Asset price bubbles: economics, mathematics and statistics

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Chapter 1

Rational Bubbles

Rational bubbles occur when the deviation of prices from some notion of fundamental values can be accounted for by rational behaviour and expectations of agents. They offer a natural starting point for the study of asset price bubbles, if only to fix the language and the notation to be employed in the study of more general notions of bubbles.

1.1 Definitions and examples

We start with the setting described in [5], namely that of a representative household maximizing expected utility of consumption $\{c_t\}_{t=0}^{\infty}$ of a single perishable good. Assume for simplicity a constant discount factor $0 < \beta < 1$ and consider the discrete-time, infinite-horizon problem

$$\sup_c E_t \left[\sum_{j=t}^{\infty} \beta^{j-t} u(c_j) \right], \tag{1.1}$$

where $E_t[\cdot]$ denotes conditional expectation with respect to a sigma algebra $\mathcal{F}_t$ that contains, at least, current and past values of all processes in the model. Suppose further that the household receives an endowment e_t of the consumption good at each period and can smooth consumption over time by holding x_t shares of a representative firm's stock, each paying a dividend d_t units of the consumption good per period. Assuming that share holdings are rebalanced at the beginning of the period at price p_t (in units of the consumption good) and held until the next period, the budget constraint for the household at time j then becomes

$$c_j \leq e_j + d_j x_j + p_j(x_j - x_{j+1}), \tag{1.2}$$

where $\{e_j, d_j\}$ are assumed to be exogenous stationary processes. The first-order condition for optimality then gives the Euler equation[1]

$$p_t u'(c_t^*) = \beta E_t \left[(p_{t+1} + d_{t+1}) u'(c_{t+1}^*)\right], \tag{1.3}$$

which states that, at the optimum, the marginal utility of selling one share for p_t back to the representative firm at time t equals the discounted value of the expected marginal utility from holding the share until time $t+1$, receiving the dividend d_{t+1}, and then selling the share for p_{t+1}.

When the utility is sufficiently regular (e.g, strictly concave, increasing, continuously differentiable), the market clearing condition implies that the budget constraint is binding. Normalizing the number of existing shares per capita to unit then leads to

$$c_t^* = e_t + d_t. \tag{1.4}$$

Substituting (1.4) into (1.3) then leads to the pricing equation

$$p_t u'(e_t + d_t) - \beta E_t \left[p_{t+1} u'(e_{t+1} + d_{t+1})\right] = \beta E_t \left[d_{t+1} u'(e_{t+1} + d_{t+1})\right]. \tag{1.5}$$

Denoting $q_t = u'(e_t + d_t) p_t$ and introducing the operator $L(X_t) = X_{t+1}$, we arrive at the following difference equation

$$E_t \left[(1 - \beta L)\, q_t\right] = \beta E_t \left[d_{t+1} u'(e_{t+1} + d_{t+1})\right]. \tag{1.6}$$

Since $0 < \beta < 1$, we can use the formal expansion

$$(1 - \beta L)^{-1} = 1 + \beta L + \beta^2 L^2 + \ldots$$

and conclude that the particular solution associated with the inhomogeneous term in (1.6) can be written as the convergent series

$$F_t = \sum_{j=1}^{\infty} \beta^j E_t \left[d_{t+i} u'(e_{t+i} + d_{t+i})\right], \tag{1.7}$$

[1] For an informal definition of this equation, consider the objective function

$$W = u(c_t) + \beta E_t[u(c_{t+1})] + \beta^2 E_t[u(c_{t+2}] + \cdots$$

If the household purchases x_{t+1} shares at time t and sells at time $t+1$, this changes to

$$W(x_{t+1}) = u(c_t - p_t x_{t+1}) + \beta E_t[u(c_{t+1} + p_{t+1} x_{t+1} + d_{t+1} x_{t+1})] + \beta^2 E_t[u(c_{t+2})] + \cdots$$

Assuming that the consumption stream $c_t, c_{t+1}, c_{t+2}, \ldots$ is optimal leads to the first order condition

$$\left.\frac{dW(x_{t+1})}{dx_{t+1}}\right|_{x_{t+1}=0} = 0,$$

which reduces the Euler equation.

provided the sequence $\{E_t[d_{t+i}u'(e_{t+i}+d_{t+i})]\}_{j=1}^{\infty}$ grows slower than β^{-j}. We call (1.7) the market fundamental value for q_t, as it is directly related to the discounted expected value of the future stream of dividends.

The general solution of (1.6), however, consists of

$$q_t = F_t + B_t \tag{1.8}$$

where B_t is a solution to the homogenous equation

$$E_t[(1-\beta L)\, q_t] = 0, \tag{1.9}$$

that is to say, any process B_t satisfying

$$E_t[B_{t+1}] = \beta^{-1} B_t. \tag{1.10}$$

It follows from iterated expectations that

$$E_t[B_{t+j}] = \beta^{-j} B_t, \qquad \text{for all } j > 0. \tag{1.11}$$

Since $\beta^{-1} > 1$ and the fundamental solution (1.7) is convergent, this implies that the sequence $\{E_t[q_{t+j}]\}_{j=1}^{\infty}$ either increases or decreases without bound. But given free disposal of assets, we conclude that a stock price cannot be *expected* to become negative at a future date, which implies that $B_t \geq 0$ for all t.

Now if $B_t = 0$ for some t, we have $E_t[B_{t+1}] = \beta^{-1} B_t = 0$. But since $B_{t+1} \geq 0$ as well, we must have that $B_{t+1} = 0$ almost surely. Therefore, if a rational bubble does not exist at $t \geq 0$ it cannot get started at $t+1$ or *any later date*. By extension, any nonzero rational bubble must start with $B_0 > 0$, which implies that before the first day of trading in the stock we must have

$$E_{-1}[q_0 - F_0] = E_{-1}[B_0] > 0. \tag{1.12}$$

Remark 1.1.1. When the agent is risk-neutral we have that $u'(c_t) = 1$ and the difference equation reduces to

$$E_t[(1-\beta L)\, p_t] = \beta E_t[d_{t+1}]. \tag{1.13}$$

In the case $\beta = (1+r)^{-1}$ leads to the familiar form

$$p_t = \frac{1}{1+r} E_t[p_{t+1} + d_{t+1}], \tag{1.14}$$

which admits a fundamental solution

$$F_t = \sum_{j=1}^{\infty} \frac{E_t[d_{t+1}]}{(1+r)^j} \tag{1.15}$$

and a bubble component satisfying

$$E_t\left[B_{t+1}\right] = (1+r)B_t, \tag{1.16}$$

so that the general solution is given by $p_t = F_t + B_t$.

Examples

(1) The simplest example of a rational bubble consists of a deterministic component $B_t = \beta^{-t}B_0$, which must grow forever and never burst.

(2) More generally, any solution to (1.10) satisfies

$$B_{t+1} = \beta^{-1}B_t + z_{t+1} \tag{1.17}$$

for some stochastic process z_t with $E_{t-j}[z_{t+1}] = 0$ for all $j \geq 0$. The general solution of (1.17) is of the form

$$B_t = \beta^{-t}B_0 + \sum_{s=1}^{t} \beta^{s-t} z_s.$$

A simple example provided in [2] is

$$z_t = (\theta_{t+1} - \beta^{-1})B_t + \varepsilon_{t+1},$$

where $(\theta_t, \varepsilon_t)$ are mutually and serially independent stochastic processes and satisfy $E_{t-j}[\theta_{t+1}] = \beta^{-1}$ and $E_{t-j}[\varepsilon_{t+1}] = 0$ for all $j \geq 0$. This leads to

$$B_{t+1} = \theta_{t+1}B_t + \varepsilon_{t+1}.$$

We can then use the property that $B_{t+1} \geq 0$ almost surely to establish that $\theta_{t+1} \geq 0$ almost surely as well, since a negative realization of θ_{t+1} with nonzero probability would imply a nonzero probability of $B_{t+1} < 0$, since e_{t+1} is independent from it θ_{t+1}. Moreover, the same argument shows that $\pi = P(\theta = 0) > 0$ implies $\varepsilon = 0$ almost surely. In this case, we see that such rational bubble can crash in each period with probability π, has expected duration π^{-1} periods and survival probability after T equal to $(1-\pi)^{T-1}$, which tends to zero as $T \to \infty$.

Put together, the results of this section show that a rational bubble satisfying (1.10) is either identically zero for all times or must have started from a strictly positive value on the first day of trading for the stock, possibly crashing at some future time, and never restarting again. It is also clear that if such rational bubble stops growing in expectation at some future time T, say because the stock stops trading at this time, then it must be zero for all times, thereby precluding the existence of rational bubbles of this form for finite-maturity securities such as bonds and derivatives.

We might suspect that these strong properties are an artefact of the representative agent framework, in particular because it does not take into account the possibility of stock trading between agents with access to different information sets, a setup to which we turn next.

1.2 Rational Expectations Equilibrium

1.2.1 Static Speculation and the no-trade theorem

Consider first a one-period market with risk-averse or risk-neutral traders $i = 1, \ldots, I$ who take positions at time $t = 0$ for a price p on a claim with random value $\widetilde{X} \in E \subset \mathbb{R}$. Think of X, for example, as the payoff of a derivative security, which is known at the end of the period $t = T$, but a random variable at the initial time $t = 0$. Suppose further that each trader receives a private signal $s^i \in S^i$ at time zero, where S^i is a discrete set, and let $s = (s^1, \ldots, s^I) \in S = \times_i S^i$ denote the collective signal. Let the states of nature be $\Omega = E \times S$ and assume that traders have the same prior ν on Ω, with $\nu^i(s^i) > 0$ being the prior probability of signal s^i.

Denote by $x^i \in \mathbb{R}$ the position of trader i and by $G^i = (X - p)x^i$ her gains. We are now ready to define a rational expectations equilibrium.

Definition 1.2.1. A static rational expectations equilibrium (REE) is a forecast function $\Phi : s \mapsto p$ and a set of positions $x^i(p, s^i, S(p))$, relative to the private information s^i and that set of signals $S(p) := \Phi^{-1}(p)$ compatible with the price p, such that

1. the market clears, that is, $\sum_i x^i = 0$,
2. x^i maximizes i's expected utility of gains G^i conditional on i's private signal s^i and the information $S(p)$ conveyed by the price p.

It follows from the first condition of such equilibrium that $\sum_i G^i = 0$, that is, the market as a whole offers a zero-sum. We say that the market is *purely speculative* if, moreover, the portfolio each trader before taking positions in the claim X are uncorrelated with the gains G^i and the set of signals [2]. In other words, traders do not have a reason to hold the claim to hedge some other prior position.

Proposition 1.2.1. *In a static REE of purely speculative market, risk-averse traders do not trade and risk-neutral traders may trade but do not expect any gain from their trade.*

Proof. A trader with a concave utility function, no hedging motive, and the option not to trade must expect a nonnegative gain, that is,

$$E\left[G^i \mid s^i, S(p)\right] \geq 0.$$

which implies that

$$E\left[G^i \mid S(p)\right] = E\left[E\left[G^i \mid s^i, S(p)\right] \mid S(p)\right] \geq 0$$

But from the market clearing conditions we have that $\sum_i G^i = 0$. Using the fact that traders have the same prior then give

$$\sum_i E\left[G^i \mid s(p)\right] = 0,$$

which implies that $E\left[G^i \mid s(p)\right] = 0$ for each i and consequently that

$$E\left[G^i \mid s^i, S(p)\right] = 0$$

for each i also. □

This proposition is a reformulation of a result by [10] on the impossibility of speculation in a static model. In particular, it negates the view that rational risk-averse or risk-neutral agents can trade on the basis of differences in information. We see that for trade to occur in a static REE of a purely speculative market, at least one of the conditions of one proposition needs to be relaxed, namely by: (i) introducing either risk-seeking or non-rational agents or (ii) relaxing the assumption of a common prior to all agents. Alternatively, one can drop the assumption of a purely speculative market by introducing correlation between the claim X and previous positions, that is to say, a hedging motive.

[2] For example, X can be a option on a stock that is independent from any stock already held by the traders

1.2.2 Dynamic Speculation

Another way to avoid the result of Proposition 1.2.1 is to consider the possibility of dynamic speculation, whereby the right to resell an asset at a later time could make agents willing to pay more for it than if they were obliged to hold the asset forever.

For this, consider a stock with fixed aggregate supply $\overline{x}$ that may trade at $t = 0, 1, 2, \ldots$ with a nonnegative dividend process d_t given exogenously. Suppose for simplicity that traders are risk-neutral with a common discount factor $\beta < 1$. Assume now that $s_t^i \in F^i \subset S^i$ and $F_t^i \subset F_{t+1}^i$.

Definition 1.2.2. A myopic REE is a sequence of forecast functions $\Phi_t : s_t \mapsto p_t$ and holdings $x_t^i(p_t, s_t^i, S_t(p_t))$, where $S_t(p_t) := \Phi_t(p_t)$ is the set of signals compatible with the price p_t, such that

1. $\sum_i x_t^i = \overline{x}$,
2. there is short-run optimization, in the sense that
   (a) if short sales are allowed
   $$p_t = \beta E\left[d_{t+1} + p_{t+1} \mid s_t^i, S_t(p_t)\right] \tag{1.18}$$
   (b) if short sales are prohibited
   $$\begin{cases} p_t = \beta E\left[d_{t+1} + p_{t+1} \mid s_t^i, S_t(p_t)\right] \Rightarrow x_t^i \in [0, \overline{x}] \\ p_t > \beta E\left[d_{t+1} + p_{t+1} \mid s_t^i, S_t(p_t)\right] \Rightarrow x_t^i = 0 \\ p_t < \beta E\left[d_{t+1} + p_{t+1} \mid s_t^i, S_t(p_t)\right] \Rightarrow x_t^i = \overline{x} \end{cases}$$

The interpretation of this definition is that, in the absence of short-sales restrictions, each trader chooses positions x^i that maximize the expected short-run gain, leading to an equilibrium price of the form (1.18) for all traders. With short-sales restrictions, a trader i that considers the stock to be overvalued, in the sense that the price p_t is strictly above the right-hand side of 1.18, will stay out of the market for this period, leading to $x_t^i = 0$. Conversely, a trader that considers the stock to be undervalued will attempt to buy the entire market for that period, leading to $x_t^i = \overline{x}$. Observe that is the undervaluation holds for more than one trader, then the price will increase until it no longer holds for all traders but one, who will then buy the entire market for that period.

