Asset price bubbles: economics, mathematics and statistics

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Outline

The optimization problem

- Consider a representative agent solving

$$\sup_c E_t \left[\sum_{j=1}^{\infty} \beta^{j-t} u(c_j) \right] \tag{1}$$

where $\{c_t\}_{t=0}^{\infty}$ is the consumption of a single perishable good and $0 < \beta < 1$ a constant discount factor.

- The agent receives an endowment e_t at each period and can smooth consumption by buying x_t shares at price p_t each paying a dividend d_t.
- The budget constraint at time j is then

$$c_j \leq e_j + d_j x_j + p_j(x_j - x_{j+1}). \tag{2}$$

- We take $\{e_t, d_t\}$ as exogenous and want to determine p_t.

The pricing equation

- From the first-order condition for optimality we find

$$p_t u'(c_t^*) = \beta E_t \left[(p_{t+1} + d_{t+1}) u'(c_{t+1}^*) \right] \tag{3}$$

- When the utility is sufficiently regular, the budget constraint is binding. Normalizing the number of shares to unit then leads to

$$c_t^* = e_t + d_t \tag{4}$$

- Inserting (4) into (3) leads to the following pricing equation

$$p_t u'(e_t + d_t) - \beta E_t \left[p_{t+1} u'(e_{t+1} + d_{t+1}) \right] = \beta E_t \left[d_{t+1} u'(e_{t+1} + d_{t+1}) \right] \tag{5}$$

The general solution

- Denoting $q_t = u'(e_t + d_t)p_t$ we have that the general solution of (5) is of the form

$$q_t = F_t + B_t,$$

where F_t is the fundamental price and B_t is a bubble term.

- We identify F_t with the particular solution associated with the inhomogeneous term in (5), that is

$$F_t = \sum_{j=1}^{\infty} \beta^j E_t \left[d_{t+j} u'(e_{t+j} + d_{t+j}) \right]. \qquad (6)$$

- The bubble term B_t is a solution to the homogeneous equation and satisfy

$$E_t[B_{t+1}] = \beta^{-1} B_t \qquad (7)$$

Examples

- The simplest bubble consists of $B_t = \beta^{-t} B_0$.
- More generally, consider

$$B_{t+1} = \beta^{-1} B_t + z_{t+1}, \quad E_t[z_{t+1}] = 0. \tag{8}$$

- The general solution of (8) is

$$B_t = \beta^{-t} B_0 + \sum_{s=1}^{t} \beta^{s-t} z_s. \tag{9}$$

- A simple example is

$$z_t(\theta_{t+1} - \beta^{-1}) B_t + \varepsilon_{t+1}, \quad E_t[\theta_{t+1}] = \beta^{-1}, E_t[\varepsilon_{t+1}] = 0,$$

  where $\{\theta_t, \varepsilon_t\}$ are mutually and serially independent.
- This leads to

$$B_{t+1} = \theta_{t+1} B_t + \varepsilon_{t+1}. \tag{10}$$

- If $\varepsilon \equiv 0$, this bubble will crash in each period with probability $\Pi = P(\theta_t = 0)$.

Positivity and inception

- Observe that it follows directly from (7) that

$$E_t[B_{t+j}] = \beta^{-j} B_t, \quad \forall j > 0. \tag{11}$$

- Since $\beta^{-1} > 1$, we see that $E_t[q_{t+j}] \to \pm\infty$.
- Given free disposal, we conclude that $B_t \geq 0$ for all t.
- But this implies that $z_{t+1} \geq -\beta^{-1} B_t$ for all t.
- Now if $B_s = 0$ for some s, then $z_{s+1} \geq 0$.
- But since $E_s[z_{s+1}] = 0$ we see that $z_{s+1} = 0$ a.s.
- Therefor any nonzero rational bubble must start with $B_0 > 0$.
- In particular, at the first day of trade we have

$$E_{-1}[q_0 - F_0] = E_{-1}[B_0] > 0$$

- The same argument shows that $\varepsilon \equiv 0$ in (10), so once a bubble burst it cannot restart either.

Diverse Information

- Consider agents $i = 1, \ldots, I$ who exchange claims with a random value $\widetilde{p}$ with realizations in $E \subset \mathbb{R}$ at price p.
- Each agent receives a private signal $s^i \in S^i$. Let $s = (s^1, \ldots, s^I) \in S = \prod_i S^i$.
- Take $\Omega = E \times S$ and assume that all agents have the same prior ν on Ω.
- Denote by x^i the transaction by agent i and by $G^i = (\widetilde{p} - p)x^i$ the corresponding gain.
- We say that the market is *purely speculative* if the initial positions of the agents (no trade) are uncorrelated with the return on the asset and the set of signals.

Static Rational Expectations Equilibrium

- A REE is a forecast function $\Phi : s \to p$ and a set of transactions $x^i(p, s^i, s(p))$ with $s(p) \in S(p) := \Phi^{-1}(p)$ such that
  1. x^i maximizes i's expected utility.
  2. $\sum_i x^i = 0$.

Proposition (Tirole,1982)

In a REE of a purely speculative market, risk-averse agents do not trade and risk-neutral agents may trade but do not expect any gain from their trade.

- The key idea for the proof is that $E[G^i | s^i, S(p)] \geq 0$. Using the market clearing condition $\sum_i G^i = 0$ and Bayes rule with identical priors gives the result.

