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Asset Price Bubbles in Complete Markets

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Summary. This paper reviews and extends the mathematical finance literature on bubbles in complete markets. We provide a new characterization theorem for bubbles under the standard no arbitrage (NFLVR) framework, showing that bubbles can be of three types. Type 1 bubbles are uniformly integrable martingales, and these can exist with an infinite lifetime. Type 2 bubbles are non-uniformly integrable martingales, and these can exist for a finite, but unbounded, lifetime. Last, type 3 bubbles are strict local martingales, and these can exist for a finite lifetime only . When one adds a no dominance assumption (from Merton [24]), only type 1 bubbles remain. In addition, under Merton's no dominance hypothesis, put-call parity holds and there are no bubbles in standard call and put options. Our analysis implies that if one believes asset price bubbles exist and are an important economic phenomena, then asset markets must be incomplete.

Key words: Bubbles; NFLVR; Complete markets; Local martingale; Put-call parity; Derivative pricing

1.1 Introduction

Although asset price bubbles, their existence and characterization, have enthralled the imagination of economists for many years, only recently has this topic been studied using the tools of mathematical finance, see in particular Loewenstein and Willard [22], Cox and Hobson [7], Jarrow and Madan [20], Gilles [15], Gilles and Leroy [16], and Huang and Werner [17]. The purpose of this paper is to review and to extend this mathematical finance literature in order to increase our understanding of asset price bubbles. In this paper, we restrict our attention to arbitrage free economies that satisfy both the no-free-lunch-with-vanishing-risk (NFLVR) and complete markets hypotheses, in order that both the first and second fundamental theorems of asset pricing apply. Equivalently, there exists a unique equivalent local martingale measure. We exclude the study of incomplete markets. (We study incomplete market asset price bubbles in a companion paper, see Jarrow, Protter, Shimbo [21].) We also exclude the study of charges, since charges require a stronger notion of no arbitrage (see Jarrow and Madan [20], Gilles [15], Gilles and Leroy [16]).

We make two contributions to the bubbles literature. First, we provide a new characterization theorem for asset price bubbles. Second, we study the effect of additionally imposing Merton's [24] no dominance assumption on the existence of bubbles in an economy. Our new results in this regard are:

(i) Bubbles can be of three types: an asset price process that is (1) a uniformly integrable martingale, (2) a martingale that is not a uniformly integrable martingale, or (3) a strict local martingale that is not a martingale. Bubbles of type 1 can be viewed as the asset price process containing a component analogous to fiat money. Type 2 bubbles are generated by the fact that all trading strategies must terminate in finite time, and type 3 bubbles are caused by the standard admissibility condition used to exclude doubling strategies.

(ii) Bubbles cannot be started - "born" - in a complete market. (In contrast, they *can* be born in incomplete markets.) They either exist at the start or not, and if they do exist, they may disappear as the economy evolves.

(iii) Bubbles in standard European call and put options can only be of type 3, because standard options have finite maturities. Under NFLVR, any assets and contingent claims can have bubbles and put–call parity does not hold in general.

(iv) Under NFLVR and no dominance, in complete markets, there can be no type 2 or type 3 asset price bubbles. Consequently, standard options have no bubbles and put-call parity holds.

The economic conclusions from this paper are three-fold. First, bubbles of type 1 are uninteresting from an economic perspective because they represent a permanent but stochastic wedge between an asset's fundamental value and its market price, generated by a perceived residual value at time infinity.

Second, type 2 bubbles are the result of trading strategies being of finite time duration, although possibly unbounded. To try to profit from a bubble of type 2 or type 3, one would short the asset in anticipation of the bubble bursting. Because a type 2 bubble can exist, with positive probability, beyond any trading strategy, these bubbles can persist as they do not violate the NFLVR assumption. Type 3 bubbles occur in assets with finite maturities. For these asset price bubbles, unprotected shorting is not feasible because due to the admissibility condition, if the short's value gets low enough, the trading strategy must be terminated with positive probability, before the bubble bursts. This admissibility condition removes downward selling pressure on the asset's price, and hence enables these bubbles to exist.

