Solutions to Problem Set 1

1 (a) $z^4 - (i+1)z^2 + i = 0.$

$$z^2 = \frac{(i+1) \pm \sqrt{(i+1)^2 - 4i}}{2} \qquad (i+1)^2 = (-1) + 2i + 1 = 2i$$

$$(i+1)^2 - 4i = -2i$$

$$= \frac{(i+1) \pm \sqrt{-2i}}{2} \qquad w^2 = -2i = 2e^{-i\pi/2} \Rightarrow$$

$$w = \sqrt{2}\, e^{-i\pi/4} \text{ or } -\sqrt{2}\, e^{-i\pi/4}$$

$$\Rightarrow z^2 = \frac{(i+1) + \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)}{2} = \frac{(i+1) + (1-i)}{2} = 1$$

or

$$z^2 = \frac{(i+1) + \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)}{2} = \frac{(i+1) + (-1+i)}{2} = i$$

$$\Rightarrow z = \pm 1 \text{ or } z = e^{i\pi/4} \text{ or } -e^{i\pi/4}.$$

(b) The roots of polynomial $z^5 + 1 = 0$ $\quad z^5 = -1 = e^{i\pi}$

$$z = e^{i\pi/5}, e^{i3\pi/5}, e^{i\pi} = -1, e^{i7\pi/5}, e^{i9\pi/5}$$

$$\Rightarrow z^5 + 1 = (z - e^{i\pi/5}) \cdots (z - e^{i9\pi/5})$$

$$\Rightarrow x^5 + y^5 = y^5\left(\left(\frac{x}{y}\right)^5 + 1\right) = y^5\left(\frac{x}{y} - e^{i\pi/5}\right) \cdots \left(\frac{x}{y} - e^{i9\pi/5}\right)$$

$$= (x - e^{i\pi/5} y) \cdots (x - e^{i9\pi/5} y).$$

(2)

(c) $\dfrac{1}{2-2i} = \dfrac{\overline{2-2i}}{|2-2i|^2} = \dfrac{2+2i}{4+4} = \dfrac{1+i}{4}$.

$\dfrac{3(\sqrt{3}+1)+3i(1-\sqrt{3})}{2-2i} = \dfrac{3}{4}\left((1+i)(\sqrt{3}+1)+i(1-\sqrt{3})\right)$

$= \dfrac{3}{4}\left(\sqrt{3}+\cancel{1}+i(\sqrt{3}+1)+i(1-\sqrt{3})\cancel{-1}+\sqrt{3}\right)$

$= \boxed{\dfrac{3}{4}(2\sqrt{3}+2i)} = \boxed{3\left(\dfrac{\sqrt{3}}{2}+i\dfrac{1}{2}\right)} = 3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$= \boxed{3e^{i\pi/6}}$.

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2 (a) Consider $f(z) = z^5$.

If $z = x+iy \Rightarrow Re(z^5) = Re(x+iy)^5 = x^5 - 10x^3y^2 + 5xy^4 = u(x,y)$

So $u(x,y)$ is the real part of a holomorphic function.

$Im((x+iy)^5) = 5x^4y - 10x^2y^3 + y^5 = v(x,y)$.

(b) z is in the region of analyticity for $\log(\sqrt{z})$ if z is in the region of analyticity for $\sqrt{z}$ i.e.

1 z is in $\mathbb{C}\setminus\{x+0i \mid x \leq 0\}$. (we use principal branch for $\sqrt{z}$).

2 $\sqrt{z}$ is in the region of analyticity for $\log(z)$ i.e. $\mathbb{C}\setminus\{x+0i \mid x \leq 0\}$. (we use principal branch for log (z)).

But image $\sqrt{z}$ consists of $re^{i\theta}$ where $-\pi/2 < \theta \leq \pi/2$, $r>0$ and hence image of $\sqrt{z} \subset \mathbb{C}\setminus\{x+0i \mid x\leq 0\}$. Thus 2 is automatically satisfied so the region of analyticity for $\log\sqrt{z}$ is $\mathbb{C}\setminus\{x+0i \mid x \leq 0\}$.

Note $f(z) = f(x+iy) = \sqrt{|xy|} + 0\,i$. ③

By def. the partial derivative $\frac{\partial f}{\partial x}(0,0)$ is

$\frac{\partial}{\partial x}\left(f(x,0)\right)\Big|_{x=0}$. But $f(x,0) = 0 \Rightarrow f'(x,0) = 0$.

so $\frac{\partial f}{\partial x}(0,0) = 0$. Similarly $\frac{\partial f}{\partial y}(0,0) = 0$.

Clearly these satisfy C-R relations.

If f is analytic at the origin then $f'(0) = 0 + i0$.

So by def. the limit $\lim_{z\to 0} \frac{f(z)-0}{z-0}$ should be $\underline{0}$.