Dynamic Speculation

- Consider now trades at $t = 0, 1, 2, \ldots$ and a (nonnegative) dividend process d_t.
- Suppose that agents are risk neutral with common discount factor $0 < \beta < 1$.
- Assume that $s_t^i \in \mathcal{F}_t^i \subset S^i$ and $\mathcal{F}_t^i \subset \mathcal{F}_{t+1}^i$.
- A myopic REE is a sequence of forecast functions $\Phi_t : s_t \to p_t$ and transactions $x_t^i(p_t, s_t^i, s_t(p_t))$ with $s_t(p_t) \in S_t(p_t) := \Phi_t^{-1}(p_t)$ such that
  1. $p_t = E[\beta d_{t+1} + \beta p_{t+1} | s_t^i, S_t(p_t)]$
  2. $\sum_i x^i = \bar{x}$.

Bubbles in myopic REE

- Given the information $(s_t^i, S_t(p_t))$, define

$$F(s_t^i, \Phi_t^{-1}(p_t)) = E\left[\sum_{j=1}^{\infty} \beta^j d_{t+j} | s_t^i, S_t(p_t)\right],$$

and a bubble as $B(s_t^i, p_t) = p_t - F(s_t^i, S_t(p_t))$.

Proposition (Tirole, 1982)

In a stock market with horizon $T < \infty$, price bubbles are all equal to zero for all agents.

Proposition (Tirole, 1982)

In the infinite horizon case, price bubbles satisfy

$$B(s_t^i, p_t) = \beta^T E[B(s_{t+T}^i, p_{t+T}) | s_t^i, S_t(p_t)].$$

Example of a myopic REE with bubble

- Let $d_t = 1$, $\beta = 1/2$, so that $p_t = \frac{1}{2}(1 + p_{t+1})$.
- The general solution is then $p_t = 1 + \alpha 2^t$.
- Consider $\alpha = 1$ and agents A and B.
- Then the following sequence of trades is a myopic REE:
  1. at $t = 0$ A sells to B for 2
  2. at $t = 1$ B sells to A for 3
  3. at $t = 2$ A sells to B for 5
  4. $\vdots$
- Observe that the corresponding gains are

$$G^A = 2 - \frac{1}{2}(3) + \frac{1}{4}(5+1) - \frac{1}{8}(9) + \frac{1}{16}(17+1) - \dots$$

$$G^B = -2 + \frac{1}{2}(3+1) - \frac{1}{4}(5) + \frac{1}{8}(9+1) - \frac{1}{16}(17) + \dots$$

- But at any moment A (resp. B) can gain 2 (resp. 0) by leaving just after selling, which is in fact a dominant strategy !

Fully Dynamic REE

- A fully dynamic REE is a sequence of forecast functions $\Phi_t : s_t \to p_t$ and transactions $x_t^i(p_t, s_t^i, s_t(p_t))$ with $s_t(p_t) \in S_t(p_t) := \Phi_t^{-1}(p_t)$ such that
  1. x_t^i maximizes i's maximizes expected present discounted gains from t onwards.
  2. $\sum_i x^i = \bar{x}$.

Proposition (Tirole, 1982)

Whether short-sales are allowed or not, price bubbles do not exist in a fully dynamic REE and

$$F(s_t^i, S_t(p_t)) = p_t.$$

- **Moral**: If all agents plan to sell in finite time, there will be nobody left to buy thereafter. Therefore there can be no bubbles with finitely many infinitely lived agents

Overlapping generations

- An alternative is to consider overlapping consumers in a Diamond (1965) growth model.
- This consists of consumers who live for two periods and have utility $u(c^y, c^o)$
- Define wages w_t, production function $Y_t = L_t f(k_t)$ (for labor force L_t and capital stock k_t), savings function $s(w_t, r_{t+1})$, and real interest rate $r_t = f'(k_t)$.
- These assumptions uniquely define an asymptotic real interest rate $\bar{r}$.
- Tirole (1985) then shows that a bubble can exist provided $0 < \bar{r} < g$, where g is the rate of growth of the economy.

The Efficient Markets Hypothesis

- Denote $R_{t+1} = \frac{p_{t+1} - p_t + d_{t+1}}{p_{t+1}}$.
- As we have seen, a first-order rational expectations condition for risk-neutral agents lead to

$$E_t[R_{t+1}] = 1 + r. \qquad (12)$$

- Solving this recursively leads to

$$p_t = \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} E_t[d_{t+j}], \qquad (13)$$

plus a possible rational bubble term satisfying $E_t[B_{t+1}] = (1+r)B_t$.

- Either (12) or (13) can be taken as an EMH.
- Statistical tests on actual returns indicate that they are not *very* forecastable, leading to the conclusion that the EMH cannot be rejected.

Alternative models (Shiller, 1984)

- Consider a model where sophisticated investors have a demand function (portion of shares) of the form

$$Q_t^i = \frac{E_t[R_{t+1}] - \alpha}{\phi}. \tag{14}$$

- In addition, suppose there are noise traders who react to fads Y_t through a demand function $Q_t^n = Y_t/p_t$.
- In equilibrium we have $Q_t + \frac{Y_t}{p_t} = 1$.
- Inserting this into (14) and solving recursively leads to

$$p_t = \sum_{j=1}^{\infty} \frac{E_t[d_{t+j}] + \phi E_t[Y_{t-1+j}]}{(1+\alpha+\phi)^j}. \tag{15}$$

- This is also consistent with prices being not very forecastable.

