MATH3XO3 (Complex Analysis) Spring 2010

Sketch of solutions for practice problems for final

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1. True or False? Justify.

- If f is an entire function and $\lim_{z\to\infty} f(z) = 1$ then f is the constant function $f = 1$.

  Solution: True. Use definition of limit to show that if f has a limit at infinity then f is bounded. Then by Liouville's theorem f should be constant equal to 1.

- If f is an entire function, for any $c \in \mathbb{C}$, the equation $f(z) = c$ has finite number of solutions.

  Solution: False. $\sin(z) = 0$ has infinite number of solutions.

- Suppose f is analytic in an open disk A except at one point $z_0 \in A$. Also suppose $\lim_{z\to z_0} f(z)$ exists. Then f has a removable singularity at z_0.

  Solution: True. If f is bounded then z_0 is not a pole or an essential singularity. Thus it is removable i.e. f can be extended to an analytic function on the whole A (see Prop. 3.3.4(1)).

- Any power series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is an analytic function inside its circle of convergence.

  Solution: True. Follows from Analytic Convergence Theorem and Power Series Converegence Theorem.

- For any power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ the circle of convergence has radius $R > 0$.

  Solution: False. Consider the power series $\sum_{n=0}^{\infty} n^n z^n$. From root test it follows that it diverges for all z.

- Let f and g be two functions which are analytic in some open region A. Suppose A contains a simple closed curve γ and $f(z) = g(z)$ for every $z \in \gamma$. Then f and g are equal at every point inside γ.

  Solution: True. Follows from the Cauchy Integral Formula applied to the curve γ and the functions f and g.

- Any closed curve in $A = \mathbb{C} \setminus \{0\}$ is homotopic to the unit circle (traversed counter-clockwise).

  Solution: False. A curve whose winding number at $z_0 = 0$ is not equal to 1 can not be homotopic to the unit circle traversed counter-clockwise which has winding number euqal to 1. Because by the Deformation Theorem the winding numbers of two homotopic curves are the same.

2. Find the singularities for the following functions:

(a) $f(z) = \frac{\cos(z)-1}{z^2}$.

(b) $f(z) = \frac{z+1}{z^2+9}$.

(c) $f(z) = (1 - z^2)e^{1/z}$.

Determine the type of singularity i.e. removable, pole or essential singularity. For poles determine their order. Find the residue at each singularity.

Solution: (a) Write the Taylor expansion of $\cos(z)$ at 0 to see that the only singularity of f is 0 which is removable. (b) $z = \pm 3i$ are the singularities which are poles of order 1. (c) 0 is an essential singularity because the Laurent expansion of $e^{1/z}$ has an infinite number of nonzero b_i terms.

3. Compute the integrals:

- $\int_\gamma \frac{\cos z}{z^2} dz$, where γ is the unit circle parametrized counterclockwise.

  Solution: Use the Caucht Integral Formula.

- $\int_\gamma \frac{z dz}{z^2-2z+2}$, where γ is the square with vertices $\pm 2 \pm 2i$ and parametrized counterclockwise.

  Solution: Factor the denominator and use partial fractions. Alternatively you can use the Cauchy Integral Formula for each of the roots of denominator.

- $\int_\gamma (1-z^2)e^{1/z}dz$ where γ is the unit circle parametrized counterclockwise.

  Solution: Use the Taylor series of e^w at $w=0$ with $w=1/z$ to find the Laurent series of $f(z)$. Then use the Residue Theorem to compute the integral.

4. Find the radius of convergence:

- $\sum_{n=0}^{\infty} \frac{z^{3n}}{27^n}$.
- $\sum_{n=0}^{\infty} \frac{z^n}{1+2^n}$.

Solution: Both can be computed using root test. For the first one, we get $|z|/3 < 1$ which implies that $R=3$. For the second one, we get $|z|/2 < 1$ and hence $R=2$.

5. For what complex values of z is $\sin(z) = \cos(z)$?

Solution: Write the definition of $\sin(z)$ and $\cos(z)$ in terms of the exponential function and collect the like terms together.

6. Find all the values of $\sqrt[4]{-6-6\sqrt{3}i}$.

Solution: Write $-6-6\sqrt{3}i$ in polar coordinates.

7. Compute the Taylor series of $f(z) = \log(3+z)$ at $z=0$, where log denotes the principal branch of the logarithm. What is the radius of convergence of this Taylor series?

Solution: Use the Taylor series of $\log(1+z)$ (see p. 209).

8. Find the maximum of $|\cos(z)|$ for $|z| \leq 2$.

Solution: Use $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and triangle inequality.

9. Use Cauchy-Riemann relations to show that the function $f(z) = \overline{z}^2$ is not analytic.

Solution: Let $z = x + iy$. Then $f(z) = u(z) + iv(z) = (x^2 + y^2) - 2xyi$, i.e. $u(z) = x^2 + y^2$ and $v(z) = -2xy$. Thus for the partial derivatives we have $u_x(z) = 2x$, $u_y(z) = 2y$ and $v_x(z) = -2y$ and $v_y(z) = -2x$. As $u_x \neq v_y$ for all z, f does not satisfy the Cauchy-Riemann relations and hence is not analytic.

10. Write the function $\sin(z)$ as $u(x, y) + iv(x, y)$ where $z = x + iy$. That is, find the real and imaginary components u and v for $\sin(z)$.

Solution: Use $\sin(x + iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$.

11. Find the Taylor series of $e^z + 1/(z - 3)$, at $z_0 = 1$.

Solution: Write of $e^z = e \cdot e^{z-1}$ and use the Taylor series of e^w at $w = 0$ then plug in $w = z - 1$. Also use the geometric series to find the Taylor series of $1/(z - 3)$.

12. Find the Laurent series of $f(z) = \frac{e^{1/z}+1}{z} + \frac{1}{z-1}$ in $0 < |z| < 1$.

Solution: For $\frac{e^{1/z}+1}{z}$ use the Taylore series of e^w at $w = 0$ and plug in $w = 1/z$. The Laurent series of $1/(z - 1)$ is straight forward.