~~But~~ Thus $\lim_{z\to 0} \left|\frac{f(z)}{z}\right| = 0$. But if $z = t+it$ then

as $t \to 0$, $z \to 0+i0 = 0$. Then $\left|\frac{f(t+it)}{(t+it)}\right| = \frac{\sqrt{|t^2|}}{\sqrt{t^2+t^2}}$

$= \frac{|t|}{\sqrt{2}\,|t|} = \frac{1}{\sqrt{2}} \neq 0$ which is a contradiction so

$f'(0)$ does not exist.

This does not contradict Thm. 1.5.8 because for this theorem to apply f should be differentiable at $(0,0)$ in the sense of real-variables but $\sqrt{|xy|}$ is not. (it only has partial derivatives).

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4) Let $z = x+iy$ be on the circle $|z| = c$.

Then $z - \frac{1}{z} = x+iy - \frac{(x-iy)}{\sqrt{x^2+y^2}}$ $\left(= z - \frac{z}{|z|}\right)$.

$= x+iy - \frac{x+iy}{c} = \frac{(c-1)}{c}x + i\frac{(c+1)}{c}y$. So $z \mapsto z - \frac{1}{z}$

on the circle $|z| = c$ is the same as the linear trans.

$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} \frac{c-1}{c}x \\ \frac{c+1}{c}y \end{bmatrix}$. This maps a circle centred at the origin to an ellipse centred at origin.

If $X = \frac{c-1}{c}x \quad Y = \frac{c+1}{c}y \Rightarrow \left(\frac{c}{c-1}\right)X = x$

$$\left(\frac{c}{c+1}\right)Y = y$$

$$x^2+y^2 = c^2 \Rightarrow \frac{c^2}{(c-1)^2}X^2 + \frac{c^2}{(c+1)^2}Y^2 = c^2 \Rightarrow \frac{X^2}{(c-1)^2} + \frac{Y^2}{(c+1)^2} = 1.$$

which equ. of an ellipse.

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5). Note that if $z = x+iy$ $\tan(\operatorname{Arg}(z)) = \frac{y}{x}$. Thus whenever $x \neq 0$ (i.e. away from the y-axis) it is continuous.

Now let us see it is not continuous at $(0,0)$.

Let $z_k = \left(\frac{1}{k} + i\frac{1}{k}\right)$. Then $z_k \longrightarrow 0+0i$ as $k \to \infty$.

Then ~~tan(Arg~~ $\operatorname{Arg}(z_k) = \pi/4$ $\forall k>0$ and

$\tan(\operatorname{Arg}(z_k)) = 1 \not\longrightarrow 0 = f(0)$.

So $\underset{1=}{f(z_k)} \not\to 0$ i.e. f not contin. at $z=0$.

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6) We use Inverse Function Theorem.

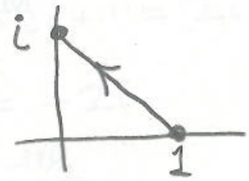
Suppose at some point $z_0 \in A$ we have $f'(z_0) \neq 0$.

Then by I.F.T. we can find small discs $U \ni z_0$ & $V \ni f(z_0)$ s.t. $f: U \longrightarrow V$ is invertible. In particular $f(U) = V$ so $V \subset \operatorname{Image}(f)$ but Image of f is contained in the line $2x=y$ which is not possible became the line $2x=y$ can not contain any disc U.

So we conclude that at every $z_0 \in A$ we have $f'(z_0) = 0$ which then implies that f is constant.

7(a) We know $\log(z)$ is an anti-derivative for $\frac{1}{z}$ in the region $\mathbb{C} \setminus \{x+i0 \mid x \leq 0\}$ (we use principal branch) & this region contains the line segment joining 1 to i.

So by Fundamental Th. of Calculus:



$$\int_\gamma \frac{1}{z}\,dz = \log(i) - \log(1)$$

$$= \log(e^{i\pi/2}) - 0 = i\pi/2 .$$

(b)

$$\int_\gamma \frac{3}{(z-i)} + e^z = \int_\gamma \frac{3}{(z-i)} + \int_\gamma e^z .$$

$(e^z)' = e^z$ i.e. it is its own anti-derivative thus by the Fundamental Th. of Calculus & since γ is a closed curve we have $\int_\gamma e^z = 0.$

Parametrize γ by: $\gamma(t) = i + 2e^{i\theta}$ $0 \leq \theta \leq 2\pi.$

$$\int_\gamma \frac{3}{z-i} = \int_0^{2\pi} \frac{3}{2e^{i\theta}}\,\gamma'(t)\,dt = \int_0^{2\pi} \left(\frac{3}{2}\right)\cancel{e^{-i\theta}} \cdot (2i)\,\cancel{e^{i\theta}}\,d\theta$$

$$= 3i \int_0^{2\pi} 1\,d\theta = 2\pi\,(3i) = 6\pi i .$$