Proposition 1.2.2. *Even if short sales are prohibited, for any trader i active at time t we have that*

$$p_t = \beta E\left[d_{t+1} + p_{t+1} \mid s_t^i, s_t(p_t)\right].$$

Proof. Variant of proof of Proposition (1.2.1). □

Now given information $(s_t^i, S_t(p_t))$, define the market fundamental value for a risk-neutral trader i as

$$F(s_t^i, s_t(p_t)) = E\left[\sum_{j=1}^{\infty} \beta^j d_{t+j} \mid s_t^i, s_t(p_t)\right] \tag{1.19}$$

and for any price p_t consistent with S_t define a price bubble as seen by trader i as

$$B(s_t^i, p_t) = p_t - F(s_t^i, s_t(p_t)). \tag{1.20}$$

Proposition 1.2.3. *In a stock market with myopic REE and finite horizon $\overline{T}$, price bubbles are all equal to zero for all traders.*

Proof. Backward induction from $p_{\overline{T}} = 0$ and Proposition (1.2.2). □

In other words, rational agents anticipate that a bubble will crash at time $\overline{T}$ (as there can be no trade on the stock after that), leading to no bubbles at any prior time. As the next proposition shows, however, price bubbles can exist in an infinite-horizon case, myopic REE , provided they satisfy that same type of growth condition we have seen in Section 1.1.

Proposition 1.2.4. *In a stock market with myopic REE and infinite time horizon:*

1. *If short sales are allowed, price bubbles satisfy*

$$B(s_t^i, p_t) = \beta^T E\left[B(s_{t+T}^i, p_{t+T}) \mid s_t^i, s_t(p_t)\right]$$

2. *If short sales are prohibited, a price bubble for trader i satisfy the martingale property above between t and $t+T$ if trader i is active in each period $t, t+1, \ldots, t+T-1$.*

Proof. Iterated conditional expectations. □

Example 1.2.1. Consider a stock market with no uncertainty, where $d_t = 1$ for all t, and $\beta = 1/2$, so that the market fundamental value is $\sum_{j=1}^{\infty}(1/2)^j = 1$ for traders A and B. In this case, a myopic REE then satisfies

$$p_t = \frac{1}{2}(1 + p_{t+1}),$$

whose general solution is $p_t = 1 + \alpha 2^t$ where $F_t = 1$ and $B_t = \alpha 2^t$. Consider $\alpha = 1$ and the trades:

$$\begin{array}{ll} \text{at time } t = 0, & \text{A sells to B at price 2} \\ \text{at time } t = 1, & \text{B sells to A at price 3} \\ \text{at time } t = 2, & \text{A sells to B at price 5} \\ \vdots & \vdots \end{array}$$

Then

$$\begin{aligned} G^A &= 2 - \frac{1}{2}(3) + \frac{1}{4}(5+1) - \frac{1}{8}(9) + \frac{1}{16}(17+1) - \dots \\ G^B &= -2 + \frac{1}{2}(3+1) - \frac{1}{4}(5) + \frac{1}{8}(9+1) - \frac{1}{16}(17) + \dots \end{aligned}$$

so the gains do not converge. But at any moment A (respectively B) can guarantee a gain of 2 (respectively 0) by leaving the market just after selling. In other words, the bubble persists in this example because traders only maximize the one-period gain, rather than considering a trading strategy that maximizes gains for all times. This motivative the concept of a fully dynamic rational expectations equilibrium explored next

1.2.3 Fully Dynamic REE

To avoid the situation depicted in the previous example, consider the following definition.

Definition 1.2.3. A fully dynamic REE is a sequence of forecast functions $\phi_t : s_t \mapsto p_t$ and holdings $x_t^i(p_t, s_t^i, S_t(p_t))$ such that

1. $\sum_i x_t^i = \overline{x}$,
2. maximizing behaviour: at each time t and for any (s_t^i, S_t), the strategy for trader i maximizes the discounted expected gain for trader i from t onwards.

Proposition 1.2.5. *Whether short sales are allowed or not, price bubbles do not exist in a fully dynamic REE, that is,*

$$F(s_t^i, S_t) = p_t. \tag{1.21}$$

Proof. Technical variation of Proposition 1.2.1 using the fact that a trader cannot expect to gain by selling x_t^i and leaving the market. □

The interpretation for this is that if agents plan to sell in finite time, then there will be no one left to buy after wards. Therefore, there can be no bubble with finitely many infinitely lived agents with rational expectations. The next section explores the possibilities of bubbles in an overlapping generations model instead.

1.3 Overlapping Generations

Consider a model with consumers who live for two periods but work only during the first, making up a labour force

$$L_t = (1+n)^t L_0,$$

where n is the rate of population growth and we take $L_0 = 1$ for simplicity. The utility of each consumer is $u(c^y, c^o)$ where c^y is consumption when young and c^o is consumption when old. The wage income of each worker is w_t and aggregate savings in the economy are $(1+n)^t s(w_t, r_{t+1})$ where r_{t+1} is the real interest rate and $s(\cdot,\cdot)$ is an increasing function on both arguments.

On the production side, assume that total output is given by a constant returns to scale technology

$$Y_t = F(K_t, L_t) = L_t f(k_t), \tag{1.22}$$

where $k_t = K_t/L_t$ is capital per worker. It follows from competition that $r_t = f'(k_t)$ and there exists a downward slopping factor price frontier of the form $w_t = \phi(r_t)$.

Investment occurs so that, at equilibrium,

$$r_{t+1} = f'(k_{t+1}) = f'\left(\frac{s(w_t, r_{t+1}) - a_t}{1+n}\right) =: \psi(w_t, a_t), \tag{1.23}$$

where

$$a_t = s(w_t, r_{t+1}) - (1+n)k_{t+1} \tag{1.24}$$

is the difference between savings per capita and the equilibrium level of capital per capita, and $\psi_w < 0, \psi_a > 0$.

Assume that there exists a point $\overline{r}$ such that

$$\overline{r} = \psi(\phi(\overline{r}, 0), \tag{1.25}$$

that is, an intersection of the two decreasing functions ϕ and $\psi(w, 0)$ in the (r, w) plane. Diamond (1965) has shown that there exists a unique competitive equilibrium if $a_t = 0$ for all t. Moreover, this equilibrium is efficient if $\overline{r} > n$ and inefficient if $\overline{r} < n$.

To investigate the possibility of bubbles, assume now that there exists an asset paying a total rent (i.e dividends) per period equal to R units of the real good. Then the market-fundamentals value for this asset is

$$F_t = R\left[\sum_{s=t+1}^{\infty} \frac{1}{(1+r_{t+1})\cdots(1+r_s)}\right]. \tag{1.26}$$

Defining $f_t = F_t/(1+n)^t$, we observe that it satisfies

$$f_{t+1} = \frac{1+r_{t+1}}{1+n} f_t - \frac{R}{(1+n)^{t+1}}.$$

In addition, there can be a bubble component whose price per capita satisfies

$$b_{t+1} = \frac{1+r_{t+1}}{1+n} b_t, \qquad b_t \geq 0. \tag{1.27}$$

Since this is the only asset in which consumers can invest, we have that

$$a_t = f_t + b_t, \tag{1.28}$$

which then corresponds to non-productive savings.

Definition 1.3.1. A perfect foresight equilibrium is a sequence of interest rates r_t, wages w_t, bubbles peer capita b_t, market fundamentals per capita f_t and non-productive savings a_t satisfying

$$s(w_t, r_{t+1}) - f_t > b_t \geq 0. \tag{1.29}$$

Moreover, such equilibrium is bubbly if there exists t such that $b_t > 0$. It is asymptotically bubbly if $\lim_{t\to\infty} b_t > 0$.

In the case $\overline{r} < n$, define $\widehat{b}$ by

$$n = f'\left(\frac{s(\phi(n), n) - \widehat{b}}{1+n}\right). \tag{1.30}$$

Proposition 1.3.1. *1. If $\overline{r} > n$, there exists a unique equilibrium, no bubbles, and $r_t \to \overline{r}$.*

2. If $0 < \overline{r} < n$, there exists a maximum feasible bubble $\widehat{b}_0$ such that

(a) *For any* $b_0 \in [0, \widehat{b}_0)$*, there exists a unique equilibrium with initial bubble* b_0*. This equilibrium is asymptotically bubbleless and* $r_t \to \overline{r}$*. The initial value* f_0 *decreases with* b_0*.*

(b) *There exists a unique equilibrium with initial bubble* $\widehat{b}_0$ *and the bubble per capita converges to* $\widehat{b}$*.*

3. *If* $\overline{r} < 0$*, there exists no bubbleless equilibrium. There exists a unique asymptotically bubbly equilibrium and* $r_t \to \overline{r}$*.*

4. *If* $\overline{r} < n$*, then the asymptotically bubbleless equilibria are inefficient and the asymptotically bubbly equilibrium is efficient.*

Remark 1.3.1.

1. In the asymptotically bubbleless cases, the total bubble $B_t = (1+n)^t b_t$ continues to grow, but becomes progressively smaller compared to the economy.

2. Bubbles lower productive savings by increasing a_t, thereby increasing the marginal productivity of capital and interest rates.

3. In an efficient economy (that is with $\overline{r} < n$), bubbles cannot exist because of wealth constraints, as they would grow faster than the economy.

4. In the inefficient case, a bubble helps transfer wealth (goods) from the younger generation to the older, similar to national debt in the Diamond (1965) model.

Chapter 2

Market Inefficiencies

2.1 Fads and the EMH

In its general form, the efficient market hypothesis (EMH) states that asset prices fully reflect all available information. Specific formulations of this statement need to make precise what it means to "fully reflect" and also what is the "available information". In statistical tests, one typically formulates this by saying that it should not be possible to forecast returns on an asset based on some well defined information available at present time, for example a specified number of previous returns. Expressed in this way, the EMH can be rejected by a statistical test if it can be shown that returns are forecastable beyond an agreed upon threshold, since there is always some small degree of forecastability in any realized series of returns. In other words, one says that the EMH has passed a statistical test (meaning that it cannot be rejected) if observed returns are not *very* forecastable, according to some criterion. For example, if the test consists of regression of returns on some number of past observations of stock prices, then the EMH can be said to pass the test provided the R^2 of the regression is sufficiently low.

Denoting returns on stock with price p_t and dividends d_t by

$$R_{t+1} = \frac{p_{t+1} - p_t + d_{t+1}}{p_t}, \tag{2.1}$$

we have seen that the first-order rational expectations condition for risk-neutral agents leads to (1.14), which is equivalent to

$$E_t\left[R_{t+1}\right] = r. \tag{2.2}$$

Solving this equation recursively leads to

$$p_t = \sum_{j=1}^{\infty} \frac{E_t[d_{t+j}]}{(1+r)^j} + B_t, \tag{2.3}$$

where the possible rational bubble B_t satisfies $E_t[B_{t+1}] = (1+r)B_t$. It is easy to see that returns satisfying (2.2) are unforecastable, so that a statistical test on prices generated according to (2.3) would result in the EMH not being rejected.

However, alternative models might also imply that returns are not very forecastable and would not lead to a rejection of the EMH either. For example, consider the model proposed in [11], where risk-averse sophisticated investors (the so-called "smart money") respond to available information through a demand function (expressed as a portion of shares outstanding) of the form

$$Q_t^s = \frac{E_t[R_{t+1}] - r}{\phi}, \tag{2.4}$$

where ϕ is a risk-aversion parameter, r is the expected return for which they will have no demand for the stock (i.e $Q_t^s = 0$), and $(r + \phi)$ is the expected return that would lead them to hold the entire market ($Q_t^s = 1$).

In addition, suppose that there are noise traders (ordinary investors) who do not respond to optimally forecasted returns, but instead react to news of fast through a demand function of the form Y_t/p_t for an exogenous random variable Y_t, so that an equilibrium is reached at

$$Q_t^s + \frac{Y_t}{p_t} = 1. \tag{2.5}$$

Inserting (2.5) into (2.4) and solving recursively gives

$$p_t = \sum_{j=1}^{\infty} \frac{E_t[d_{t+j}] + \phi E_t[Y_{t-1+j}]}{(1+r+\phi)^j}. \tag{2.6}$$

Observe that the limit of this expression as $\phi \to 0$ is (2.3), whereby sophisticated investors are risk-neutral and dominated the market by requiring no risk-premium for owning the stock. Conversely, as ϕ increases, sophisticated investors take on a progressively smaller fraction of the market. As $\phi \to \infty$, the market is completely dominated by noise traders and the price converges to Y_t.

For moderate values of ϕ, however, it can be shown that both (2.3) and (2.6) lead to prices that are not very forecastable. They are therefore both equally consistent with findings that news announcements have instant effects on returns and little predictable effect thereafter. Nevertheless, (2.6) allows for a hump-shaped path in Y_t (as predicted, for example, by theories of diffusion of opinions through social interactions) to have an effect on p_t. The strength of the effect, however, will depend of how quickly the hump builds up and fades away.