Third, modulo type 1 bubbles, under both the NFLVR and no dominance hypotheses, there can be no asset pricing bubbles in complete markets. This implies that, if one believes asset pricing bubbles exist and are an important economic phenomena, and if one accepts Merton's "no dominance" assumption, then asset markets must be *incomplete*.

An outline of this paper is as follows. Section 2 presents our model structure and defines an asset price bubble. Section 3 characterizes the properties of asset price bubbles. Section 4 provides the economic intuition underlying the mathematics, while section 5 extends the analysis to contingent claims bubbles. Finally, section 6 concludes.

1.2 Model Description

This section presents the details of our economic model.

1.2.1 No Free Lunch with Vanishing Risk (NFLVR)

Traded in our economy is a risky asset and a money market account. For simplicity, and without loss of generality, we assume that the spot interest rate is 0 in our economy, so that the money market account has constant unit value. Let τ be a maturity (life) of the risky asset. Let $\{D_t\}_{0\leq t<\tau}$ be a càdlàg semimartingale representing the cumulative dividend process of the risky asset with X_τ its terminal payoff or liquidation value at time τ. We assume that $X_\tau,\ D_t \geq 0$. The market price of the risky asset is given by a non-negative càdlàg semimartingale $S=(S_t)_{t\geq 0}$ defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with filtration $\mathbb{F}$. We assume that the filtration $\mathbb{F}$ satisfies the usual hypotheses. Note that for t such that $\triangle D_t > 0$, S_t denotes a price *ex-dividend*, since S_t is càdlàg . Let W_t be a wealth process from owning the asset, given by

$$W_t = S_t + \int_0^{t\wedge\tau} dD_u + X_\tau 1_{\{\tau\leq t\}}. \tag{1.1}$$

A key notion in our economy will be an equivalent local martingale measure.

Definition 1 (Equivalent Local Martingale Measure). *Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ such that the wealth process W_t is a $\mathbb{Q}$-local martingale. We call $\mathbb{Q}$ an Equivalent Local Martingale Measure (ELMM). We denote the set of ELMMs by $\mathcal{M}^e_{loc}(W)$.*

A trading strategy is defined to be a pair of predictable processes $\{\pi_t,\eta_t\}_{t\geq 0}$ representing the number of units of the risky asset and money market account held at time t with $\{\pi_t\}_{t\geq 0} \in L(W)$.(See Protter [25] for the definition of the space of integrable processes $L(W)$.) The wealth process of the trading strategy $\{\pi_t, \eta_t\}_{t\geq 0}$ is given by $V_t^\pi = \pi_t S_t + \eta_t$. Assume temporarily that π is a semimartingale. Then a **self-financing trading strategy** with $V_0^\pi = 0$ is a trading strategy $\{\pi_t, \eta_t\}_{t\geq 0}$ such that the associated wealth process $V_t = V_t^{\pi,\eta}$ is given by

$$\begin{aligned} V_t^{\pi,\eta} &= \int_0^t \pi_u dW_u \\ &= \int_0^t \pi_u dS_u + \int_0^{t\wedge\tau} \pi_u dD_u + \pi_\tau X_\tau 1_{\{\tau\leq t\}} \\ &= \left(\pi_t S_t - \int_0^t S_{u-} d\pi_u - [\pi^c, S^c]_t\right) + \int_0^{t\wedge\tau} \pi_u dD_u + \pi_\tau X_\tau 1_{\{\tau\leq t\}} \\ &= \pi_t S_t + \eta_t \end{aligned} \tag{1.2}$$

where we have used integration by parts, and where

$$\eta_t = \int_0^{t\wedge\tau} \pi_u dD_u + \pi_\tau X_\tau 1_{\{\tau\leq t\}} - \int_0^t S_{u-} d\pi_u - [\pi^c, S^c]_t. \tag{1.3}$$