A model with noise traders (DeLong, Shleifer, Summers and Waldmann, 1990)

- Consider a safe asset (s) with perfectly elastic supply paying a dividend leading to a constant price 1 and an unsafe asset (u) with fixed unit supply and the same dividend rate.
- Suppose that a proportion μ of the agents are noise traders.
- According to their beliefs when young, all agents want to maximize the expected values of an identical utility $u(w) = -e^{-2\gamma w}$, where w is their wealth when old.
- Sophisticated investors accurately perceive the distribution of (u), whereas noise traders young at t misperceives its expected value by an i.i.d random variable

$$\rho_t \sim N(\rho^*, \sigma_\rho^2)$$

Demand functions

- It follows that each group maximizes $\bar{w} - \gamma\sigma_w^2$.
- Accordingly,

$$\begin{aligned}\bar{w}^i - \gamma\sigma_w^2 &= c_0^i + Q_t^i(r + E_t[p_{t+1}] - p_t(1+r)) \\ &\qquad - \gamma\left(Q_t^i\right)^2 \mathrm{Var}_t[p_{t+1}] \\ \bar{w}^n - \gamma\sigma_w^2 &= c_0^n + Q_t^n(r + E_t[p_{t+1}] - p_t(1+r) + \rho_t) \\ &\qquad - \gamma\left(Q_t^n\right)^2 \mathrm{Var}_t[p_{t+1}]\end{aligned}$$

- The corresponding optimal demands are

$$Q_t^i = \frac{r + E_t[p_{t+1}] - p_t(1+r)}{2\gamma \mathrm{Var}_t[p_{t+1}]} \tag{16}$$

$$Q_t^n = \frac{r + E_t[p_{t+1}] - p_t(1+r)}{2\gamma \mathrm{Var}_t[p_{t+1}]} + \frac{\rho_t}{2\gamma \mathrm{Var}_t[p_{t+1}]} \tag{17}$$

Equilibrium price

- At equilibrium we have $(1-\mu)Q_t^i + \mu Q_t^n = 1$.
- This leads to the pricing equation

$$p_t = \frac{1}{1+r}\left(r + E_t[p_{t+1}] + \mu\rho_t - 2\gamma \mathrm{Var}_t[p_{t+1}]\right).$$

- Assuming stationary unconditional distributions, we find the steady state solution

$$p_t = \underbrace{1}_{\text{fundamental}} + \overbrace{\frac{\mu(\rho_t - \rho*)}{1+r}}^{\text{misconceptions at } t} + \underbrace{\frac{\mu\rho^*}{r}}_{\text{price pressure}} - \overbrace{\frac{2\gamma\mu^2\sigma_\rho^2}{(1+r)^2}}^{\text{compensation}} \qquad (18)$$

Expected returns

- The difference in returns between the two groups of investors is

$$\Delta R_{n-i} = (Q_t^n - Q_t^i)(r + p_{t+1} - p_t(1+r)) = \frac{(1+r)^2 \rho_t}{2\gamma\mu^2\sigma^2\rho}$$

- Using the pricing equation and taking expectations leads to

$$E[[\Delta R_{n-i}] = \underbrace{\rho^*}_{\text{hold more}} - \frac{\overbrace{(1+r)^2(\rho^*)^2}^{\text{price pressure}} + \overbrace{(1+r)^2\sigma_r^2 ho}^{\text{Friedman}}}{\underbrace{2\gamma\mu\sigma^2\rho}_{\text{create space}}}$$

- **Moral:** Noise traders create their own space and can earn higher return by holding the risk they create themselves.

A model for arbitrage funds (Shleifer and Vishny, 1997)

- Consider now three types of agents: noise traders, arbitrageurs and investors in arbitrage funds who do not trade.
- Let $t = 1, 2, 3$ and suppose that the fundamental value of an asset is revealed to be V to all investors at $t = 3$.
- Arbitrageurs know V at all times and bet against noise traders, who receive pessimistic shocks S_t determining their demand as
$$Q_t^n = \frac{V - S_t}{p_t}. \tag{19}$$
- In addition, arbitrageurs have limited funds F_t received from investors.

Equilibrium prices

- Assume everything is known at $t = 1$.
- If $S_2 = 0$, the price recovers to V and arbitrageurs invest in cash.
- Otherwise, the demand function for arbitrageurs is $Q_2^a = F_2/p_2$.
- In equilibrium

$$p_2 = V - S_2 + F_2,$$

  where we assume that $F_2 < S_2$.
- At $t = 1$, arbitrageurs might decide to invest $D_1 \leq F_1$, so that $Q_1^a = D_1/p_1$ and the equilibrium price is

$$p_1 = V - S_1 + D_1,$$

  where we again assume that $F_1 < S_1$.

Performance based arbitrage

- To complete the model, assume that

$$F_2 = F_1 - aD_1 \left(1 - \frac{p_2}{p_1}\right) \tag{20}$$

- The parameter $a \geq 0$ measures the fund sensitivity to performance.
- In the absence of PBA, $a = 0$ and funds are constant.
- When $a > 1$, investors penalized for losing money.

Optimal investment at $t = 1$

- We assume that arbitrageurs choose D_1 to maximize profits at $t = 3$.
- For concreteness, let

$$S_2 = \begin{cases} S > S_1 & \text{with prob } q \\ 0 & \text{with prob } 1 - q \end{cases} \tag{21}$$

- The expected payoff at $t = 3$ is

$$E[W] = (1-q)\left[F_1 - aD_1\left(1 - \frac{V}{p_1}\right)\right] + q\left[F_1 - aD_1\left(1 - \frac{p_2}{p_1}\right)\right]\frac{V}{p_2}$$

- The first-order condition is

$$(1-q)\left(\frac{V}{p_1} - 1\right) + q\left(\frac{p_2}{p_1} - 1\right)\frac{V}{p_2} \geq 0, \tag{22}$$

with equality at the interior solution $D_1 < F_1$ and a strict inequality at the corner solution $D_1 = F_1$.