For example, suppose $d \equiv 1$, so that the fundamental price in (2.3) is given by a rational bubble B_t only, which is unforecastable. On the other hand, let

$$Y_t = \epsilon_{t-1} + \epsilon_{t-2} + \cdots + \epsilon_{t-n}, \tag{2.7}$$

where ϵ_j are i.i.d. $N(0, \sigma_\epsilon^2)$ innovations. Suppose further that the information available to sophisticated investors corresponds to the vector of observations $(\epsilon_{t-n}, \ldots, \epsilon_t)$, that is, current and lagged values of the innovations. Then $(Y_{t+1} - Y_t) = \epsilon_t$ is perfectly forecastable given available information up to time t, but $(p_{t+1} - p_t)$ is not. For instance, the R^2 of a regression of $(p_{t+1} - p_t)$ on p_t is 0.015 for $n = 20$, $r = 0$ and $\phi = 0.2$. Using all lagged values of the innovations in the regression raises this to $R^2 = 0.151$.

To summarize, when a statistical test on observed prices fails to reject the efficient market hypothesis, in the sense that the forecastability of returns is observed to be small, it could be because prices are given (2.3), but it could also be that prices are generated by rumours or social behaviour that might or might not have anything to do with expected future dividends (or rational bubbles).

2.2 Noise trader risk

The previous section shows that in the presence of traders who react to irrelevant information (i.e noise), the price of an asset might differ from its fundamental value in a way that is still consistent with low forecastability of returns. An argument dating back to [6], however, states that sophisticated investors can take advantage of these uninformed traders and eventually drive them out of the market, at which point any price anomalies should fade away and the asset should be traded at its fundamental value (plus a possible rational bubble). The essence of the argument is that, because noise traders take irrelevant information into account, they are more likely to "buy high and sell low" than sophisticated investors, or arbitrageurs, who by being aggressive enough in these trades can quickly deplete noise traders from their funds.

To investigate this argument in more detail, consider the overlapping generations model proposed in [4], which consists of a safe asset (s) in perfectly elastic supply paying a dividend rate $R = r$, where r is the risk-free interest rate, and an unsafe asset (u) with fixed quantity normalized to 1,also paying the same dividend rate. Suppose further that there are two groups of agents: arbitrageurs (a) with rational expectations and noise traders (n). Let μ be the proportion of noise traders in the population and suppose that all agents in the same group are identical and have utility function $u(x) = -e^{-2\gamma x}$.

As in Section 1.3, agents in each group live for two periods, but with the simplifying assumption that consumption only takes place when old. In addition, the model assumes that there is no labour supply decision and no bequest at death. Consequently, the only decision that agents make is to choose their portfolio when young to maximize perceived expected utility of wealth when old. Furthermore, the representative young arbitrageur is assumed to accurately perceive the distribution of returns from holding the risky asset. Conversely, the representative young noise trader misperceives the expected *price* of the risky asset at time $t+1$ by an i.i.d random variable

$$\rho_t \sim N(\rho^*, \sigma_\rho^2),$$

where ρ^* is the average misperception and measures the "bullishness" of noise traders. Since this is the only source of randomness in the model, the return

$$R_{t+1} = \frac{p_{t+1} + r - p_t}{p_t}, \tag{2.8}$$

conditioned on the price p_t, is also normally distributed. As a result, maximizing expected utility is equivalent to maximizing $\overline{w} - \gamma\sigma_w^2$, where $\overline{w}$ is the expected wealth at time $t+1$ and σ_w^2 is the one-period ahead variance of wealth.

Accordingly, arbitrageurs choose hold Q_t^i units of the risky asset at time t to maximize

$$\overline{w}^a - \gamma\sigma_{w^a}^2 = c_0 + Q_t^a\big(r + E_t[p_{t+1}] - p_t(1+r)\big) - \gamma\,(Q_t^a)^2\,\mathrm{var}_t[p_{t+1}],$$

where $(r + E_t[p_{t+1}])$ is the expected payoff from holding the risky asset and $-p_t(1+r)$ is the payment that needs to be made for funds used to buy the risky asset at time t (i.e short-selling the safe asset), c_0 is a constant related to labour income, and

$$\mathrm{var}_t[p_{t+1}] = E_t\left[(p_{t+1} - E_t[p_{t+1}])^2\right]$$

is the one-period variance of p_{t+1}. Conversely, noise traders choose Q_t^n to maximize

$$\bar{w}^n - \gamma\sigma^2_{w^n} = c_0 + Q_t^n\left(r + E_t[p_{t+1}] - p_t(1+r) + \rho_t\right) - \gamma\left(Q_t^n\right)^2 \mathrm{var}_t[p_{t+1}],$$

where the only difference compared to the previous expression is that additional wealth $Q_t^n \rho_t$ that noise traders expect to have because of their misperception ρ_t of the expected price of the risky asset at $t+1$. This leads to

$$Q_t^a = \frac{r + E_t[p_{t+1}] - p_t(1+r)}{2\gamma \mathrm{var}_t[p_{t+1}]} \tag{2.9}$$

$$Q_t^n = \frac{r + E_t[p_{t+1}] - p_t(1+r) + \rho_t}{2\gamma \mathrm{var}_t[p_{t+1}]}. \tag{2.10}$$

When old, traders convert (s) into a consumption good, sell (u) to the new young at price p_{t+1}, and consume all their wealth. At equilibrium,

$$(1-\mu)Q_t^i + \mu Q_t^n = 1, \tag{2.11}$$

which leads to

$$p_t = \frac{r + E_t[p_{t+1}] + \mu\rho_t - 2\gamma \mathrm{var}_t[p_{t+1}]}{1+r}. \tag{2.12}$$

One can obtain a steady-state equilibrium by further imposing stationarity of the unconditional distribution of p_t, which then leads to

$$p_t = 1 + \frac{\mu(\rho_t - \rho^*)}{r} + \frac{\mu\rho^*}{r} - \frac{2\gamma\mu^2\sigma_\rho^2}{r(1+r)^2}, \tag{2.13}$$

where we have used $\mathrm{var}_t[p_{t+1}] = \mu^2\sigma_\rho^2/(1+r)^2$.

The first term above is the fundamental value of the risky asset. The second term corresponds to fluctuations due to variations in misconceptions: if a young generation of noise traders is more bullish than the average ρ^* the push up the price. The third term, on the other hand, represents a permanent price pressure (i.e bias) created by the systematic misconception ρ^*, that is, the average bullishness of noise traders. The last term is crucial for the model and represents a compensation that risk-averse agents (both sophisticated investors and noise traders) require to bear the risk created by the noise traders.

The difference in returns between the two groups of investors is

$$\begin{aligned}\Delta R_t^{n-i} &= \left(Q_t^n - Q_t^i\right)\left(r + p_{t+1} - p_r(1+r)\right) \\ &= \frac{(1+r)^2 \rho_t}{2\gamma\mu^2\sigma_\rho^2}\left(r + p_{t+1} - p_r(1+r)\right). \end{aligned} \tag{2.14}$$

Observe that it follows from the pricing equation that

$$E_t\left[r + p_{t+1} - p_r(1+r)\right] = 2\gamma \mathrm{var}[p_{t+1}] - \mu\rho_t = \frac{2\gamma\mu^2\sigma_\rho^2}{(1+r)^2} - \mu\rho_t.$$

Substituting back into (2.14) we find

$$E_t\left[\Delta R_t^{n-i}\right] = \rho_t - \frac{(1+r)^2\rho_t^2}{2\gamma\mu\sigma_\rho^2}.$$

Taking expectations on both sides of this equation and using the distribution of ρ_t leads to

$$E\left[\Delta R_t^{n-i}\right] = \rho^* - \frac{(1+r)^2(\rho^*)^2 + (1+r)^2\sigma_\rho^2}{2\gamma\mu\sigma_\rho^2}. \tag{2.15}$$

In the expression above, the first term contributes to a higher return for noise traders arising from holding more of the risky asset, since a higher average bullishness leads to higher demand Q_t^n in (2.10) (as ρ_t is distributed around a higher value). On the other hand, the numerator in the second term tends to lower the return for noise traders and is composed of two effect. The first is a price pressure due to high ρ^*, as a higher price p_t tends to lead to a lower return. The second term in the numerator is what one might call the Friedman effect: the higher the variance in misperception σ_ρ^2, the more likely it is for noise traders to buy and sell at the wrong moment. Finally, the denominator of the second term also favours higher returns for noise traders: the higher the variance in misperception σ_ρ^2, the higher the risk posed by noise trader, which lead to risk-averse arbitrageurs to hold less of the asset than they should in order to take advantage of the uninformed traders. In other words, as put by the authors, "noise traders can earn higher expected returns solely by bearing more of the risk that they themselves create".

Despite sometimes being able to earn higher expected returns, noise traders must necessarily have lower expected utility than sophisticated investors, since they maximize their utility based on an incorrect distribution. In fact, the average cash amount that must be given to old noise traders to give them the *ex ante* expected utility of sophisticated investors is

$$\frac{(1+r)^2}{4\gamma\mu^2}\left(1 + \frac{(\rho^*)^2}{\sigma_\rho^2}\right). \tag{2.16}$$

2.3 Limits of Arbitrage

One for he possible modifications of the previous model to deal with more realistic trading restrictions is the following agency model for limited arbitrage proposed by Shleifer and Vishny (1997). Consider three types of agents: noise traders, arbitrageurs and investors in arbitrage funds, who do not trade directly. Suppose that noise traders and arbitrageurs trade at times $t = 1, 2, 3$ on an asset with fundamental value V revealed to all agents at time $t = 3$. Suppose further that arbitrageurs know V at all times and try to trade at $t = 1, 2$ against noise traders, who receive pessimistic shocks S_t determining a demand function of the form

$$Q_t^n = \frac{V - S_t}{p_t}, \tag{2.17}$$

where p_t is the price for the asset, which needs to be determined by the equilibrium condition

$$1 = Q_t^a + Q_t^n \tag{2.18}$$

In addition, arbitrageurs have limited funds F_t to invest in the asset. Assume that, conditional on receiving a negative shock $S_1 > 0$ at time $t = 1$, noise traders receive a shock of the form

$$S_2 = \begin{cases} S > S_1 & \text{with probability } q \\ 0 & \text{with probability } 1 - q \end{cases}. \tag{2.19}$$

If $S_2 = 0$, it follows from the equilibrium condition that any strictly positive demand $Q_2^a > 0$ from arbitrageurs will lead to an equilibrium price $p_2 > V$ and therefore to a loss for arbitrageurs, who know that the price will converge to V at $t = 3$. Consequently, if $S_2 = 0$, the arbitrageurs demand for the asset is $Q_2^a = 0$ (they invest in cash), leading to $p_2 = V$. Alternatively, if $S_2 = S > 0$, arbitrageurs will want to invest as much as possible on the asset, since they know that the true value V will be revealed at time $t = 3$ to be higher than that perceived by noise traders. This leads to a demand of the form $Q_2^a = F_2/p_2$, from which we obtain

$$\begin{aligned} 1 = Q_2^a + Q_2^n = \frac{F_2}{p_2} + \frac{V - S_2}{p_2} = 1 \\ \Rightarrow p_2 = V - S + F_2, \end{aligned} \tag{2.20}$$

where we assume that $F_2 < S$, thetas to say, the resources available to arbitrageurs are not enough to bring the price up to the fundamental value at time $t = 2$. At $t = 1$, arbitrageurs might decide to invest $D_1 \leq F_1$ in the asset and the remainder $(F_1 - D_1)$ in cash in case the price drops even further at time $t = 2$, so they can invest more in the even more undervalued asset. In this case, the demand of arbitrageurs is $Q_1^a = D_1/p_1$ and the equilibrium price is

$$p_1 = V - S_1 + D_1, \tag{2.21}$$

where we again assume that $F_1 < S_1$, so that the resources available to arbitrageurs are not enough to bring the price up to the fundamental value at $t = 1$ either. Finally, assume that

$$F_2 = F_1 - aD_1\left(1 - \frac{p_2}{p_1}\right), \tag{2.22}$$

corresponding to a performance-based arbitrage (PBA) fund with sensitivity $a \geq 0$.

To complete the model, assume that arbitrageurs aim to maximize expected profit at time $t = 3$, which under competition in the market for funds corresponds to maximizing wealth (i.e funds) under management. As we have seen, when $S_2 = 0$, arbitrageurs liquidate their position in the risky asset and invest in cash, leading to a wealth at time $t = 3$ of the form

$$W = F_2 = F_1 - aD_1\left(1 - \frac{V}{p_1}\right).$$

Conversely, if $S_2 = S$, the wealth of arbitrageurs at time $t = 3$ is

$$W = Q_2^a \cdot V = F_2\frac{V}{p_2} = \left[F_1 - aD_1\left(1 - \frac{V}{p_1}\right)\right]\frac{V}{p_2}.$$

The optimization problem faced by arbitrageurs at time $t = 1$ is therefore

$$\max_{0 \leq D_1 \leq F_1} E[W] = \max_{0 \leq D_1 \leq F_1} \left\{(1-q)\left[F_1 - aD_1\left(1 - \frac{V}{p_1}\right)\right] \right. \\ \left. + q\left[F_1 - aD_1\left(1 - \frac{V}{p_1}\right)\right]\frac{V}{p_2}\right\} \tag{2.23}$$

The first order condition for optimality for this problem is

$$(1-q)\left(\frac{V}{p_1} - 1\right) + q\left(\frac{p_2}{p_1} - 1\right)\frac{V}{p_2} \geq 0, \tag{2.24}$$

with equality at an interior solution $D_1 < F_1$ and a strict inequality at a corner solution $D_1 = F_1$.