If we now discard the temporary assumption that π is a semimartingale, we simply define a **self-financing trading strategy** (π, η) to be a pair of processes, with π predictable and η optional and such that:

$$V_t^{\pi,\eta} = \pi_t S_t + \eta_t = \int_0^t \pi_u dW_u.$$

As noted, a self-financing trading strategy starts with zero units of the money market account, $\eta_0 = 0$, and it reflects proceeds from purchases/sales of a risky asset which accumulate holdings in the money market account as the cash flows from the risky asset are deposited. In particular, equation (1.3) shows that η is uniquely determined by π if a trading strategy is self-financing. Therefore without loss of generality, we represent (π, η) by π.

To avoid doubling strategies, we further restrict the class of self-financing trading strategies.

Definition 2 (Admissibility). *Let $V_t^{\pi,\eta}$ be the wealth process given by (1.2). We say that the trading strategy π is $a-$admissible if it is self-financing and $V_t^{\pi,\eta} \geq -a$ a.s. We say a trading strategy is admissible if it is self-financing and $V_t^{\pi,\eta} \geq -a$ for some $a \in \mathbb{R}_+$.*

The notion of admissibility corresponds to a lower bound on the wealth process, an implicit inability to borrow if one's debt becomes too large. (For example, see Loewenstein and Willard [22, Equation (5) on page 23]). There are several alternative definitions of admissibility that could be employed and these are discussed in Section 1.4.3. However, all of our results are robust to these alternative formulations.

We want to explore the existence of bubbles in arbitrage free markets, hence, we need to define the no free lunch with vanishing risk (NFLVR) hypothesis. Let

$$\mathcal{K} = \{W^{\pi}_{\infty} = (\pi \cdot W)_{\infty} : \pi \text{ is admissible}\}$$
$$\mathcal{C} = (\mathcal{K} - L^0_+) \cap L_{\infty}$$

where $\overline{\mathcal{C}}$ denotes the closure of $\mathcal{C}$ in the sup-norm topology of L^{∞}.

We say that a semi-martingale S satisfies the No Free Lunch with Vanishing Risk (NFLVR) condition with respect to admissible integrands, if

$$\bar{\mathcal{C}} \cap L^{\infty}_{+} = \{0\}. \tag{1.4}$$

Given NFLVR, we impose the following assumption.

Assumption 1 *The market satisfies NFLVR hypothesis.*

By the first fundamental theorem of finance [9], this implies that the market admits an equivalent σ-martingale measure. By Proposition 3.3 and Corollary 3.5 [1, pp. 307, 309], a σ-martingale bounded from below is a local martingale. (For the definition and properties of σ-martingales, see [25], [14], [9], [19, Section III.6e].) Thus we have the following theorem:

Theorem 1 (First Fundamental Theorem). *A market satisfies the NFLVR condition if and only if there exists an ELMM.*

Theorem 1 holds even if the price process is not locally bounded due to the assumption that W_t is non-negative. (In [9], the driving semimartingale (price process) takes values in $\mathbb{R}^d$ and is not locally bounded from below.)

We are interested in studying the existence and characterization of bubbles in complete markets. A market is complete if for all $X_{\infty} \in L^2(\Omega, \mathcal{F}_{\infty}, P)$, there exists a self-financing trading strategy $\{\pi_t, \eta_t\}_{t \geq 0}$ and $c \in \mathbb{R}$ such that

$$X_{\infty} = c + \int_0^{\infty} \pi_u dW_u. \tag{1.5}$$

For the subsequent analysis, we also assume that the market is complete, hence by the second fundamental theorem of asset pricing (see Harrison and Pliska [18]), the ELMM is unique.

Assumption 2 *Given the market satisfies NFLVR, the ELMM is unique.*

This assumption will be key to a number of the subsequent results. For the remainder of the paper we assume that both Assumptions 1 and 2 hold, i.e. that the markets are arbitrage free and complete.