Properties

Proposition

For any given V, S_1, S, F_1, a, there exits a unique q^* such that

1. *if $q > q^*$, then $D_1 < F_1$, $\frac{dp_1}{dS_1} < 0$, $\frac{dp_2}{dS} < 0$ and $\frac{dp_1}{dS} < 0$.*
2. *if $q < q^*$, then $D_1 = F_1$, $\frac{dp_1}{dS_1} < 0$, $\frac{dp_2}{dS} < -1$, and $\frac{dp_1}{dS} = 0$. Moreover, if $S_2 = S$ and $a > 1$, then $F_2 < D_1$ and $\frac{F_2}{p_2} < \frac{D_1}{p_1}$.*

- **Morals:**
  1. Larger shocks lead to less efficient pricing.
  2. For large q, arbitrageurs spread the risk of a deeper shock at $t = 2$ and prices at $t = 1$ fall even further.
  3. Markets lose resilience under PBA.
  4. Arbitrageurs bail out when opportunities are best.

A model for credit (Allen and Gale, 2000)

- Suppose there is a continuum of small, risk-neutral investors with no wealth of their own and a continuum of small, risk-neutral banks with $B > 0$ funds to lend at rate r trading at $t = 1, 2$.
- Consider a safe asset (s) with return $(1 + r)$ and a risky asset (R) with price at $t = 2$ given by a random variable p_2 with density $h(p_2)$ on $[0, p_2^{\max}]$ and mean $\overline{p_2}$.
- In addition, there is a production function $f(x)$ for the economy and an investment cost $c(x)$.

The optimization problem

- A representative investor needs to choose quantities Q_1^s and Q_1^R of the safe and unsafe assets at time $t = 1$ at prices 1 and p_1, respectively.
- In other words, the investor needs to borrow an amount

$$Q_1^s + Q_1^R p_1.$$

- Since the investor can default on the loan, the optimization problem is

$$\max_{Q_1^R \geq 0} \int_{(1+r)p_1}^{p_2^{\max}} (p_2 - (1+r)p_1) Q_1^R h(p_2) dp_2 - c(Q^R), \quad (23)$$

subject to the market clearing conditions

$$Q_1^R = 1 \quad (24)$$

$$Q^s + p_1 Q_1^R = B \quad (25)$$

$$1 + r = f'(Q_1^s) \quad (26)$$

Equilibrium

- The first-order condition for (23) with $Q_1^R = 1$ then gives

$$\int_{(1+r)p_1}^{p_2^{\max}} (p_2 - (1+r)p_1)h(p_2)dp_2 = c'(1), \tag{27}$$

with the remaining market clearing condition being

$$1 + r = f'(B - p_1). \tag{28}$$

- Solving (27) and (28) for (r, p_1) gives an equilibrium.
- A sufficient condition for existence is $\overline{p_2} > c'(1)$.

Fundamental value

- Define the fundamental value as the price that an investor would pay if he had to use his own money $B > 0$.
- In other words, he would solve

$$\max_{Q_1^s, Q_1^R} \int_0^{p_2^{\max}} [(1+r)Q_1^s + p_2 Q_1^R] h(p_2) dp_2 - c(Q^R), \quad (29)$$

subject to $Q_1^s + Q^R p_1^F \leq B$.

- The first-order condition for (29) with $Q_1^R = 1$ now gives

$$\int_0^{p_2^{\max}} p_2 h(p_2) dp_2 - c'(1) - (1+r) p_1^F = 0. \quad (30)$$

Existence of bubbles

- Observe that (27) can be rewritten as

$$p_1 = \frac{1}{1+r}\left[\frac{\int_{(1+r)p_1}^{p_2^{\max}} p_2 h(p_2) dp_2 - c'(1)}{\text{Prob}[p_2 \geq (1+r)p_1]}\right]. \qquad (31)$$

- Similarly, (28) can be rewritten as

$$p_1^F = \frac{\overline{p_2} - c'(1)}{1+r}. \qquad (32)$$

Proposition

$p_1 \geq p_1^F$ *with strict inequality iff* $Prob[p_2 < (1+r)p_1] > 0$.

Uncertainty in the financial sector

- Consider now $t = 0, 1, 2$ and suppose that $p_2 = \overline{p_2}$.
- Assume that central banks can alter the amount of credit available, so that B_1 is a random variable with density $k(B_1)$ on $[0, B_1^{\max}]$ and $B_0 > 0$ is known at $t = 0$.
- Since there is no risk of default at $t = 2$, we have that the equilibrium price at $t = 1$ satisfies

$$\begin{cases} p_1 = \frac{\overline{p_2} - c'(1)}{1+r} \\ 1 + r_1 = f'(B_1 - p_1) \end{cases} \tag{33}$$

Equilibrium at $t = 0$

- An investor at $t = 0$ now needs to solve

$$\max_{Q_0^R \geq 0} \int_{B_1^*}^{B_1^{\max}} (p_1(B_1) - (1 + r_0)p_0) Q_0^R k(B_1) dB_1 - c(Q_0^R), \tag{34}$$

  where $p_1(B_1^*) = (1 + r_0)p_0$.

- As before, an equilibrium is a pair (r_0, P_0) solving

$$\begin{cases} p_0 = \frac{1}{1+r_0} \left[\frac{\int_{B_1^*}^{B_1^{\max}} p_1(B_1)k(B_1)dB_1 - c'(1)}{\mathrm{Prob}[B_1 \geq B_1^*]} \right] \\ 1 + r_0 = f'(B_0 - p_0) \end{cases} \tag{35}$$

Bubbles and financial instability

- Defining again the fundamental value as the price that an investor would pay if he had to use his own money, we find

$$p_0^F = \frac{\overline{p_1(B_1)} - c'(1)}{1 + r_0} \tag{36}$$

Proposition

$p_0 \geq p_0^F$ *with strict inequality iff* $Prob[B_1 < B_1^*] > 0$.