Proposition 2.3.1. *For given parameters (V, S_1, S, F_1, a), there exists a q^* such that for $q < q^*$, $D_1 = F_1$ and for $q > q^*$, $D_1 < F_1$.*

Proposition 2.3.2. *At the corner solution, we have that*

$$\frac{\partial p_1}{\partial S_1} < 0, \frac{\partial p_2}{\partial S} < 0, \frac{\partial p_1}{\partial S} = 0,$$

whereas at the interior solution we have that

$$\frac{\partial p_1}{\partial S_1} < 0, \frac{\partial p_2}{\partial S} < 0, \frac{\partial p_1}{\partial S} < 0.$$

That is, larger pessimistic shocks lead to less efficient pricing in general. Moreover, at the interior solution, arbitrageurs spread the effect of a deeper shock at $t = 2$ and, consequently, prices at $t = 1$ fall further.

Proposition 2.3.3. *If arbitrageurs are fully invested at $t = 1$ and there is a deeper shock at $t = 2$, then for all $a > 1$ we have that $F_2 < D_1$ and $\frac{F_2}{p_2} < \frac{D_1}{p_1}$.*

That is, arbitrageurs pull out of the market when opportunities are best. Observe that, in this case

$$p_2 = V - S + F_2 = \frac{V - S + (1-a)F_1}{1 - a\frac{F_1}{p_1}}. \tag{2.25}$$

Proposition 2.3.4. *At the fully invested equilibrium we have that $\frac{\partial p_2}{\partial S} < -1$ and $\frac{\partial^2 p_2}{\partial a \partial S} < 0$.*

If we interpret $\frac{\partial p_2}{\partial S}$ as a measure of resilience of the market, being zero for efficient markets and -1 when $a = 0$, then the market becomes less resilient under performance-based arbitrage.

2.4 Financial Intermediation

We turn now to a model proposed in [1] where banks and credit play an explicit role in the formation of asset price bubbles. Consider times $t = 1, 2$ and two assets: a safe one, in variable supply and payoff $(1 + r)$ per unit at time $t = 2$ and a risky one, with unit supply and payoff per unit at time $t = 2$ equal to a random variable p_2 with density $h(p_2)$ supported on $[0, p_2^{\max}]$ and mean $\overline{p}_2$. The safe asset can be interpreted as debt issued by the corporate sector in order to finance production at $t = 1$, whereas the risky asset can be interpreted as an asset held for speculative purposes, such as real estate or existing stocks (i.e issued prior to time $t = 1$).

As in Section 1.3, the return on the safe asset is determined by the marginal productivity of capital. For this, consider further a production function that turns x units of the consumption good at time $t = 1$ into $f(x)$ units at time $t = 2$ and satisfies $f'(x) > 0$, $f''(x) < 0$, $f'(0) = \infty$ and $f'(\infty) = 0$. In addition, there is transaction cost $c(x)$ incurred at time $t = 1$ for investing in the risky asset, which is assumed to be increasing and convex.

The model also assumes that there is a continuum of small, risk-neutral investors with no wealth of their own and a continuum of risk-neutral banks with a total $B > 0$ units of the consumption good to lend to investors, who then invest in the safe and risky assets. Because of competition, the rate of interest on loans must be the safe as the return r on the safe asset.

Let Q^s and Q^R be the number of units of the safe and risky assets held by the representative investor at time $t = 1$, purchased at prices 1 and p_1 respectively. That is, at time $t = 1$ the investor borrows an amount $Q^s + Q^R p_1$ from the bank and has to repay $(1 + r)(Q^s + Q^R p_1)$ at time $t = 2$. Since the investor can default, his profit at time $t = 2$ is

$$\left[(1 + r)Q^s + p_2 Q^R - (1 + r)(Q^s + p_1 Q^R)\right]^+ = \left[p_2 - (1 + r)p_1\right]^+ Q^R.$$

Therefore, the optimization problem faced by the investor is

$$\max_{Q^R} \left(\int_{(1+r)p_1}^{p_2^{\max}} \left[p_2 - (1 + r)p_1\right] Q^R h(p_2) dp_2 - c(Q^R) \right). \tag{2.26}$$

On the other hand, the market clearing conditions for the risky asset, loan market, and capital market are

$$Q^R = 1 \tag{2.27}$$

$$Q^s + p_1 = B \tag{2.28}$$

$$r = f'(Q^s) \tag{2.29}$$

An equilibrium for this model is given by (r, p_1, Q^s, Q^R) such that Q^R solves (2.26) for given parameters (r, p_1) and the market clearing conditions (2.27) to (2.29) are satisfied. It is easy to see that a sufficient condition for an equilibrium to exist is

$$\overline{p}_2 > c'(1). \tag{2.30}$$

At equilibrium, the first-order condition for (2.26) with $Q^R = 1$ gives

$$\int_{(1+r)p_1}^{p_2^{\max}} \left[p_2 - (1+r)p_1\right] h(p_2)dp_2 = c'(1), \tag{2.31}$$

with the two remaining market-clearing conditions reducing

$$(1+r) = f'(B - p_1). \tag{2.32}$$

Solving (2.31) and (2.32) for (r, p_1) and setting $Q^s = B - p_1$ completes the specification of the equilibrium.

Observe that we can rewrite (2.31) as

$$p_1 = \frac{1}{1+r}\left[\frac{\int_{(1+r)p_1}^{p_2^{\max}} p_2 h(p_2)dp_2 - c'(1)}{\text{Prob}\left[p_2 \geq (1+r)p_1\right]}\right] \tag{2.33}$$

By contrast, let us define the fundamental value for the traded asset as the price that investors would pay if they had to use their own funds B. In other words, investors would then solve

$$\max_{Q^s, Q^R}\left(\int_0^{p_2^{\max}} \left[(1+r)Q^s + p_2 Q^R\right] h(p_2)dp_2 - c(Q^R)\right), \tag{2.34}$$

subject to $Q^s + p_1^F Q^R \leq B$. The first-order condition for (2.34) with $Q^R = 1$ now reads

$$\int_0^{p_2^{\max}} p_2 h(p_2)dp_2 - c'(1) - (1+r)p_1^F = 0, \tag{2.35}$$

which can be rewritten as

$$p_1^F = \frac{\overline{p}_2 - c'(1)}{1+r}. \tag{2.36}$$

Proposition 2.4.1. *We have that $p_1 \geq p_1^F$ with strict inequality provided $Prob\left[p_2 < (1+r)p_1\right] > 0$.*

Let us know extend the model to incorporate uncertainty coming from the banking sector itself. Consider times $t = 0, 1, 2$ and let $p_2 = \overline{p}_2 > c'(1)$ to simplify the notation. Assume that B_0 is known at time $t = 0$ and that the central bank can alter the amount of credit available in the economy in such a way that B_1 is a random variable with density $\kappa(B_1)$ supported on $[0, B_1^{\max}]$.

Since there is no risk of default at $t = 2$, the equilibrium price for the risky asset and the interest rate at $t = 1$ satisfy

$$p_1 = \frac{\overline{p}_2 - c'(1)}{1 + r} \tag{2.37}$$

$$r_1 = f'(B_1 - p_1), \tag{2.38}$$

An investor at $t = 0$ needs to solve

$$\max_{Q_0^R} \left(\int_{B_1^*}^{B_1^{\max}} [p_1(B_1) - (1 + r_0)p_0] \, Q_0^R \kappa(B_1) dB_1 - c(Q_0^R) \right), \tag{2.39}$$

where $p_1(B_1)$ is the increasing function obtained by solving (2.37)–(2.38) and B_1^* satisfies $p_1(B_1^*) = (1 + r_0)p_0$. The first-order condition for (2.39) with $Q_0^R = 1$ then gives

$$\int_{B_1^*}^{B_1^{\max}} [p_1(B_1) - (1 + r_0)p_0] \, \kappa(B_1) dB_1 = c'(1), \tag{2.40}$$

whereas the market clearing conditions reduce to $r_0 = f'(B_0 - p_0)$. As before, the equilibrium price for the risky asset at $t = 0$ can then be rewritten as

$$p_0 = \frac{1}{1 + r_0} \left[\frac{\int_{B_1^*}^{B_1^{\max}} p_1(B_1)\kappa(B_1) dB_1 - c'(1)}{\text{Prob}\,[B_1 \geq B_1^*]} \right]. \tag{2.41}$$

By contrast, the fundamental value, defined as the valued paid by investors if they had to use their own funds, is given by

$$p_0^F = \frac{\overline{p_1(B_1)} - c'(1)}{1 + r_0}. \tag{2.42}$$

Proposition 2.4.2. *We have that $p_0 \geq p_0^F$ with strict inequality provided Prob $[B_1 < B_1^*] > 0$.*

To investigate what happens when markets expect an expansion of credit, suppose that the market for the risky asset becomes more liquid, corresponding to a smaller value for $c'(1)$. Using (2.40), we see that the range of values of B_1 for which there is no default at time $t = 1$ becomes smaller. In fact, as the next proposition shows, in the limit of flat transaction costs, credit needs to expand to its maximum value to prevent a default by investors.

Proposition 2.4.3. *As $c'(1) \to 0$, $B_1^* \to B_1^{\max}$.*

Chapter 3

Heterogeneous Beliefs

3.1 Static Model

Let us start with an argument given by Miller (1997) in a model with $t = 0, 1$, where the liquidation value of a risky asset at time $t = 1$ is

$$\widetilde{f} = \mu + \varepsilon \tag{3.1}$$

for an unknown constant μ and noise $\varepsilon \sim N(0, \sigma^2)$. Suppose that there is a continuum of investors with beliefs parametrized by μ_i, uniformly distributed in $[\mu - k, \mu + k]$. At time $t = 0$, each investor chooses Q^i to solve

$$\max_{Q^i} E\left[-e^{-\gamma Q^i(\widetilde{f} - p_0)}\right], \tag{3.2}$$

for a given market price p_0, with market-clearing condition $\int_i Q^i = Q$. It is easy to see that, in the absence of short-sale constraints, the optimal demand for each investor is

$$Q^i = \frac{\mu_i - p_0^F}{\gamma\sigma^2}, \tag{3.3}$$

so that the market clearing condition becomes

$$\int_{\mu-k}^{\mu+k} \frac{\mu_i - p_0^F}{\gamma\sigma^2} \frac{d\mu_i}{2k} = Q, \tag{3.4}$$

which implies a fundamental value

$$p_0^F = \mu - \gamma\sigma^2 Q \tag{3.5}$$

for the asset.

When short sales are prohibited, the demand for investor i becomes

$$Q^i = \max\left\{\frac{\mu_i - p_0^F}{\gamma\sigma^2}, 0\right\} \tag{3.6}$$

leading to the modified market-clearing condition

$$\int_{\max\{p_0,\mu-k\}}^{\mu+k} \frac{\mu_i - p_0}{\gamma\sigma^2}\frac{d\mu_i}{2k} = Q, \tag{3.7}$$

which implies an equilibrium price of the form

$$p_0 = \begin{cases} \mu - \gamma\sigma^2 Q & \text{if } k < \gamma\sigma^2 Q \\ \mu + k - 2\sqrt{k\gamma\sigma^2 Q} & \text{if } k \geq \gamma\sigma^2 Q \end{cases}. \tag{3.8}$$

It follows that, when the dispersion of beliefs is large enough *and* there are short-sale restrictions, asset prices reflect the opinion of optimistic investors and exhibit a bubble.

3.2 A dynamic model in discrete time

Consider now a model with two groups, A and B, of risk-neutral agents with constant discount rate β, each viewing a dividend stream d_t, for $t = 1, 2, \ldots$, as stochastic process on a probability space $(\Omega, \mathcal{F}, P^g)$, $g \in \{A, B\}$ with $P^A \sim P^B$, and let $\mathcal{F}_t$ be the sigma algebra generated by $(d_s)_{1\leq s\leq t}$. Assuming that there is a fixed unit supply for the asset and that short sales are prohibited, competition will lead to an equilibrium price of the form

$$p_t = \max_g \sup_{\tau > t} E^g\left[\sum_{i=t+1}^{\tau} \beta^{i-t} d_i + \beta^{\tau-t} p_\tau \big| \mathcal{F}_t\right]. \tag{3.9}$$

Since $\tau = \infty$ is a feasible strategy, we must have that

$$p_t \geq \max_g E^g\left[\sum_{i=t+1}^{\infty} \beta^{i-t} d_i \big| \mathcal{F}_t\right]. \tag{3.10}$$

Proposition 3.2.1. *Suppose that $F \in \mathcal{F}_t$ is a set of outcome in which A realizes the maximum in* (3.9)*. Suppose further that for some $t' > t$, there exists another set of outcomes $F' \in \mathcal{F}_{t'}$, with $F' \subset F$ and $P^A(F') > 0$ such that*

$$E^B\left[\sum_{i=t+1}^{\infty} \beta^{i-t} d_i \big| \mathcal{F}_t\right](\omega) > E^A\left[\sum_{i=t+1}^{\infty} \beta^{i-t} d_i \big| \mathcal{F}_t\right](\omega), \quad \text{for } \omega \in F'. \tag{3.11}$$

Then a strict inequality holds in (3.10) *for $\omega \in F$.*

3.3 Overconfidence in continuous time

Consider a risky asset with cumulative dividend process

$$dD_t = f_t dt + \sigma_D dW_t^D \tag{3.12}$$

where f is not observable but satisfies

$$df_t = -\lambda(f_t - \overline{f})dt + \sigma_f dW_t^f. \tag{3.13}$$

Suppose there are two groups, A and B, of risk-neutral agents, each observing D_t and a pair of signals

$$ds_t^A = f_t dt + \sigma_s dW_t^A \tag{3.14}$$

$$ds_t^B = f_t dt + \sigma_s dW_t^B. \tag{3.15}$$

Assume that in the real world all four Brownian motions are uncorrelated, but agents in group A believes that

$$ds_t^A = f_t dt + \sigma_s \left(\phi dW_t^f + \sqrt{1-\phi^2} dW_t^A\right), \tag{3.16}$$

whereas agents in group B believe that

$$ds_t^B = f_t dt + \sigma_s \left(\phi dW_t^f + \sqrt{1-\phi^2} dW_t^B\right), \tag{3.17}$$

while correctly believing that the innovations of the other signal are uncorrelated with dW^f.