1.2.2 No Dominance

In addition to assumption 1, we will also study the imposition of Merton's [24] *no dominance assumption*. To state this assumption in our setting, assume that there are two assets or contingent claims characterized by the pair of cash flows $(\{D_t^1\}_{t\geq 0}, X_\tau^1)$, $(\{D_t^2\}_{t\geq 0}, X_\tau^2)$. Let V_t^1, V_t^2 denote their market prices at time t.

Assumption 3 (No Dominance) *For any stopping time σ, if*

$$D_{\sigma+u}^2 - D_\sigma^2 \geq D_{\sigma+u}^1 - D_\sigma^1 \text{ and } X_\tau^2 1_{\{\tau>\sigma\}} \geq X_\tau^1 1_{\{\tau>\sigma\}} \text{ for } u > 0 \tag{1.6}$$

then $V_\sigma^2 \geq V_\sigma^1$. Furthermore, if for some stopping time σ:

$$E\{1_{(\{D_\infty^2 - D_\sigma^2 > D_\infty^1 - D_\sigma^1\} \cup \{X_\tau^2 1_{\{\tau>\sigma\}} > X_\tau^1 1_{\{\tau>\sigma\}}\})} | \mathcal{F}_\sigma\} > 0 \tag{1.7}$$

with positive probability, then $V_\sigma^2 > V_\sigma^1$.

Note that (1.6) implies that $X_\tau^2 1_{\{\tau>\sigma\}} \geq X_\tau^1 1_{\{\tau>\sigma\}}$ for any stopping time σ.

This assumption rephrases Assumption 1 of Merton [24] in modern mathematical terms, we believe for the first time. In essence, it codifies the intuitively obvious idea that, all things being equal, financial agents prefer more to less. Assumption 3 is violated only if there is an agent who is willing to buy a dominated security at the higher price.

Assumption 3 is related to Assumption 1, but they are not equivalent.

Lemma 1 *Assumption 3 implies Assumption 1. However, the converse is not true.*

Proof. Assume that W allows for a free lunch with vanishing risk. There is $f \in L_+^\infty(\mathbb{P})\backslash\{0\}$ and sequence $\{f_n\}_{n=0}^\infty = \{(H^n \cdot W)_\infty\}_{n=0}^\infty$ where H^n is a sequence of admissible integrands and $\{g_n\}$ satisfying $g_n \leq f_n$ such that

$$\lim_n \|f - g_n\|_\infty = 0 \tag{1.8}$$

In particular the negative part $\{(f_n)^-\}$ tends to zero uniformly (See [10, page 131]). Applying Assumption 3 to two terminal payoff f and 0, we have $0 = V_0(f) > 0$, a contradiction. Therefore Assumption 3 implies Assumption 1. For the converse, see Example 1. □

The domain of Assumption 3 contains a domain of Assumption 3, $\bar{C} \cap L_+^\infty(\mathbb{P})$. This explains why Assumption 3 implies Assumption 1.

The following is an example consistent with Assumption 1 but excluded by Assumption 3.

Example 1. Consider two assets maturing at τ with payoffs X_τ and Y_τ, respectively. Suppose that $X_\tau \geq Y_\tau$ almost surely. Then,

$$X_t^* = E_{\mathbb{Q}}[X_\tau|\mathcal{F}_t]1_{\{t<\tau\}} \geq E_{\mathbb{Q}}[Y_\tau|\mathcal{F}_t]1_{\{t<\tau\}} = Y_t^*. \tag{1.9}$$

Let $\{\beta_\cdot\}$ be a non-negative local martingale such that $\beta_\tau = 0$ and $\beta_t > X_t^* - Y_t^*$ for some $t \in (0,\tau)$. (The existence of such a process follows, for example, from Example 3.) Suppose further that the prices of asset $X_t = X_t^*$ and $Y_t = \beta_t + Y_t^*$. Then, Assumption 3 is violated because $Y_t > X_t$.

