly simplify this by noticing that $i = e^{i\pi/2}$ and hence $\log(i) = (\pi/2)i$ and $\cos(i) = (e^{-1} + e)/2$.

b) Note $z^2 - 1 = (z-1)(z+1)$. By solving $\frac{A}{z-1} + \frac{B}{z+1} = \frac{1}{z^2-1}$, we get $A = 1/2$ and $B = -1/2$. Thus the integral in question is equal to $(1/2)\int_\gamma 1/(z-1)dz + (-1/2)\int_\gamma 1/(z+1)dz$. The only point where $1/(z+1)$ is not analytic is $z = -1$ and hence $1/(z+1)$ is analytic on and inside of the closed curve γ. Now by Cauchy's theorem $\int_\gamma 1/(z+1)dz = 0$. And we already know that $\int_\gamma 1/(z-1)dz = 2\pi i$. Putting this together the total integral is equal to $(2\pi i)/2 = \pi i$.

2. Compute the integral $\int_\gamma \frac{dz}{1+z^2}$, where γ is the rectangle with vertices $(3+3i)$, $(3-3i)$, $(-3+3i)$, $(-3-3i)$ and traversed counter-clockwise. Hint: find the points at which $1/(1+z^2)$ is not analytic and compute the integral of $1/(1+z^2)$ around small circles (say of radius 1) centered at these points. Cite any theorems you use from the text.

Solution: Note $1 + z^2 = (z - i)(z + i)$. Now solving $\frac{A}{z-i} + \frac{B}{z+i} = \frac{1}{1+z^2}$ we get $A = 1/2i$ and $B = -1/2i$. Thus the integral in question is equal to

$$\int_\gamma \frac{1/2i}{z-i} + \frac{-1/2i}{z+i} dz.$$

The points where the function $g(z) = \frac{1/2i}{z-i} + \frac{-1/2i}{z+i}$ is not analytic are i and $-i$. Let γ_1, γ_2 be the unit circles centered at i and $-i$ respectively (traversed counter-clockwise). Since the function $g(z)$ is analytic in the area between the closed curves γ, γ_1 and γ_2, by the deformation theorem, we conclude that

$$\int_\gamma g(z)dz = \int_{\gamma_1} g(z)dz + \int_{\gamma_2} g(z)dz.$$

Now by Cauchy's theorem

$$\int_{\gamma_1} g(z)dz = \int_{\gamma_1} \frac{1/2i}{z-i} dz = \pi,$$

$$\int_{\gamma_2} g(z)dz = \int_{\gamma_2} \frac{-1/2i}{z+i} dz = -\pi.$$

Thus the total integral adds up to 0.

3.

(a) Give a parametrization of the ellipse $(x - 1)^2 + y^2/4 = 1$.

(b) Give a homotpoy between this ellipse and the point $z_0 = (1, 0)$ (inside some domain G containing the ellipse).

Solution: (a) The ellipse in question is obtained from the unit circle by stretching by 2 in the direction of y-axis and then shifting by 1 in the direction of x-axis. Thus the parametrization of it is

$$\gamma(t) = (\cos(2\pi t) + 1, 2\sin(2\pi t)),$$

where t ranges in the interval $[0, 1]$. (Similarly you can define

$$\gamma(t) = (\cos(t) + 1, 2\sin(t)),$$

where t ranges in the interval $[0, 2\pi]$.

(b) A homotopy $F : [0, 1] \times [0, 1] \to \mathbb{C}$ is given by:

$$F(s, t) = (s\cos(t) + 1, 2s\sin(t)).$$

This is a continuous function since its components are continuous functions. It is easy to see that when $s = 0$ we get the point $(1, 0)$ and when $s = 1$ we get the ellipse.

4. Prove using definition (Definition 2.3.9) that the set $G = \{z \mid 1/4 < |z - 1/4| < 3$ is not convex. Use Cauchy's theorem (Theorem 2.3.14) to show that the unit circle centered at the origin is not homotopic to a point in G (and hence G is not simply connected).

Solution: Both of the points 2 and -2 are in G but the line segment joining them does not completely lie in G, e.g. the point $1/4$ is not in G. Thus G is not convex. We know that the integral of $1/(z - 1/4)$ over the unit circle with center at $1/4$ is equal to $2\pi i$. Thus by the deformation theorem, the integral of $1/(z - 1/4)$ over the unit circle with center at the origin is also equal to $2\pi i$. On the other hand, as $1/(z - 1/4)$ is analytic on G, if it is homotopic to a point in G then by (homotopy form of) Cauchy's theorem its integral should be 0, which is not. Thus it follows that it is not homotpoic to a point in G.

5. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Moreover, assume that for any $z \in \mathbb{C}$ we have $f(z) = f(z+1) = f(z+i)$. Use Liouville's theorem (Theorem 2.4.8) to prove that f is necessarily a constant function. (Cite any other theorem that you use from the text.)

Solution: Since $f(z) = f(z+1) = f(z+i)$ we see that for any $z = a+ib$ we have $f(z) = f(w)$ where $w = (a - [a]) + (b - [b])i$. Note $[x]$ represent the *integer part* of x. So for example $[2.4] = 2$ and $2.4 - [2.4] = 0.4$. So the value of f at any z is equal to the value of f at some w where w lies in the unit square $[0, 1] \times [0, 1]$. But as f is entire it should be continuous. So it should be bounded on the compact set $[0, 1] \times [0, 1]$. Thus f is bounded on the whole $\mathbb{C}$. From Liouville's theorem it then follows that f is constant.

6. Evaluate $\int_\gamma \frac{\sin(e^z)}{z} dz$, where γ is the unit circle (centered at the origin).

Solution: It follows directly from Cauchy's integral formula that the integral in question is equal to $2\pi i \sin(e^0) = 2\pi i \sin(1)$.