Assuming that all of the above is public information (including the beliefs of each group of agents), it follows that the estimates of the process f_t have stationary distributions with conditional means $\widehat{f}_t^A$ and $\widehat{f}_t^B$ that follow relatively simple mean-reverting processes. It follows that the dynamics of the difference in belief

$$b^A = \widehat{f}^B - \widehat{f}^A \tag{3.18}$$

is given by

$$db_t^A = -\rho b_t^A dt + \sigma_b dW_t^{A,b} \tag{3.19}$$

where both σ_b and the mean-reversion speed $\frac{-\rho}{2\sigma_b^2}$ are increasing functions of the the overconfidence parameter ϕ.

As in the discrete-time model, it follows that the equilibrium price in the presence of short-sales, fixed supply and an infinite number of agents is

$$p_t^g = \sup_{\tau \geq 0} E_t^g \left[\int_t^{t+\tau} e^{-r(s-t)} dD_s + e^{-r\tau}\left(p_{t+\tau}^{\overline{g}} - c\right)\right], \tag{3.20}$$

where $g \in \{A, B\}$ denotes the group of the current owner of the asset, $p_{t+\tau}^{\overline{g}}$ is the reservation price of a buyer from the other group $\overline{g}$ at the time $t+\tau$ of a future transaction, and c is a selling cost. Using the equations for the dividend process and the conditional means of beliefs, this reduces to

$$p_t^g = \sup_{\tau \geq 0} E_t^g \left[\int_t^{t+\tau} e^{-r(s-t)} [\overline{f} + e^{-\lambda(s-t)} (\widehat{f}_s^g - \overline{f})] ds + e^{-r\tau} \left(p_{t+\tau}^{\overline{g}} - c \right) \right].$$

Because of the Markovian structure of the model, it is natural to consider a price of the form

$$p_t^g = \frac{\overline{f}}{r} + \frac{\widehat{f}_t^g - \overline{f}}{r+\lambda} + q\left(b_t^g\right), \tag{3.21}$$

where the first two terms combined represent a fundamental value for the asset, corresponding to the expected present value of dividends from the point of view of the current owner of the asset, and the last term corresponds to the value of the option to sell it later, which in turn depends on the current difference b_t^g between the beliefs of the other group and those of the current owner.

Inserting this into (3.3) then leads to

$$q\left(b_t^g\right) = \sup_{\tau \geq 0} E_t^g \left[e^{-r\tau} \left(\frac{b_{t+\tau}^g}{r+\lambda} + q\left(-b_{t+\tau}^g\right) - c \right) \right] \tag{3.22}$$

Using standard arguments from optimal control, it follows that the function q must satisfy the variational problem

$$\begin{cases} \frac{1}{2}\sigma_g^2 q'' - \rho x q' - rq \leq 0 \\ q(x) \geq \frac{x}{r+\lambda} + q(-x) - c \end{cases} \tag{3.23}$$

One can then find semi-analytic solutions in terms of Kummar functions and characterize the size of the bubble by

$$b = q(-x^*) \tag{3.24}$$

where x^* is the exercise threshold for the resale option.

Chapter 4

Local martingales

4.1 NFLVR and No Dominance

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F})_{t \geq 0}$ satisfying the usual conditions.

Consider a riskless money market account as a numeraire and a risk asset paying a cumulative dividend process $D_t \geq 0$ given by a càdlàg semimartingale adapted to $\mathbb{F}$ and liquidation value $0 \leq X_\tau \in \mathcal{F}_\tau$, where τ is a stopping time.

Let the market price of this asset be given by a nonnegative càdlàg semimartingale S_t, so that the wealth process from owning the asset from time 0 is given by

$$W_t = S_t + \int_0^{t \wedge \tau} dD_u + X_\tau 1_{\{\tau \leq t\}}. \tag{4.1}$$

Observe that the càdlàg condition implies that S_t is the *ex-dividend* price, so that $S_\tau = 0$ and

$$W_\tau = \int_0^\tau dD_u + X_\tau. \tag{4.2}$$

A trading strategy consists of an adapted process (π, η) corresponding to the number of units $\pi \in L(W)$ of the risky asset and the number of units η of the money market account, where $L(W)$ denotes the set of integrable processes with respect to W. The strategy is said to be *self-financing* if its value $V_t^{\pi,\eta} := \pi_t S_t + \eta_t$ satisfies

$$V_t^{\pi,\eta} = \int_0^t \pi_u dW_u. \tag{4.3}$$

for a predictable process π. It follows that a self-financing strategy can be completely characterized by π, so that we denote its value simply by V_t^π. A self-financing strategy is *admissible* if, in addition, $V_t^{\pi,\eta} \geq -a$ for some $a \geq 0$.

Define the set of attainable claims by

$$K = \left\{ V_\infty^\pi = \int_0^\infty \pi_u dW_u : \pi \text{ admissible} \right\} \tag{4.4}$$

and the cone of bounded claims that can be superreplicated by attainable claims by $C = (K - L_0^+) \cap L_\infty$. We say that the market satisfies the No Free Lunch with Vanishing Risk (NFLVR) condition if

$$\overline{C} \cap L_\infty^+ = \{0\}, \tag{4.5}$$

where the closure in the expression above is taken with respect to the L^∞-norm. Moreover, a probability measure Q, equivalent to P, is called an equivalent local martingale measure (ELMM) if W_t is a Q-local martingale. We denote this set by $\mathcal{M}_{loc}(W)$. The following theorem is a consequence of the First Fundamental Theorem of Asset Pricing (FTAP) of [3] adjusted for the setting of this chapter, namely using the fact that $W_t \geq 0$.

Theorem 4.1.1. *NFLVR* $\iff \mathcal{M}_{loc}(W) \neq \emptyset$

A market is said to be complete if for all claims $X_\infty \in L^2(\Omega, \mathcal{F}_\infty, P)$, there exists a self-financing trading strategy (π, η) and $c \in \mathbb{R}$ such that

$$X_\infty = c + \int_0^\infty \pi_s dW_s. \tag{4.6}$$

Under NFLVR, the Second Fundamental Theorem of Asset Pricing (see [7]) that a market is complete if and only if the set $\mathcal{M}_{loc}(W)$ consists of a single measure Q.

We now introduce the concept of No Dominance. Let $\phi = (\Delta, \Xi^\nu)$ be the payoff of an asset, where $\Delta \geq 0$ is cádlǵ process representing a cumulative dividend stream and $\Xi^\nu \geq 0$ is a terminal payoff at time $\nu \in \mathbb{R}_+$. Denote by Φ be the set of assets such that

$$\Delta_\nu + \Xi^\nu \leq a + V_\nu^\pi \tag{4.7}$$

for some admissible strategy π. That is, Φ is the set of assets with bounded termination, positive cumulative dividends and positive terminal payoff that can be super-replicated by trading on the risky asset and the money market account. It is easy to prove that Φ is a cone (see [9]). Moreover, if $\phi \in \Phi$ then foreach $Q \in \mathcal{M}_{loc}(W)$ we have

$$E_Q[\Delta_\nu + \Xi^\nu \leq a + E_Q[V_\nu^\pi] \leq a, \tag{4.8}$$

since V_ν^π is a nonnegative Q-local martingale and therefore a Q-supermartingale.

Denote the market price of $\phi \in \Phi$ by $\Lambda_t(\phi)$ and the net gain from purchasing it at time σ and selling at time $\mu \leq \nu$ by

$$G_{\sigma,\mu}(\phi) = \Lambda_\mu(\phi) + \int_\sigma^\mu d\Delta_s + \Xi^\nu 1_{\{\nu=\mu\}} - \Lambda_\sigma(\phi). \tag{4.9}$$

We then say that asset ϕ^2 dominates asset ϕ^1 at σ if there exists a pair of stopping times $\sigma < \mu \leq \nu$ such that $G_{\sigma,\mu}(\phi^2) \geq G_{\sigma,\mu}(\phi^1)$ almost surely and

$$E_P\left[1_{\{G_{\sigma,\mu}(\phi^2) > G_{\sigma,\mu}(\phi^1)\}} | \mathcal{F}_\sigma\right] > 0.$$

Accordingly, we say that there is No Dominance (ND) if the pricing functions $\Lambda_t : \Phi \to \mathbb{R}_+$ is such that there are no dominated assets in the market. Intuitively, this means that if two assets provide the same cash flows ϕ, then there cannot be a time $\sigma \leq \nu$ for which the market price of one asset is lower than the other, since otherwise the cheaper asset would dominate the more expensive one at this time.

It is relatively straightforward to show that ND implies NFLVR, but the converse is not true. For instance, [8] offer an example that does not violate NFLVR but nevertheless violates ND.

4.2 Bubbles in Complete Markets

Assuming NFLVR, let $\mathcal{M}_{loc}(W) = \{Q\}$ and define the fundamental price for the risky asset as

$$S_t^* = E_Q\left[\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}} | \mathcal{F}_t\right] 1_{\{t<\tau\}} \tag{4.10}$$

and the corresponding wealth process by

$$W_t^* = S_t^* + \int_0^{t\wedge\tau} dD_u + X_\tau 1_{\{\tau \leq t\}}. \tag{4.11}$$

Observe that it follows from this definition that

$$W_\tau^* = \int_0^\tau dD_u + X_\tau. \tag{4.12}$$

Lemma 4.2.1. *The fundamental price is well defined. Furthermore, we have that* $S_t \xrightarrow{a.s.} S_\infty \in L^1(Q)$, $S_t^* \xrightarrow{a.s.} 0$, *and* W_t^* *is a uniformly integrable* Q*-martingale closed by*

$$W_\infty^* = \int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}. \tag{4.13}$$

Proof. Since W_t is a nonnegative supermartingale, it follows from the martingale convergence theorem that there exists a random variable $W_\infty \in L^1(Q)$ such that $W_t \xrightarrow{\text{a.s.}} W_\infty$. But

$$\begin{aligned} W_\infty = \lim_{t\to\infty} W_t &= \lim_{t\to\infty}\left(S_t + \int_0^{t\wedge\tau} dD_u + X_\tau 1_{\{\tau\le t\}}\right) \\ &= \lim_{t\to\infty} S_t + \int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}. \end{aligned}$$

Therefore, there exist $S_\infty \in L^1(Q)$ and $\left(\int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}\right) \in L^1(Q)$, which implies that S_t^* is well defined. Moreover, since

$$E_Q\left[\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right] = -\int_0^t dD_u + E_Q\left[\int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right],$$

we see that

$$\begin{aligned} \lim_{t\to\infty} S_t^* &= -\int_0^\infty dD_u 1_{\{\tau=\infty\}} + 1_{\{\tau=\infty\}} E_Q\left[\int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_\infty\right] \\ &= -\int_0^\infty dD_u 1_{\{\tau=\infty\}} + 1_{\{\tau=\infty\}} E_Q\left[\int_0^\tau dD_u|\mathcal{F}_\infty\right] = 0. \end{aligned}$$

We then have

$$\begin{aligned} W_\infty^* := \lim_{t\to\infty} W_t^* &= \lim_{t\to\infty}\left(S_t^* + \int_0^{t\wedge\tau} dD_u + X_\tau 1_{\{\tau\le t\}}\right) \\ &= \int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}. \end{aligned}$$

Therefore, $W_\infty^* + S_\infty = W_\infty$ which implies that $W_\infty^* \in L^1(Q)$. Finally

$$\begin{aligned} E_Q\left[W_\infty^*|\mathcal{F}_t\right] &= E_Q\left[\int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right] \\ &= \int_0^t dD_u 1_{\{t<\tau\}} + E_Q\left[\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right] 1_{\{t<\tau\}} \\ &+ \int_0^\tau dD_u 1_{\{\tau\le t\}} + X_\tau 1_{\{\tau\le t\}} = \int_0^{t\wedge\tau} dD_u + X_\tau 1_{\{\tau\le t\}} + S_t^* = W_t^*. \end{aligned}$$

□

Definition 4.2.1. An asset price bubble for S is defined as

$$B_t = S_t - S_t^*. \tag{4.14}$$

The next theorem establishes that, in general, an asset price bubble is a Q-local martingale. Imposing a restriction on the liquidation time τ, namely that τ is P-almost surely finite, prevents the bubble from being a uniformly integrable Q-martingale. A further restriction on the liquidation time, namely that it is bounded, prevents the bubble from being a Q-martingale. In [8], this last situation is called a *strict local martingale*, namely a local martingale that is not a martingale.

Theorem 4.2.2. *If $B_t \neq 0$, then B_t is a Q-local martingale. If, in addition, $P(\tau < \infty) = 1$, then B_t is not a uniformly integrable Q-martingale. Furthermore, if τ is bounded, then B_t is not a Q-martingale.*

Proof. It follows from (4.1) and (4.11) that

$$B_t = S_t - S_t^* = W_t - W_t^*, \tag{4.15}$$

which shows that B_t is a Q-local martingale, being the sum of the Q-local martingale W_t (by definition of $Q \in \mathcal{M}_{loc}(W)$) and the uniformly integrable Q-martingale W_t^* (by the previous proposition). This establishes the first assertion in the theorem.

