

MATH 2R03: LINEAR ALGEBRA II
HOMEWORK 2
DUE: FRIDAY 25 SEPTEMBER 2009

Required problems (to be handed in):

1. Let $V = C[0, 1]$ be the (infinite-dimensional) real vector space of continuous functions on the interval $[0, 1]$. Let $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ be an inner product on V defined by

$$\langle f, g \rangle := \int_0^1 f(x)g(x)dx. \quad (0.1)$$

- (a) What is $\|\sin(\pi x)\|$ in this inner product space?
(b) As in \mathbb{R}^n , the distance between two vectors u and v in an inner product space is defined to be $\|u - v\|$. Compute the distance between the functions $x + 1$ and e^x in this inner product space.
(c) Prove: the functions $\sin(2\pi x)$ and $\cos(2\pi x)$ are orthogonal in this inner product space.
2. Anton-Rorres, Chapter 6, Section 2, Exercise 17.
3. Anton-Rorres, Chapter 6, Section 2, Exercise 23.

Recommended problems (not to be handed in):

1. (Challenge.) Let $V = \mathbb{R}^n$. Let A be an $n \times n$ real matrix. Define a function $\langle \cdot, \cdot \rangle_A : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\langle v, w \rangle_A := v^T A w.$$

Give conditions on A so that: $\langle \cdot, \cdot \rangle_A$ is a valid inner product on \mathbb{R}^n if and only if the conditions you give are satisfied by A . (Hint: think about eigenvalues and eigenvectors.) Prove/justify your answer.

2. Let W be a subspace of \mathbb{R}^n , considered with the usual (standard) inner product. Define the **orthogonal complement** of W to be the subspace

$$W^\perp := \{v \in \mathbb{R}^n : \langle w, v \rangle = 0, \text{ for all } w \in W\}.$$

- (a) Consider the subspace $W = \text{span}\{(1, 1, 1)^T, (2, 0, -1)^T\}$ of \mathbb{R}^3 . Find a vector that spans W^\perp . Justify your answer.
(b) Express the vector $(2, 1, -3)$ in the form $w + w^\perp$, where $w \in W$ and $w^\perp \in W^\perp$. Justify your computations. Is your answer unique? Why or why not? Justify.
3. In an inner product space V , the **unit sphere** is the set of all vectors v for which $\|v\| = 1$. Note that this 'sphere' may not be the conventional sphere that you imagine, if the dimension of V is different from 3. Describe the unit spheres of each of the following inner product spaces.
- (a) $V = \mathbb{R}^1$ with the inner product given by multiplication of real numbers.
(b) $V = \mathbb{R}^2$ with the usual dot product.
(c) $V = \mathbb{R}^3$ with the usual dot product.
(d) $V = P_3$ (the vector space of polynomials in one variable of degree ≤ 3 : see Anton-Rorres Chapter 5, Section 3, Example 5) with the inner product given in Required Problem 1 above in equation (0.1).