Proposition

As $c'(1) \to 0$, $B_1^* \to B_1^{\max}$.

- **Moral:** As investment costs go to zero, crashes occur whenever the credit does not occur at its maximum value.

Optimists, pessimists, and bubbles (Miller, 1977)

- Let $t = 0, 1$ and consider a risky asset with value at $t = 1$ given by

$$\widetilde{f} = \mu + \varepsilon, \quad \mu \in \mathbb{R}, \quad \varepsilon \sim N(0, \sigma^2) \tag{37}$$

- Suppose there is a continuum of investors with beliefs parametrized by μ_i uniformly distributed on the interval $[\mu - k, \mu + k]$.
- At $t = 0$, each investor chooses Q^i according to

$$\max_{Q^i} E\left[-e^{-\gamma(w_0 + Q^i(\widetilde{f} - p_0))}\right]. \tag{38}$$

where p_0 is the market price and the market clearing condition is

$$\int_i Q^i(\mu_i) = Q \tag{39}$$

Fundamental value

- In the absence of short-sale constraints, the optimal demand for each investor is

$$Q^i = \frac{\mu_i - p_0^F}{\gamma\sigma^2}.$$

and the market clearing condition gives

$$\int_{\mu-k}^{\mu+k} \frac{\mu_i - p_0^F}{\gamma\sigma^2} \frac{d\mu_i}{2k} = Q$$

- The fundamental price is then

$$p_0^F = \mu - \gamma\sigma^2 Q \tag{40}$$

The effect of no short-sales

- Imposing no short-sales leads to

$$Q^i = \max\left\{\frac{\mu_i - p_0}{\gamma\sigma^2}, 0\right\}$$

- Using the market clearing condition, this leads to an equilibrium price

$$p_0 = \begin{cases} \mu - \gamma\sigma^2 Q & \text{if } k \leq \gamma\sigma^2 Q \\ \mu + k - 2\sqrt{k\gamma\sigma^2 Q} & \text{if } k > \gamma\sigma^2 Q \end{cases}$$

- **Moral:** When the divergence of beliefs is large enough and there are short-sale constraints, asset prices reflect the opinion of optimists and exhibit a bubble.

Dynamic speculation in discrete-time (Harrison and Kreps, 1978)

- Consider now two groups A and B of risk-neutral agents with constant discount rate $0 < \beta < 1$.
- Each group regards a dividend stream d_t, $t = 1, 2, \ldots$, as a process on $(\Omega, \mathcal{F}, P^h)$, $h \in \{A, B\}$, with $P^A \sim P^B$.
- Let $\mathcal{F}_t$ be generated by $(d_s)_{1 \leq s \leq t}$.
- Assuming a fixed unit supply for the asset and no short-sales, competition leads to an equilibrium price of the form

$$p_t = \max_h \sup_{\tau > t} E^h \left[\sum_{j=t+1}^{\tau} \beta^{j-t} d_j + \beta^{\tau - t} p_\tau \middle| \mathcal{F}_t \right] \qquad (41)$$

A sufficient condition for speculative bubbles

- Since $\tau = \infty$ is a feasible strategy in (41), we must have

$$p_t \geq \max_h E^h \left[\sum_{j=t+1}^{\infty} \beta^{j-t} d_j \middle| \mathcal{F}_t \right] \tag{42}$$

Proposition

Let $F \in \mathcal{F}_t$ be a set where A realizes the max in (41). Suppose that for some $u > t$ and $G \in \mathcal{F}_u$ with $G \subset F$ and $P^A(G) > 0$ we have that

$$E^B \left[\sum_{j=u+1}^{\infty} \beta^{j-u} d_j \middle| \mathcal{F}_u \right](\omega) > E^A \left[\sum_{j=u+1}^{\infty} \beta^{j-u} d_j \middle| \mathcal{F}_u \right](\omega),$$

for all $\omega \in G$. Then a strict inequality (bubble) holds in (42) for $\omega \in F$.

A model for overconfidence (Scheinkman and Xiong, 2003)

- Let the cumulative dividend process for a risky asset be

$$dD_t = f_t dt + \sigma_D dW_t^D,$$

where f_t is not observable and satisfies

$$df_t = \lambda(\bar{f} - f_t)dt + \sigma_f dW_t^f$$

- Suppose there are two groups of risk-neutral agents who observe D_t and a pair of signals

$$ds_t^A = f_t dt + \sigma_s dW_t^A$$
$$ds_t^B = f_t dt + \sigma_s dW_t^B$$

- In the real world, all four Brownian motions are uncorrelated.

Difference in beliefs

- Agents in group A believe that

$$ds_t^A = f_t dt + \sigma_s(\phi dW_t^f + \sqrt{1-\phi^2} dW_t^A)$$

- Agents in group B believe that

$$ds_t^B = f_t dt + \sigma_s(\phi dW_t^f + \sqrt{1-\phi^2} dW_t^B)$$

- It follows that the estimates of the process f_t can be obtained from a linear Gaussian filter and have a stationary distribution with variance γ and conditional means $\hat{f}_t^A$ and $\hat{f}_t^B$.
- Moreover, the difference in beliefs $g^A = \hat{f}_t^B - \hat{f}_t^A$ is given by

$$dg_t^A = -\rho g_t^A dt + \sigma_g dW_t^{A,g} \tag{43}$$

with a similar expression holding for $g^B = \hat{f}_t^A - \hat{f}_t^B$.