For the second assertion, observe that it follows from (4.2) and (4.12) that

$$B_\tau = S_\tau - S_\tau^* = W_\tau - W_\tau^* = 0.$$

Therefore, if B were a uniformly integrable martingale, then by Doob's optional sampling theorem we would have that

$$B_{\tau_0} = E_Q\left[B_\tau | \mathcal{F}_{\tau_0}\right] = 0,$$

for any $\tau_0 \leq \tau$, which implies that $B = 0$ on $[0, \tau]$.

Finally for the last assertion, let

$$K_t = W_t - E_Q\left[W_\infty | \mathcal{F}_t\right]. \tag{4.16}$$

Then K_t is a Q-local martingale, being the sum of the Q-local martingale W_t and the uniformly integrable Q-martingale $E_Q\left[W_\infty | \mathcal{F}_t\right]$. Moreover, since W_t itself is a Q-supermartingale (by virtue of being a Q-local martingale bounded from below by zero), we have that $W_t \geq E_Q[W_\infty | \mathcal{F}_t]$, so that

$K_t \geq 0$ and therefore also a Q-supermartingale. On the other hand, by the previous lemma, we have that

$$E_Q\left[W_\infty|\mathcal{F}_t\right] = E_Q\left[W^*_\infty|\mathcal{F}_t\right] + E_Q\left[S_\infty|\mathcal{F}_t\right] = W^*_t + E_Q\left[S_\infty|\mathcal{F}_t\right]. \tag{4.17}$$

Now if $\tau < T$ for some $T \in \mathbb{R}_+$, then $S_\infty = 0$ and $B_t = K_t$ for all t. But for $t \geq \tau$,

$$\begin{aligned} K_t &= W_t - E_Q\left[\int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right] \\ &= W_t - \left(\int_0^\tau dD_u + X_\tau\right) = S_t = 0. \end{aligned}$$

In particular, $B_T = 0$. If B were a martingale we would have

$$B_t = E_Q\left[B_T|\mathcal{F}_t\right] = 0,$$

for all $t \leq T$, contradicting the hypothesis that $B_t \neq 0$. □

Theorem 4.2.3. *S_t admits a unique decomposition*

$$S_t = S^*_t + B_t = S^*_t + B^1_t + B^2_t + B^3_t \tag{4.18}$$

where $B_t \geq 0$ is a Q-local martingale and

1. *$B^1_t \geq 0$ is a uniformly integrable Q-martingale with $B^1_t \xrightarrow{a.s.} X_\infty$.*
2. *$B^2_t \geq 0$ is a non-uniformly integrable Q-martingale with $B^2_t \xrightarrow{a.s.} 0$.*
3. *$B^3_t \geq 0$ is a strict Q-local martingale with $E_Q\left[B^3_t\right] \xrightarrow{a.s.} 0$ and $B^3_t \xrightarrow{a.s.} 0$.*

Proof. Let

$$B^1_t = E_Q\left[S_\infty|\mathcal{F}_t\right].$$

Recalling the definition of K_t in (4.16) and using the identity (4.17), we have that

$$B_t = K_t + E_Q\left[S_\infty|\mathcal{F}_t\right] = K_t + B^1_t,$$

where K_t is given by (4.16). Furthermore, using the Riesz decomposition, the positive supermartingale K_t can written as

$$K_t = B^2_t + B^3_t \tag{4.19}$$

where B_t^2 is a martingale and B^3 is a potential, that is to say, a positive supermartingale with $E_Q[B_t^3] \to 0$, so that $B_t^3 \xrightarrow{\text{a.s.}} 0$ as well. But since

$$K_t = W_t - E_Q[W_\infty|\mathcal{F}_t] \xrightarrow{\text{a.s.}} W_\infty - E_Q[W_\infty|\mathcal{F}_\infty] = 0,$$

we conclude that

$$B_t^2 = K_t - B_t^3 \xrightarrow{\text{a.s.}} 0.$$

Moreover, by the martingale property for B^2, we have that

$$B_t^2 = E_Q\left[B_{t+u}|\mathcal{F}_t\right] = E_Q\left[K_{t+u}|\mathcal{F}_t\right] - E_Q\left[B_{t+u}^3|\mathcal{F}_t\right]$$

which implies that

$$B_t^2 = \lim_{u\to\infty} E_Q\left[K_{t+u}|\mathcal{F}_t\right] \geq 0.$$

□

Corollary 4.2.4. *Any asset price bubble satisfies*

1. $B_t \geq 0$
2. $B_\tau 1_{\{\tau<\infty\}} = 0$
3. *If* $B_t = 0$ *then* $B_u = 0$ *for all* $u \geq t$.

Using the decomposition in Theorem 4.2.3, we say that an asset has a bubble of Type 1 if $B^1 \neq 0$, a bubble of Type 2 if $B^1 = 0$ but $B^2 \neq 0$, and a bubble of Type 3 if $B^1 = B^2 = 0$ but $B^3 \neq 0$. The next proposition shows that No Dominance is enough to rule out bubbles of Types 2 and 3.

Proposition 4.2.5. *Assume that* $\tau < \infty$, *so that* $B_t^1 = 0$. *Under No Dominance and complete markets, we have that* $B_t^2 = B_t^3 = 0$.

Proof. By market completeness, there exists an admissible π^1 such that

$$W_t^* = W_0^* + \int_0^t \pi_u^1 dW_u.$$

Since $B_\infty = 0$, W_t and W_t^* have the same payoff, so it follows from No Dominance that $W_0^* = W_0$, which implies that $B_0 = 0$. □

Examples:

(1) *Uniformly integrable martingale bubble*

Consider an asset with $D_t = 0$, $\tau = \infty$ and $X_\infty = 1$. We then have

$$S_t^* = E_Q\left[\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right]1_{\{t<\tau\}} = 0.$$

If $S_t = 1$, then $B_t = S_t - S_t^* = 1$ is a uniformly integrable martingale bubble. This corresponds, for example, to fiat money.

(2) *Martingale bubble*

For the next example, consider the process

$$B_t = \frac{1 - I_t}{Q(\tau > t)},$$

where $I_t = 1_{\{\tau \le t\}}$ for a random variable $0 \le \tau < \infty$ such that $Q(\tau > t) > 0$ for all t. If $\mathcal{F}_t$ is the filtration generated by I_t, it follows that

$$E_Q[1_{\{\tau>t\}}|\mathcal{F}_s] = Q(\tau > t|\mathcal{F}_s) = 1_{\{\tau>s\}}Q(\tau > t|\tau > s) = 1_{\{\tau>s\}}\frac{Q(\tau > t)}{Q(\tau > s)}.$$

Therefore

$$E_Q[B_t|\mathcal{F}_s] = E\left[\frac{1_{\{\tau>t\}}}{Q(\tau > t)}|\mathcal{F}_s\right] = \frac{1_{\{\tau>s\}}}{Q(\tau > s)} = B_s,$$

which shows that B is a Q-martingale. On the other hand, $B_t = 0$ on $\{t \ge \tau\}$, which implies that $B_t \to B_\infty = 0$ a.s., since $\tau < \infty$. If B were uniformly integrable, then $B_t = E[B_\infty|\mathcal{F}_t] = 0$ for all t, which is not true.

Now consider an asset with $D_t = 0$ and $X_\tau = 1$, so that

$$S_t^* = E_Q\left[\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right]1_{\{t<\tau\}} = 1_{\{t<\tau\}}.$$

Then $S_t = S_t^* + B_t$ is an example of an asset price with a non-uniformly integrable martingale bubble.

(3) *Strict Local Martingale Bubble*

For the next example, consider the process

$$B_t = \int_0^t \frac{\beta_u}{\sqrt{T-u}}dZ_u, \tag{4.20}$$

where Z_t is a standard Brownian motion. Observe first that

$$L_t = \int_0^t \frac{1}{\sqrt{T-u}} dZ_u$$

is a local martingale with

$$A_u := [L, L]_u = -\log \left[1 - \frac{u}{T}\right].$$

It then follows from the Dubins-Schwartz theorem that

$$dB_u = B_u d\widetilde{Z}_{A_u}$$

for a Brownian motion $\widetilde{Z}_t$. Therefore

$$B_u = B_0 e^{\widetilde{Z}_{A_u} - \frac{1}{2} A_u}, \tag{4.21}$$

from which it follows that B_s is a martingale on $[0, u]$ for $u < T$. However, since $A_u \to +\infty$ monotonically as $u \to T$ and

$$\lim_{t \to +\infty} e^{\widetilde{Z}_t - \frac{1}{2} t} = 0,$$

we have that

$$\lim_{u \to T} B_u = 0.$$

Defining $B_T = 0$ we see that B_t is continuous on $[0, T]$ but $E_Q[B_T] = 0 < B_0$, so that B is not a martingale. We can then set $\tau = T$, $X_\tau = 1$, $S_t^* = 1_{[0,T)}$ and observe that $S_t = S_t^* + B_t$ is an example of an asset price with a strict local martingale bubble B_t.

The economic intuition behind these three types of bubbles is as follows. Uniformly integrable martingale bubbles, that is, B_t^1 in the notation of Theorem 4.2.3, are related to assets that are infinitely lived with positive probability. They arise because of a component X_∞ of the payoff that is obtained at time $\tau = \infty$ and correspond to a permanent (albeit stochastic) wedge between the market price and the fundamental value of the asset.

Martingale bubbles that are not uniformly integrable, that is, B_t^2 in the notation of Theorem 4.2.3, are related to assets with finite but unbounded lives. To take advantage of such bubble one would go long the fundamental value (which is possible because of market completeness) and short the asset itself, that is to say, adopt the strategy $\pi_t = -1_{(0,\tau]}$. If held until τ, the combined position would generate a free-lunch once both the fundamental value and the asset price drop to zero. But such strategy is not admissible, since S_t is unbounded (otherwise B_t^2 would be uniformly integrable) and therefore the value

$$V_t^\pi = \int_0^t \pi_s dW_s = -S_t$$

cannot be guaranteed to remain above $-a$ for any $a \in \mathbb{R}_+$.

Finally, strict local martingale bubbles, that is, B_t^3 in the notation of Theorem 4.2.3, are related to assets with bounded lives $\tau = T$. The same argument regarding admissibility holds here, namely that $\pi_t = -1_{(0,T]}$ is not an admissible trading strategy, even for a known and finite T. To see this, assume otherwise, namely that $(W_0 - W_T) \geq -a$ for some $a \in \mathbb{R}_+$. Then

$$E_Q\left[\left(\int_0^T \pi_u dW_u\right)^-\right] = E_Q\left[(W_T - W_0)^+\right] \leq E[W_T] \leq W_0 < \infty.$$

But this implies that $\int_0^t \pi_u dW_u = W_0 - W_t$ is a Q-supermartingale, which in turn means that there exists a Q-martingale M_t such that

$$(W_0 - W_t)^- \leq M_t$$

for all $0 \leq t \leq T$. Therefore $W_t \leq M_t + W_0$ which implies that W_t itself is a martingale. But since $0 \leq B_t \leq W_t$, we have that B_t is also a martingale and hence not a strict local martingale.

4.2.1 Contingent Claims

We now consider European contingent claims H with payoffs of the form $H(S)_T := H(S_u, 0 \leq u \leq T) \in L^1(Q)$, that is to say, integrable claims that can depend on the entire path of the underlying asset S_t up to a fixed maturity $T \in \mathbb{R}_+$. We assume that $D_t = 0$, that is, the underlying asset pays no dividend, and that $\tau > T$, namely the liquidation date for the underlying asset itself occurs after the maturity of any contingent claim, at which point it pays $X_\tau \geq 0$ as before. It then follows from the definitions (4.1) and (4.10) that, for $0 \leq t \leq T$, we have $W_t = S_t$ and

$$S_t^* = E_Q\left[X_\tau 1_{\{\tau<\infty\}} | \mathcal{F}_t\right]. \tag{4.22}$$

The contingent claim itself can be viewed as an asset with dividends $D_t^H = 0$, liquidation date $\tau^H = T$, and liquidation value $X_\tau^H = H(S)_T$. For consistency with the general setup of this section, we consider only contingent claims with $H(S)_T \geq 0$, but notice that the results hold for any claim $\overline{H} \geq -a$ for $a \in \mathbb{R}_+$ by considering $H = \overline{H} + a$ instead. Following (4.10), we define the fundamental value for the contingent claim H as

$$V_t^*(H) = E_Q\left[H(S)_T|\mathcal{F}_t\right]. \tag{4.23}$$

Notice that this is defined in terms of the market price S_t for the underlying asset instead of its fundamental value S_t^*, since contingent claims are written on market prices. In this context, there are two related but different effects of bubbles on contingent claims. The first is that a bubble $B_t = S_t - S_t^*$ on the underlying asset can affect the valuation of the contingent claim because

$$H(S)_T = H(S_u, 0 \leq u \leq T) \neq H(S_u^*, 0 \leq u \leq T). \tag{4.24}$$

The second effect is that trading in the contingent claim itself with a market price $V_t(H)$ can give rise to a bubble, which we now denote by

$$b_t^H := V_t(H) - V_t^*(H). \tag{4.25}$$

In particular, observe that if the underlying asset has a bubble $B_T > 0$ at time T (and therefore $B_t > 0$ for all $0 \leq t \leq T$ by Corollary (4.2.4)), then $S_T > S_T^* = E_Q\left[X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_T\right]$. In this case, viewing the underlying asset as a contingent claim with payoff $H(S)_T = S_T$ leads to a fundamental value

$$V_t^*(S) = E_Q\left[S_T|\mathcal{F}_t\right] > E_Q\left[S_T^*|\mathcal{F}_t\right] = S_t^*, \tag{4.26}$$

so that the second notion of a bubble in (4.25) gives

$$b_t^S = V_t(S) - V_t^*(S) = S_t - E_Q\left[S_T|\mathcal{F}_t\right] < S_t - S_t^* = B_t, \tag{4.27}$$

that is, the bubble b_t^S in the underlying asset viewed as a derivative with payoff S_T at T is smaller than the original bubble B_t.