To see that this is not an NFLVR, consider a strategy that would attempt to take advantage of this mis-pricing. One would want to sell Y and to buy X, say at time t. Then, if held until maturity, this would generate a cash flow equal to $\beta_t - (X_t^* - Y_t^*) > 0$ at time t and $X_\tau - Y_\tau \geq 0$ at time τ. However, for any u with $t < u \leq \tau$, the market value of this trading strategy is $-Y_u + X_u = -\beta_u + (X_u^* - Y_u^*)$. Since $-\beta_t$ is negative and unbounded, this strategy is inadmissible and not a NFLVR. We will discuss issues related to admissibility further in Section 1.4.3.

One situation Assumption 3 is meant to exclude is often called a suicide strategy (see Harrison and Pliska [18] for the notion of a suicide strategy). An alternative approach for dealing with suicide strategies is to restrict the analysis to the set of maximal assets. An outcome $(\pi \cdot S)_\infty$ of an admissible strategy π is called *maximal* if for any admissible strategy π' such that $(\pi' \cdot S)_\infty \geq (\pi \cdot S)_\infty$, then $\pi' = \pi$.

1.2.3 Bubbles

This section provides the definition of an asset pricing bubble in our economy. To do this, we must first define the asset's fundamental price.

The Fundamental Price

We define the fundamental price as *the expected value of the asset's future payoffs* with respect to the ELMM $\mathbb{Q} \in \mathcal{M}_{loc}^e(W)$.(Recall that we assume that the market is complete, Assumption 2.)

Definition 3 (Fundamental Price). *The fundamental price S_t^* of an asset with market price S_t is defined by*

$$S_t^* = E_{\mathbb{Q}}\left[\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right] 1_{\{t<\tau\}}. \tag{1.10}$$

Note that the fundamental price is just the conditional expected value of the asset's cash flows, under the valuation measure $\mathbb{Q}$. (Note that since the random variable is positive, the conditional expectation is always defined; however in Lemma 2 which follows we show that it is actually in L^1 and thus is classically defined.) Also, note that if the asset has a payoff at $\tau = \infty$, then this payoff $X_\tau 1_{\{\tau=\infty\}}$ does not contribute to the fundamental price S_t^*. We do this because an agent cannot consume the payoff $X_\tau 1_{\{\tau=\infty\}}$ by employing an admissible trading strategy. Indeed, although unbounded in time, for a given $\omega \in \Omega$ all such admissible trading strategies must terminate in finite time.

Lemma 2 *The fundamental price in (1.10) is well defined. Furthermore, S_t converges to $S_\infty \in L^1(\mathbb{Q})$ almost surely and S^*_t converges to 0 almost surely.*

Proof. Fix $\mathbb{Q} \in \mathcal{M}^e_{loc}(W)$. To show that S^*_t is well defined, it suffices to show that $\int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}} \in L_1(\mathbb{Q})$ since for all t,

$$0 \le \int_t^\tau dD_u + X_\tau 1_{\{t<\tau\}} \le \int_0^\tau dD_u + X_\tau 1_{\{t<\tau\}} \tag{1.11}$$

By hypothesis, W_t is a non-negative supermartingale. By the martingale convergence theorem (see [11, VI.6 in page 72]), there exists $W_\infty \in L^1(\mathbb{Q})$ such that $W_t \to W_\infty$ almost surely. To show the convergence of S_t, observe that

$$\begin{aligned} W_\infty &= \lim_{t\to\infty} W_t = \lim_{t\to\infty} \left(S_t + \int_0^{t\wedge\tau} dD_u + X_\tau 1_{\{\tau\le t\}} \right) \\ &= \lim_{t\to\infty} S_t + \int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}} \qquad \text{a.s..} \end{aligned} \tag{1.12}$$