Equilibirum price

- As in the discrete-time model, no short-sales, fixed supply and an infinite number of agents ensure that in equilibrium we have

$$p_t^o = \sup_{\tau \geq 0} E_t^o \left[\int_t^{t+\tau} e^{-r(s-t)} dDs + e^{-r\tau}(p_{t+\tau}^{\bar{o}} - c) \right] \quad (44)$$

  where o denotes the current owner, $p_{t+\tau}^{\bar{o}}$ is the reservation price of the buyer at a future selling time $(t+\tau)$, and c is a selling cost.

- Because of the Markovian structure, it is natural to look for solutions of the form

$$p_t^o = \underbrace{\frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r+\lambda}}_{\text{fundamental}} + \underbrace{q(g_t^o)}_{\text{resale value}} \quad (45)$$

The solution

- Inserting (45) into (44) leads to

$$q(g_t^o) = \sup_{\tau \geq 0} E_t^o \left[e^{-r\tau} \left(\frac{g_{t+\tau}^o}{r+\lambda} + q(-g_{t+\tau}^o) - c \right) \right]. \quad (46)$$

- Using optimal control techniques we find that q satisfies

$$\begin{cases} q(x) \geq \frac{x}{r+\lambda} + q(-x) - c \\ \frac{\sigma_g^2}{2} q'' - \rho x q' - rq \leq 0 \end{cases} \quad (47)$$

- One can find a solution in terms of Kummer functions and characterize the size of the bubble by $q(-x^*)$ where x^* is the exercise threshold for the resale option.

A model for bubbles in continuous time (Jarrow, Protter and Shimbo, 2007, 2010)

- Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions.
- Consider a riskless money market account as a numerarie.
- Let an adapted càdlàg semimartingale $D_t \geq 0$ be the cumulative dividends paid by a risky asset with liquidation value $0 \leq X_\tau \in \mathcal{F}_\tau$, where τ is a stopping time.
- The market price of the risky asset is given by an adapted càdlàg semimartingale S_t and the wealth from owning the asset by

$$W_t = S_t + \int_0^{t \wedge \tau} dD_u + X_\tau 1_{\{\tau \leq t\}} \tag{48}$$

- Observe that S_t is the *ex-dividend* price, so in particular $S_\tau = 0$ and

$$W_\tau = \int_0^\tau dD_u + X_\tau$$

NFLVR and FTAP

- A trading strategy $\pi \in L(W)$ is said to be self–financing if its corresponding wealth satisfies

$$V_t^\pi = \int_0^t \pi_u dW_u$$

and admissible if in addition $V_t^\pi \geq -a$ for some $a \geq 0$.

- Define

$$\mathcal{K} = \{ V_\infty^\pi = \int_0^\infty \pi_u dW_u : \pi \text{ admissible } \}$$

and $C = (\mathcal{K} - L_0^+) \cap L_\infty$.

- The market satisfies NFLVR if $\overline{C} \cap L_\infty^+ = \{0\}$.
- We say that $Q \sim P$ is an ELMM if W_t is a Q-local martingale and denote this set by $\mathcal{M}_{\text{loc}}(W)$.

Theorem (FTAP)

NFLVR $\Leftrightarrow \mathcal{M}_{loc}(W) \neq \emptyset$.

Fundamental value in complete markets

- Let $\mathcal{M}_{\text{loc}}(W) = \{Q\}$ and define the fundamental value for the risky asset as

$$S^*_t = E_Q\left[\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}} | \mathcal{F}_t\right] 1_{\{t<\tau\}} \qquad (49)$$

and the corresponding wealth process

$$W^*_t = S^*_t + \int_0^{t\wedge\tau} dD_u + X_\tau 1_{\{\tau\leq t\}}. \qquad (50)$$

Lemma

The fundamental price is well defined. Furthermore, $S_t \xrightarrow{a.s} S_\infty \in L_1(Q)$, $S^*_t \xrightarrow{a.s} 0$, and W^*_t is a uniformly integrable Q-martingale closed by

$$W^*_\infty = \int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}.$$

Bubbles in complete markets

Define a bubble as $\beta_t = S_t - S_t^*$.

Theorem

If $\beta \neq 0$, then we have three and only three possibilities:

1. *β_t is a Q-local martingale (which could be a uniformly integrable Q-martingale) if $P(\tau = \infty) > 0$.*
2. *β_t is a Q-local martingale but* not *a uniformly integrable Q-martingale if $P(\tau < \infty) = 1$.*
3. *β_t is a Q-local martingale but* not *a martingale if τ is bounded.*

Bubbles decomposition

Theorem

A bubble in a complete market admits the unique decomposition

$$\beta_t = \beta_t^1 + \beta_t^2 + \beta_t^3, \tag{51}$$

where

- $\beta_t^1 \geq 0$ *is unif. int. Q-mg with* $\beta_t^1 \xrightarrow{a.s} X_\infty$.
- $\beta_t^2 \geq 0$ *is non-unif. int. Q-mg with* $\beta_t^2 \xrightarrow{a.s} 0$.
- $\beta_t^3 \geq 0$ *is a strict Q-local mg with* $E_Q[\beta_t^3] \to 0$ *and* $\beta_t^3 \xrightarrow{a.s} 0$.

Consequently it satisfies

1. $\beta_t \geq 0$
2. $\beta_\tau 1_{\{\tau<\infty\}} = 0$
3. *if* $\beta_t = 0$ *then* $\beta_u = 0$ *for all* $u \geq t$.

No dominance

- Let $\varphi = (\Delta, \Xi^{\nu})$ be the payoff of an asset and the Φ be the set of assets satisfying

$$\Delta_{\nu} + \Xi^{\nu} \leq a + V^{\pi}_{\nu},$$

for some $a \in \mathbb{R}$ and admissible π.

