

MATH3XO3 (Complex Analysis) Spring 2010

Problem Set 1

Due on February 11 in the tutorial.

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1.

- (a) Determine all the complex solutions of the equation $z^4 - (i+1)z^2 + i = 0$.
- (b) Decompose the polynomial $X^5 + Y^5$ into a product of linear polynomials (with complex coefficients).
- (c) Write the complex number $\frac{3(\sqrt{3}+1)+3i(1-\sqrt{3})}{2-2i}$ in the form $x + iy$ and the form $r(\cos \theta + i \sin \theta)$.

2.

- (a) Show that the function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, $u(x, y) = x^5 - 10x^3y^2 + 5xy^4$ is harmonic. Find a harmonic conjugate $v(x, y)$ for it.
- (b) Differentiate and give the appropriate region of analyticity for $f(z) = \log(\sqrt{z})$.

3. Let $z = x + iy$. Show that the function $f(z) = \sqrt{|xy|}$ is continuous and satisfies the Cauchy-Riemann relations at the origin but is not analytic at the origin. (Why doesn't this contradict the Cauchy-Riemann theorem (Theorem 1.5.8)?)

4. For any $c > 0$, what is the image of the circle $\{z \mid |z| = c\}$ under the map $z \mapsto z - 1/z$? Justify/prove your answer.

5. Consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(z) = 0$ when $\operatorname{Re}(z) = 0$ and $f(z) = \tan(\operatorname{Arg}(z))$ when $\operatorname{Re}(z) \neq 0$ (where Arg is

the principal branch of argument). Show that f is continuous on $\mathbb{C} \setminus y$ -axis. Also show that f is discontinuous at the origin as follows: find a sequence z_i converging to 0 such that $f(z_i)$ does not converge to 0.

6. Let $f : A \rightarrow \mathbb{C}$ be analytic (where A is a connected open subset of \mathbb{C}). Show that if $f(A)$ is contained in the line $\{x+iy \mid y = 2x\}$ then f is constant.

7.

(a) Evaluate $\int_{\gamma} dz/z$, where γ is the line segment joining 1 and i .

(b) Evaluate $\int_{\gamma} \frac{3}{(z-i)} + e^z dz$, where γ is the circle of radius 2 centred at i and traversed counter-clockwise.