Observe further that the analogue of Lemma 4.2.1 automatically holds for a contingent claim, as the fundamental value $V_t^*(H)$ given in (4.23) is well-defined, $V_t(H) \xrightarrow{\text{a.s.}} V_\infty(H) = 0 \in L^1(Q)$ and $V_t^*(H) \xrightarrow{\text{a.s.}} 0$ (since $V_t^*(H) = 0$ for $t > T$), and the fundamental wealth process

$$W_t^*(H) = V_t^*(H)$$

is a uniformly integrable Q-martingale on $[0,T]$ closed by $W_T^*(H) = H(S)_T$.

The next theorem presents a characterization of the fundamental value of a contingent claim in terms of super-replication trading strategies.

Theorem 4.2.6. *Under both NFLVR and market completeness, the fundamental value $V_0^*(H) = E_Q[H(S)_T]$ is the smallest initial cost of an admissible trading strategy π with $V_T^\pi \geq H(S)_T$.*

Proof. Let

$$\mathcal{V} = \left\{ V_t^\pi = v_0 + \int_0^t \pi_u dS_u : \pi \text{ admissible }, V_T^\pi \geq H(S)_T \right\}, \tag{4.28}$$

be the set of super-replicating strategies. Because $H(S)_T \geq 0$, it follows from NFLVR that V_t^π is a nonnegative Q-supermartingale. Therefore there exists a decomposition

$$V_t^\pi = M_t + C_t, \tag{4.29}$$

where M_t is a uniformly integrable Q-martingale and C_t is a potential (that is, a nonnegative Q-supermartingale with $C_t \to 0$). On the other hand, it is easy to verify that

$$V_t^*(H) = E_Q[H(S)_T|\mathcal{F}_t], \tag{4.30}$$

is also a uniformly integrable martingale. Moreover, it follows from market completeness that

$$V_t^* = v_0^* + \int_0^t \pi *_u dS_u$$

for a self-financing trading strategy π^*. Because $H(S)_T \geq 0$, it follows that π^* is admissible and, therefore, $V_t^* \in \mathcal{V}$. It is then easy to see that

$$\begin{aligned}\inf_{\mathcal{V}} V_t^\pi &= \inf\{M_t + C_t : M_t \ Q\text{-martingale}, M_T \geq 0, C_t \text{ potential}, C_T = 0\} \\ &= E_Q[H(S)_T|\mathcal{F}_t] = V_t^*\end{aligned}$$

□

Observe that the existence of a bubble $b_t^H > 0$ for a contingent claim H implies that $V_t(H) > V_t^*(H)$, which according to the last theorem means that that market price is higher than the super-replication price. At first sight this appears to be a misplacing that violates the NFLVR condition, as being short on the contingent claim and long the super-replicating trading strategy seems to lead to a free-lunch. However, the same argument used to explain why bubbles on the underlying asset are consistent with NFLVR also applies here. Namely, being short a contingent claim H with a nonzero bubble b^H is not an admissible trading strategy.

For the next result, consider the following payoffs

$$C_T(K) = (S_T - K)^+ \geq 0 \tag{4.31}$$
$$P_T(K) = (K - S_T)^+ \geq 0 \tag{4.32}$$
$$F_T(K) = S_T - K \geq -K, \tag{4.33}$$

corresponding to a call option with strike K, a put option with strike K and a forward contract with forward price K, all with maturity T, with the respective fundamental values denotes by $C_t^*(K)$, $P_t^*(K)$ and $F_t^*(K)$.

Lemma 4.2.7. *The fundamental values for a call, put, and forward contract with payoffs* (4.31)-(4.33) *satisfy put-call parity relationships*

$$C_t^*(K) - P_t^*(K) = F_t^*(K) = V^*(S)_t - K. \tag{4.34}$$

Proof. The result follows from the definition of fundamental value and the identity

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K.$$

□

Regarding put-call parity for market prices, we have that

$$\begin{aligned} C_t(K) - P_t(K) &= C_t^*(K) + b_t^C - P_t^*(K) - b_t^P \\ &= F_t^*(K) + b_t^C - b_t^P \\ &= F_t(K) - b_t^F + b_t^C - b_t^P \end{aligned} \tag{4.35}$$

We therefore see that

$$C_t(K) - P_t(K) = F_t(K) \iff b_t^F = b_t^C - b_t^P. \tag{4.36}$$

Similarly, inserting the relationship

$$F_t^*(K) = E_Q[S_T - K|\mathcal{F}_t] = V^*(S)_t - K = S_t + b_t^S - K \tag{4.37}$$

in (4.35) gives

$$C_t(K) - P_t(K) = S_t - K \iff b_t^S = b_t^C - b_t^P. \tag{4.38}$$

For the next results, observe that, for any contingent claim with maturity $T < \infty$, we only need to consider Type 3. Recall that Type 3 bubbles do not exist for the underlying asset S under no dominance, but they might exist for contingent claim.

Lemma 4.2.8. *Assume no dominance and let H' be a contingent claim such that $V_t(H') = V_t^*(H')$. Then for every H such that $H(S)_T \leq H'(S)_T$, we have $V_t(H) = V_t^*(H)$.*

Proof. It follows from no dominance that

$$V_\sigma(H) \leq V_\sigma(H)', \ \sigma \in [0, T]$$

But since $V_t(H') = V_t^*(H)' = E_Q[H'(S)_T|\mathcal{F}_t]$ is uniformly integrable a martingale and in class (D) on $[0, T]$, it follows that $V_t(H)$ is a uniformly integrable martingale, which excludes the possibility of a Type 3 bubble (and therefore any bubble) for this claim. □

Corollary 4.2.9. *If $H(S)_T$ is bounded, then no dominance implies that $V_t(H) = V_t^*(H)$. In particular, a put option does not have a bubble if we assume no dominance.*

Proof. Use the previous lemma with $H \leq K = H'$ for a constant K, for which no dominance implies that $V_t(H') = V_t^*(H)' = E_Q[K|\mathcal{F}_t] = K$. □

Theorem 4.2.10. *Under no dominance, $C_t(K) - C_t^*(K) = S_t - E_Q[S_T|\mathcal{F}_t]$.*

Proof. From the definition of fundamental value we have

$$\begin{aligned} C_t^*(K) &= P_t^*(K) + F_t^*(K) \\ &= P_t^*(K) + E_Q[S_T - K|\mathcal{F}_t] \\ &= P_t^*(K) + V_t^*(S) - K \\ &= P_t^*(K) + S_t - K - b_t^S \end{aligned} \tag{4.39}$$

Moreover, it also follows from no dominance that

$$C_t(K) = P_t(K) + F_t(K) = P_t(K) + S_t - K \tag{4.40}$$

Subtracting (4.39) from (4.40) we find

$$C_t(K) - C_t^*(K) = b_t^S = S_t - E_Q[S_T|\mathcal{F}_t],$$

where we have used the fact that $P_t(K) = P_t^*(K)$ since put options have no bubbles under no dominance. □

Corollary 4.2.11. *Under no dominance, $b_t^C = b_t^F = b_t^S = 0$ and put-call parity holds.*

Proof. Type 3 bubbles for S do not exist under no dominance. □

4.3 Bubbles In Incomplete Markets

Recall that $\mathcal{M}_{loc}(W) = \{Q \sim P,\ W \text{ is a Q-local martingale}\}$ is the set of equivalent local martingale measures (ELMM) for the asset S. Define the sets

$$\mathcal{M}_{UI}(W) = \{Q \in \mathcal{M}_{loc}(W) : W \text{ is a uniformly integrable Q-martingale}\},$$

and

$$\mathcal{M}_{NUI}(W) = \mathcal{M}_{loc}(W) \setminus \mathcal{M}_{UI}(W).$$

In general, $\mathcal{M}_{UI}(W)$ is a proper subset and $\mathcal{M}_{NUI}(W)$ is non-empty.

Let $(\sigma_i)_{i\geq 0}$ be an increasing sequence of random times with $\sigma_0 = 0$ and $\sigma_i \to \infty$ as $i \to \infty$ and $(Y^i)_{i\geq 0}$ be a sequence of random variables independent from (σ_i). Assume further that (Y^i) and (σ_i) are independent of $\mathbb{F}$, the filtration with respect to which we assumed that the underlying asset S is adapted. Define the processes

$$N_t = \sum_{i\geq 1} 1_{\{t\geq\sigma_i\}}$$

and

$$Y_t = \sum_{i\geq 0} Y^i 1_{\{\sigma_i\leq t<\sigma_{i+1}\}}.$$

We interpret N_t as the process that counts the number of regime switches that occurred since time $t = 0$ at the switching times σ_i and Y_t as the current state of the random variable Y_i that characterizes the regime in the time interval $\sigma_i \leq t < \sigma_{i+1}$.

Let $\mathbb{H}$ be the filtration generated by N_t and Y_t, and $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. Then σ_i are $\mathbb{G}$ stopping times, but not necessarily $\mathbb{F}$ stopping times. Notice that it follows from that fact that Y and N are independent of $\mathbb{F}$ that every $(Q,\mathbb{F})$-local martingale is also a $(Q,\mathbb{G})$-local martingale. In other words, $\mathcal{M}^{\mathbb{F}}_{loc}(W) \subset \mathcal{M}^{\mathbb{G}}_{loc}(W)$. For $Q \in \mathcal{M}^{\mathbb{F}}_{loc}(W)$, define the Radon-Nikodym derivative $Z_\infty = \frac{dQ}{dP}|_{\mathcal{F}_\infty}$ and let

$$Z_t = E[Z_\infty|\mathcal{F}_t]. \tag{4.41}$$

Suppose that $N_t = i$, and denote by $Q^i = Q(N_t, Y_t) \in \mathcal{M}^{\mathbb{F}}_{loc}(W)$ be the ELMM "selected by the market" at t (assume that there are enough derivatives so that this is unique).

Definition 4.3.1. Let $\phi = (\Delta, \Xi^\nu)$ be a contingent claim. The fundamental value of ϕ is

$$V_t^*(\phi) = \sum_{i=0}^{\infty} E_{Q^i}\left[\int_t^\nu d\Delta_u + \Xi^\nu 1_{\{v<\infty\}}|\mathcal{F}_t\right] 1_{\{t<\nu\}} 1_{\{t\in[\sigma_t,\sigma_{t+1}]\}}$$

Observe that $\lim_{t\to\infty} V_t^*(\phi) = 0$ (same proof as that $S_t^* \to 0$ in the previous section). In particular,

$$S_t^* = \sum_{i=0}^{\infty} E_{Q^i}\left[\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right] 1_{\{t<\tau\}} 1_{\{t\in[\sigma_i,\sigma_{i+1}]\}}$$

Theorem 4.3.1. *There exists $Q^{t^*} \sim P$ such that*

$$V_t^*(\phi) = E_{Q^{t^*}}\left[\int_t^v d\Delta_u + \Xi^\nu 1_{\{\nu<\infty\}}|\mathcal{F}_t\right] 1_{\{t<\nu\}}$$

Proof. Let $Z_\infty^i = \frac{dQ^i}{dP} \in \mathcal{F}_\infty$ and $Z_t^i = E[Z_\infty|\mathcal{F}_t]$. Define

$$Z_\infty^{t\,*} = \sum_{i=0}^{\infty} Z_\infty^i 1_{\{t\in[\sigma_i,\sigma_{i+1}]\}} \in \mathcal{G}_\infty.$$

Then $Z_\infty^{t\,*} > 0$ and

$$\begin{aligned} E[Z_\infty^{t\,*}] &= E[\sum_{i=0}^{\infty} Z_\infty^i 1_{\{t\in[\sigma_i,\sigma_{i+1}]\}}] \\ &= \sum_{i=0}^{\infty} E[Z_\infty^i 1_{\{t\in[\sigma_i,\sigma_{i+1}]\}}] \\ &= \sum_{i=0}^{\infty} E[Z_\infty^i] E[1_{\{t\in[\sigma_i,\sigma_{i+1}]\}}] \\ &= \sum_{i=0}^{\infty} P(\sigma_i \le t < \sigma_{i+1}) = 1 \end{aligned}$$

Therefore, we can use $Z_\infty^{t^*}$ to define $Q^{t^*} \sim P$ by $\frac{dQ}{dP} = E[Z_\infty^{t^*}|\mathcal{F}_\infty] \in \mathcal{F}_\infty$. The result then follows from more algebra with conditional expectations. □

Definition 4.3.2. We call Q^{t^*} the *valuation measure* at t and $(Q^{t^*})_{t\ge0}$ the *valuation system*. If $N_t = 1$ for all t, then $Q^{t^*} = Q^0 \in \mathcal{M}_{loc}^{\mathbb{F}}(W)$. We refer to this as a *static* valuation system. Otherwise, we call is a *dynamic* valuation system.

In general, observe that

$$\begin{aligned} W_t^* =& S_t^* + \int_0^{t\wedge\tau} dD_u + X_\tau 1_{\{\tau\leq t\}} \\ =& \sum_{i=0}^{\infty} E_{Q^i}\left[\int_0^{\tau} dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right] 1_{\{t\in[\sigma_i,\sigma_{i+1}]\}} \\ =& \sum_{i=0}^{\infty} E_{Q^i}[W_\infty^*|\mathcal{F}_t] 1_{\{t\in[\sigma_i,\sigma_{i+1}]\}} \end{aligned}$$

where $W_\infty^* = \int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}$

Lemma 4.3.2. *Suppose $\tau < \infty$ a.s. If $Q^i, R^i \in \mathcal{M}_{UI}^{\mathbb{F}}(W)$, then the fundamental price S_t^* and the fundamental wealth W_t^* with respect to Q^i and R^i conincide.*

Lemma 4.3.3. *Suppose $\tau < \infty$, let $Q^i \leq \mathcal{M}_{UI}$, $R^i \in \mathcal{M}_{NUI}$, then*

$$W_t^{R^*} \leq W_t^{Q^*}$$

on $\{\sigma_i \leq t \leq \sigma_{i+1}\}$.