- Denote the market price of $\varphi \in \Phi$ by $\Lambda_t(\varphi)$ and the net gain from buying it at σ and selling at $\mu \leq \nu$ by

$$G_{\sigma,\nu}(\varphi) = \Lambda_{\mu}(\varphi) + \int_{\sigma}^{\mu} d\Delta_u + \Xi^{\nu} 1_{\{\nu=\mu\}} - \Lambda_{\sigma}(\varphi).$$

- We say that φ^2 dominates φ^1 at σ if $G_{\sigma,\nu}(\varphi^2) \geq G_{\sigma,\nu}(\varphi^1)$ a.s and $E_Q[1_{\{G_{\sigma,\nu}(\varphi^2) > G_{\sigma,\nu}(\varphi^1)\}} | \mathcal{F}_{\sigma}] > 0$.

Proposition

Assume that $\tau < \infty$ so that $\beta^1_t = 0$ and that the market is complete and has no dominance. Then $\beta^2_t = \beta^3_t = 0$.

Direct estimates for rational bubbles

- Necessarily must test a joint hypothesis that a bubble exist and a model for intrinsic value.
- One way is to test a specification which includes a bubble term (Flood and Garber (1980))
- This is not robust due to a growing regressor problem.
- Alternatively (West (1987)), one can estimate r from

$$p_t = \frac{p_{t+1} + p_{t+1}}{1+r} + \varepsilon_t$$

and separately estimate an autoregressive dividend process and a present–value relation

$$d_t = ad_{t-1} + \nu_t$$
$$p_t = gd_t$$

Then a no-bubbles hypothesis implies that $g = d/(1+r-d)$, whereas if there is a bubble, the first estimate for r will be consistent, but the second will not.

Indirect estimates for rational bubbles

- Nonstationarity under differencing (Diba and Grossman (1982))
- Positive autocorrelation (Blanchard and Watson (1982))
- Negative skewness and high kurtosis (Fama (1976))
- Nonzero median in price changes (Evans (1986))
- Inconclusive - all could be due to market fundamentals having these properties.

Volatility bounds

- Suppose that $p_t = E_t[p_t^*]$, where p_t^* is a perfect foresight price.
- Then $p_t^* = p_t + \varepsilon_t$, where ε_t is the forecast error and is uncorrelated with p_t.
- It follows that $\sigma(p_t) \leq \sigma(p_t^*)$.
- This, however, is found to be dramatically violated by data (Shiller (1981)).
- Methodological problems: excess volatility could be do to risk–aversion, non-stationary dividends, rare events.
- Inconclusive - null hypothesis of rationality cannot be rejected, but neither can irrationality (fads).

Violation of Volatility Bounds

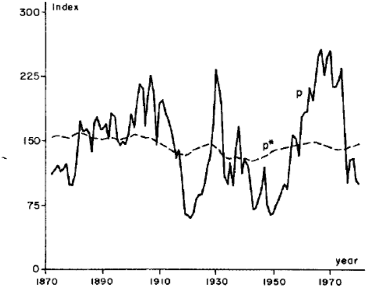


FIGURE 1

Note: Real Standard and Poor's Composite Stock Price Index (solid line p) and *ex post* rational price (dotted line p^*), 1871–1979, both detrended by dividing a long-run exponential growth factor. The variable p^* is the present value of actual subsequent real detrended dividends, subject to an assumption about the present value in 1979 of dividends thereafter. Data are from Data Set 1, Appendix.

FIGURE 2

Note: Real modified Dow Jones Industrial Average (solid line p) and *ex post* rational price (dotted line p^*), 1928-1979, both detrended by dividing by a long-run exponential growth factor. The variable p^* is the present value of actual subsequent real detrended dividends, subject to an assumption about the present value in 1979 of dividends thereafter Data are from Data Set 2, Appendix.

Figure: Source: Shiller (1981)

Some famous examples

- Tulipmania, 1634 – 1637
- Mississippi bubble, 1719 – 1720
- South Sea bubble, 1720
- The great crash, 1929
- The second great contraction, 2007 –

Tulip prices

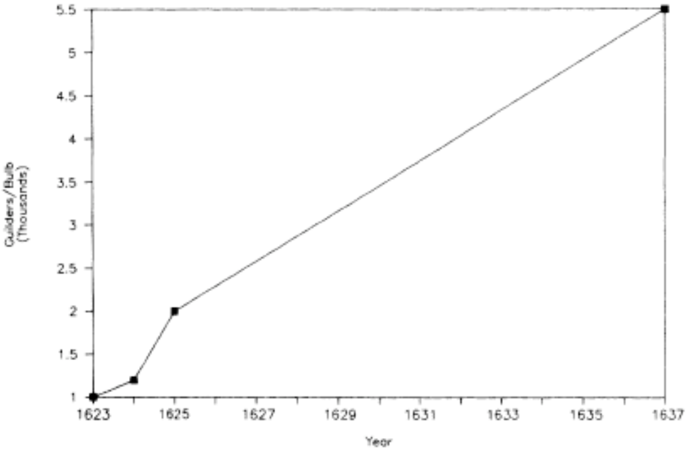


FIG. 1.—Semper Augustus

Figure: Source: Garber (1989). Note: 1 guilder = US$ 10 (gold equivalent in 1989)

Tulip prices

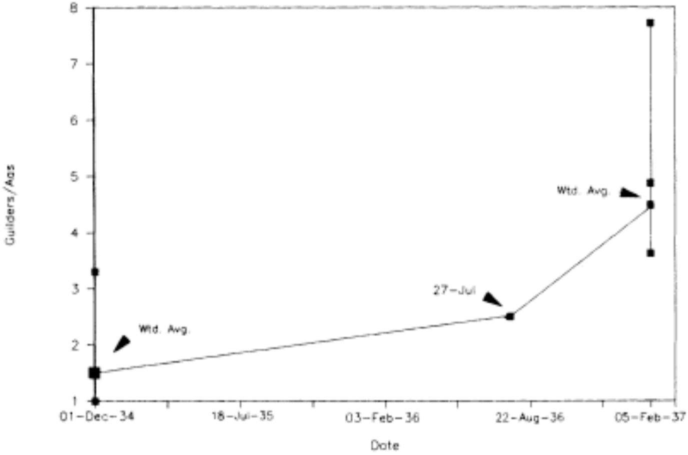


FIG. 2.—Admirael van der Eyck

Figure: Source: Garber (1989)

Tulip prices

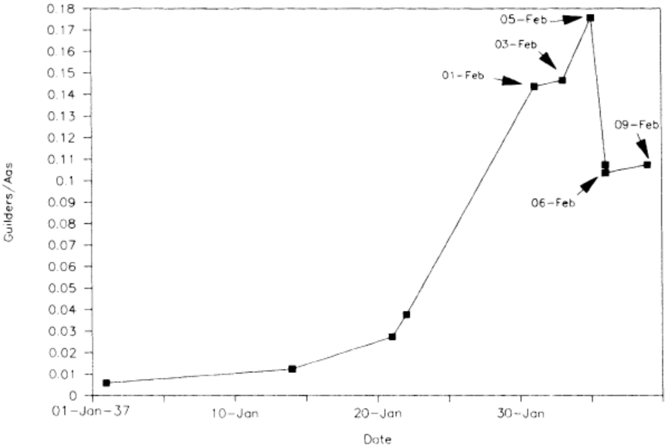
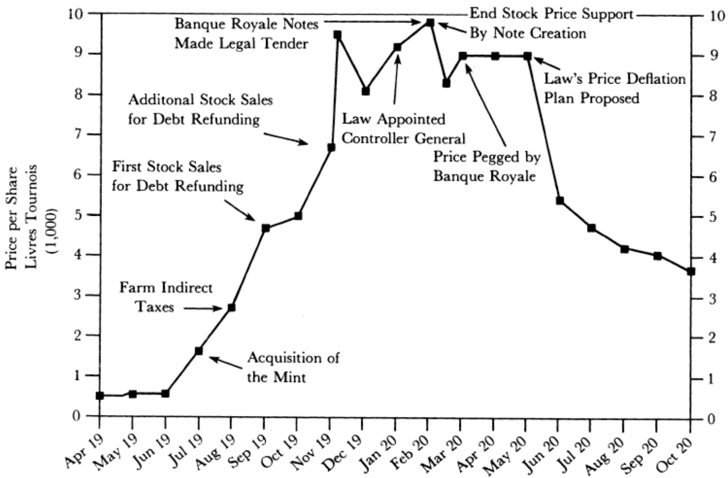


FIG. 5.—Switsers

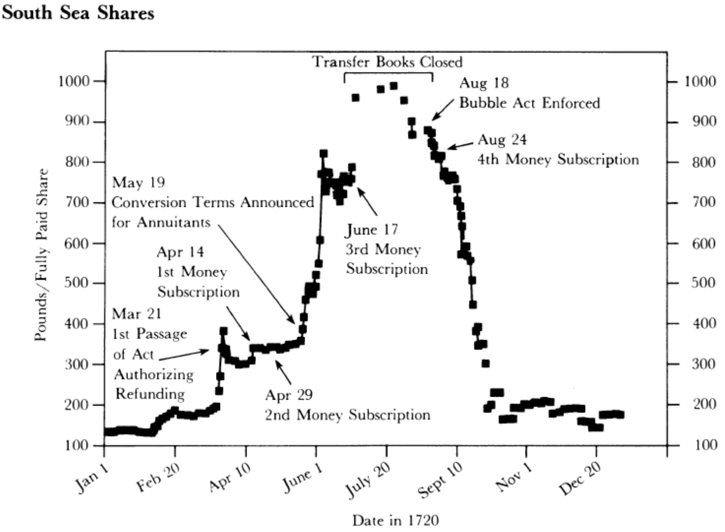
Figure: Source: Garber (1989)

Mississippi bubble

Compagnie des Indes Stock Price



South Sea bubble



The crash of 1929

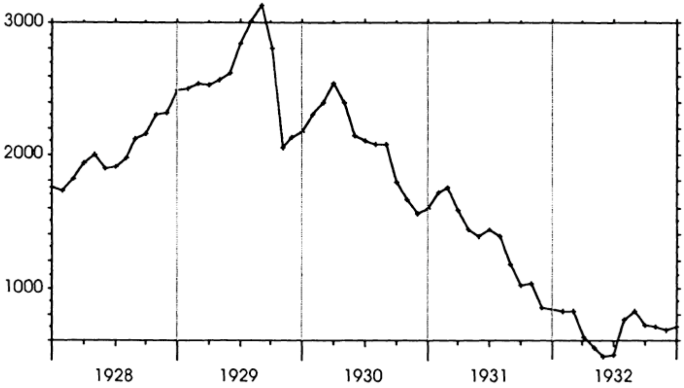


FIGURE 1

THE NOMINAL S & P COMPOSITE STOCK PRICE INDEX, 1928–1932 (1941–1943 = 1000)

Figure: Source: DeLong and Shleifer (1991)

The second great contraction, 2007–