As before, define a bubble on the underlying asset as

$$B_t = S_t - S_t^* = W_t - W_t^*$$

4.3.1 Static valuation

Suppose that $Q^{t^*} = Q^* \in \mathcal{M}_{loc}(W)$. In this case,

$$\begin{aligned} W_t^* &= S_t^* + \int_0^{t\wedge\tau} dD_u + X_\tau 1_{\{\tau\leq t\}} \\ &= E_{Q^*}\left[\int_t^{\tau} dD_u + X_\tau 1_{\{T<\infty\}}|\mathcal{F}_t\right] 1_{\{t<\tau\}} + \int_0^{t\wedge\tau} dD_u + X_\tau 1_{\{\tau\leq t\}} \\ &= E_{Q^*}\left[\int_t^{\tau} dD_u + X_\tau 1_{\{T<\infty\}}|\mathcal{F}_t\right] 1_{\{t<\tau\}} + \int_0^{t} dD_u 1_{\{t<\tau\}} + \left(\int_0^{\tau} dD_u + X_\tau\right) 1_{\{\tau\leq t\}} \\ &= E_{Q^*}\left[\int_0^{\tau} dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right] 1_{\{t<\tau\}} + \left(\int_0^{\tau} dD_u + X_\tau\right) 1_{\{\tau\leq t\}} \\ &= E_{Q^*}\left[\int_0^{\tau} dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right] \end{aligned}$$

Therefore, $B_t = W_t - W_t^*$ is a Q^*-local martingale, being the sum of the Q^*-local martingale W_t and the uniformly integrable Q^* martingale W_t^*. The next results are generalization of the results for a complete market.

Theorem 4.3.4. *If $B_t \neq 0$, then B_t is a Q^*-local martingale. If in addition $P(\tau < \infty) = 1$, then B_t is not uniformly integrable. Moreover, if τ is bounded, then B_t is not a Q^*-martingale.*

Theorem 4.3.5. *$S_t = S_t^* + B_t = S_t^* + B_t^1 + B_t^2 + B_t^3$, where*

(1) $B^1 \geq 0$ is a uniformly integrable Q^* martingale with $B_t^1 \to X_\infty$ a.s.

(2) $B^2 \geq 0$ is a non-uniformly integrable Q^* martingale with $B_t^2 \to 0$ a.s.

(3) $B^3 \geq 0$ is a strict Q^* local martingale with $E_{Q^*}[B_t^3] \to 0$ and $B_t^3 \to 0$ a.s.

Corollary 4.3.6. *(1) $B \geq 0$,*

(2) $B_\tau 1_{\{\tau < \infty\}} = 0$,

(3) If $B_t = 0$, for some t, then $B_u = 0$ for all $u \geq t$,

(4) If $D_t = 0$, $S_t = E_{Q^*}[S_T|\mathcal{F}_t] + B_t^3 - E_{Q^*}[B_T^3|\mathcal{F}_t]$.

Observe that the key difference between a static valuation system and a complete market is that No Dominance is no longer sufficient to rule out the existence of Type 2 and Type 3 bubbles, since the fundamental value is not guaranteed to be replicable.

4.3.2 Dynamic Valuation

In this case, since the valuation measure Q^{t*} changes with time and, moreover, does not need to be an ELMM, there is no guarantee that the bubble B_t is a Q-local martingale for some $Q \in \mathcal{M}_{loc}(W)$. Nevertheless, we can still prove that it is positive.

Theorem 4.3.7. *$B_t \geq 0$.*

Proof. Fix $t \geq 0$. Then

$$\begin{aligned} S_t^* 1_{\{\sigma_t \leq t < \sigma_{i+1}\}} &= E_{Q^i}\left[\int_t^T dD_u + X_\tau 1_{\tau<\infty}|\mathcal{F}_t\right] 1_{\{t<\tau\}} 1_{\{\sigma_t \leq t < \sigma_{i+1}\}} \\ &= S_t^{*i} 1_{\{\sigma_t \leq t < \sigma_{i+1}\}}, \end{aligned}$$

where $S_t^{*i} = E_{Q^i}\left[\int_t^T dD_u + X_\tau 1_{\tau<\infty}|\mathcal{F}_t\right] 1_{\{t<\tau\}}$. Therefore,

$$S_t^* = \sum_{i=0}^{\infty} S_t^{*i} 1_{\{\sigma_t \leq t < \sigma_{i+1}\}}.$$

and

$$B_t = \sum_i B_t^i 1_{\{\sigma_t \leq t < \sigma_{i+1}\}},$$

where $B_t^i = S_t - S_t^{*i} \geq 0$ by previous Corollary. □

Example 4.3.1. (Birth of a bubble) Suppose that $Q^i \in \mathcal{M}_{UI}(W)$ and $R^{i+1} \in \mathcal{M}_{NUI}(W)$. By Lemma 4.3.3, it can happen that

$$W^{*Q^i}_{\sigma_{i+1}} - W^{*R^{i+1}}_{\sigma_{i+1}} \geq 0$$

with strict inequality with positive probability. Then

$$W_t^* = W_t^{Q^*} 1_{\{t < \sigma_{i+1}\}} + W_t^{R^*} 1_{\{t \geq \sigma_{i+1}\}}$$

and it can happen that $B_t = B_t^R 1\{\sigma_{i+1} \leq t\}$, that is, a bubble is born at the switching time σ_{i+1}.

4.3.3 Black-Scholes Economies

(1) Static valuation, finite horizon $T \in \mathbf{R}_+$. Let

$$S_t = e^{(\mu - \frac{\sigma^2}{2})t + \sigma M_t}$$

where M_t is a Brownian motion. Then there exist a unique ELMM measure Q defined by the Radn Nikodym derivative

$$Z_T = \frac{dQ}{dP}|_{\mathcal{F}_T} = e^{-\frac{\mu}{\sigma} M_T - \frac{\mu^2}{\sigma^2} tT}$$

and there are no bubbles, since S_t is a Q-martingale and

$$S_t^* = E_Q[S_T|\mathcal{F}_t] = S_t.$$

(2) Static valuation, infinite horizon. Suppose $D_t = 0$ and $X_\infty = 0$, so that $S_t^* = 0$ and $B_t = S_t - S_t^* = S_t$, in other words, the entire asset price is a bubble! This happens because Q is not an ELMM, since Q and P are not equivalent at $\mathcal{F}_\infty$ and $S_t \to 0$ Q a.s. Therefore, $S_t > E_Q[S_\infty] = 0$ and S_t is a strict Q-local martingale.

(3) Dynamic valuation, infinite horizon. Let (M_t^1, M_t^2) are two independent Q-BM. Fix $k > 1$ and define

$$\sigma = \inf\{e^{M_t^2 - \frac{t}{2}} = k\}$$

Define $Z_t = e^{M^2_{t\wedge\sigma} - \frac{t\wedge\tau}{2}}$, $S_t = e^{M^1_{t\wedge\sigma} - \frac{t\wedge\sigma}{2}}$. We interpret S_t as paying no dividends ($D_t = 0$) up to a default time σ when it pays $X_\tau = S_\sigma$.

It then follows that $E_Q[S_\infty] = Q(\sigma < \infty) = \frac{1}{k} \neq 1 = S_0$. So S is not a Q-martingale.On the other hand, $\frac{dR}{dQ}|_{\mathcal{F}_t} = Z_t$ then S is a R uniformly integrable martingale on $\{t < \sigma\}$.

4.3.4 Derivatives In Incomplete Markets

For a derivative H we define the fundamental value

$$\Lambda^*_t(H) = E_{Q^{t*}}[H(S)_T|\mathcal{F}_t]$$

and $b^H_t = \Lambda_t(H) - \Lambda^*_t(H)$. As before, in general all that we can establish is that $b^H_t \geq 0$.

Lemma 4.3.8.

$$C^*_t(K) - P^*_t(K) = V^{*f}_t(K) = E_{Q^{t*}}[(S_T - K)|\mathcal{F}_t]$$

Proof. Follow from $(S_T - K)^+ - (K - S_T)^+ = S_T - K$ and the definition of fundamental values. □

Lemma 4.3.9. *Under no dominance, $C_t(K) - P_t(K) = V^f_t(K)$.*

Proof. Follows from the same identity. □

Lemma 4.3.10. *Let H and H' be derivatives with $H(S)_T \leq H'(S)_T$. Then if $\Lambda_t(H') = \Lambda^*_t(H')$ and there is no dominance, then $\Lambda_t(H) = \Lambda^*_t(H)$.*

Corollary 4.3.11. *Under no dominance, bounded derivatives have no bubbles. In particular, put options, bonds and Arrow-Debreu derivatives.*

Theorem 4.3.12. *Under no dominance,*

$$C_t(K) - C^*_t(K) = S_t - E_{Q^{t*}}[S_T|\mathcal{F}_t].$$

Proof.

$$\begin{aligned} C_t(K) - C^*_t(K) &= V^f_t(K) + P_t(K) - (V^f_t(K) + P^f_t(K)) \\ &= S_t - K - E^{t^*}_Q[(S_T - K)|\mathcal{F}_t] \\ &= S_t - E_{Q^{t*}}[S_T|\mathcal{F}_t] \end{aligned}$$

□

Observe that, in the dynamic valuation case, even if $S_t = S_t^*$ we can still have $E_{Q^{t*}}[S_T^*|\mathcal{F}_t] = E_{Q^{t*}}[E_{Q^{T*}}[S_\infty^*|\mathcal{F}_T]|\mathcal{F}_t] \neq E_{Q^{t*}}[S_\infty^*|\mathcal{F}_t] = S_t^*$, since $Q^{t^*} \neq Q^{T^*}$ and the tower property for conditional expectations might not hold. Therefore it could still happen that,

$$S_t \neq E_{Q^{t*}}[S_T|\mathcal{F}_t]$$

Corollary 4.3.13. *In a static market, no dominance implies that*

$$b_t^c = B_t^3 - E_{Q^*}[B_T^3|\mathcal{F}_t]$$

Proof. Use the decomposition $S_t = S_t^* + B_t^1 + B_t^2 + B_t^3$. Therefore,

$$S_t - E_{Q^*}[S_T|\mathcal{F}_t] = B_t^3 - E_{Q^*}[B_T^3|\mathcal{F}_t],$$

since B_1 and B_2 are Q^*-martingales. □

4.3.5 American Options

Assume a static valuation system and denote the bank account numeraire by $A_t = e^{\int_0^t r_u du}$.

Definition 4.3.3. If H is an American option with maturity T, then its fundamental value is

$$\Lambda_t^{A^*}(H) = \sup_{t \leq \eta \leq T} E_{Q^*}[H(S)_\eta|\mathcal{F}_t]$$

Theorem 4.3.14. *Under no dominance and sufficient regularity on ΔS, we have $C_t(K) = C_t^A(K) = C_t^{A^*}(K)$.*

Proof. If there is a bubble in C^A, it must be a strict Q^*-local martingale. Now

$$\begin{aligned} C_t^{A^*}(K) &= \sup E_{Q^*}\left[(S_\eta - \frac{K}{N_\eta})^+\right] \\ &= E_{Q^*}[(S_T - \frac{K}{N_T})|\mathcal{F}_t] + (S_t - E_{Q^*}[S_T|\mathcal{F}_t]) \\ &= C_t^*(K) + B_t^3 - E_{Q^*}[B_T^3|\mathcal{F}_t] \qquad (4.42) \\ &= C_t(K) \end{aligned}$$

This shows $C_t^{A^*}(K) = C_t(K)$. Moreover, using (4.42) and the fact that $C_t^A \leq S_t$, which follows from No Dominance, we have that

$$C_t^*(K) + B_t^3 - E_{Q^*}[B_T^3|\mathcal{F}_t] + C_t^A - C_t^{A^*} = C_t^A \leq S_t = S_t^* + B_t^1 + B_t^2 + B_t^3.$$

Therefore

$$0 \leq b^{C^A} = C_t^A - C_t^{A^*} \leq (S_t^* + B_t^1 - C_t^*) + (B_t^2 + E[B_T^3|\mathcal{F}_t])$$

which is the sum of two uniformly integrable Q^*-martingales. Therefore, b_t^A is a positive strict Q^*-local martingale dominated by a uniformly integrable Q^*-martingale, which implies that $b_t^A = 0$.

□

Corollary 4.3.15. $C_t^{A^*} - C_t^* = B_t^3 - E_{Q^*}[B_T^3|\mathcal{F}_t]$.

4.3.6 Forward Prices

Let $F_{t,T}$ be the forward price such that the value at time t for a forward contract with payoff $S_T - F_{t,T}$ satisfies $V_{t,T}^f(K) = 0$. Define the price of a zero coupon bond with maturity T as $p(t,T)$.

Theorem 4.3.16. *Under no dominance,* $F_{t,T} \cdot p(t,T) = S_t A_t$.

Corollary 4.3.17. *Under no dominance, we have that*

1. $F_{t,T} \geq 0$.
2. $\frac{F_{t,T} p_{t,T}}{N_t}$ *is a* Q*-local martingale for any* $Q \in \mathcal{M}_{loc}(W)$
3. $F_{t,T} p_{t,T} = (S_t^* + B_t) A_t$.

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