It follows that there exists $S_\infty \in L^1(\mathbb{Q})$ and $\int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}} \in L^1(\mathbb{Q})$ since $S_t \ge 0$. Therefore S^*_t is well defined for all $t \ge 0$. Observe that

$$\begin{aligned} &E_{\mathbb{Q}}\left[\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}} | \mathcal{F}_t\right] \\ &= -\int_0^t dD_u + E_{\mathbb{Q}}\left[\left(\int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}\right) | \mathcal{F}_t\right] \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} &E_{\mathbb{Q}}\left[\left(\int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}\right) | \mathcal{F}_t\right] 1_{\{t<\tau\}} \\ &= \left(E_{\mathbb{Q}}\left[\left(\int_0^\tau dD_u + X_\tau\right) 1_{\{\tau<\infty\}} | \mathcal{F}_t\right] + E_{\mathbb{Q}}\left[1_{\{\tau=\infty\}} \int_0^\infty dD_u | \mathcal{F}_t\right]\right) 1_{\{t<\tau\}} \end{aligned} \tag{1.14}$$

Substituting (1.14) into (1.13) and then into (1.10),

$$\begin{aligned} \lim_{t\to\infty} S^*_t &= -\int_0^\infty dD_u 1_{\{\tau=\infty\}} + 1_{\{\tau=\infty\}} E_{\mathbb{Q}}\left[\int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}} | \mathcal{F}_\infty\right] \\ &= -\int_0^\infty dD_u 1_{\{\tau=\infty\}} + 1_{\{\tau=\infty\}} E_{\mathbb{Q}}\{\int_0^\infty dD_u | \mathcal{F}_\infty\} \\ &= 0 \end{aligned} \tag{1.15}$$

Note that, in general, $\int_0^\infty dD_u + X_\tau 1_{\{\tau<\infty\}}$ need not be $\mathbb{P}$-integrable. In this regard, Lemma 2 shows that the existence of $\mathbb{Q}$ implies that $\int_0^\infty dD_u + X_\tau 1_{\{\tau<\infty\}}$ is $\mathbb{Q}$-integrable.

Lemma 3 *The fundamental wealth process* $W_t^* = S_t^* + \int_0^{\tau \wedge t} dD_u + X_\tau 1_{\{\tau \leq t\}}$ *is a uniformly integrable martingale under* $\mathbb{Q} \in \mathcal{M}_{loc}^e(W)$ *closed by*

$$W_\infty^* = \int_0^\tau dD_u + X_\tau 1_{\{\tau < \infty\}}. \tag{1.16}$$

Proof. By Lemma 2,

$$\begin{aligned} W_\infty^* &:= \lim_{t \to \infty} W_t^* \\ &= \lim_{t \to \infty} \left(S_t^* + \int_0^{t \wedge \tau} dD_u + X_\tau 1_{\{\tau \leq t\}} \right) \\ &= \int_0^\tau dD_u + X_\tau 1_{\{\tau < \infty\}} \qquad \text{a.s.} \end{aligned} \tag{1.17}$$

W_∞^* is in $L^1(\mathbb{Q})$ since $S_\infty \geq 0$, $W_\infty \in L^1$ and $W_\infty^* + S_\infty = W_\infty$. Observe that

$$\begin{aligned} E\left[W_\infty^* | \mathcal{F}_t\right] = E &\left[\left(\int_t^\tau dD_u + X_\tau \right) | \mathcal{F}_t \right] 1_{\{t < \tau\}} \\ &+ \left(- \int_\tau^t dD_u + X_\tau \right) 1_{\{\tau \leq t\}} + \left(\int_0^t dD_u \right) 1_{\{t < \tau\}} \\ &= S_t^* + \int_0^{t \wedge \tau} dD_u + X_\tau 1_{\{\tau \leq t\}} \\ &= W_t^*. \end{aligned} \tag{1.18}$$

It follows that W_t^* is a closable and hence uniformly integrable martingale.

The Asset Price Bubble

Definition 4 (Bubble). *The asset price bubble* β_t *for* S_t *is given by*

$$\beta_t = S_t - S_t^*. \tag{1.19}$$

As indicated, the asset price bubble is the asset's market price less the asset's fundamental price.

1.3 Properties of Bubbles

In this section, we analyze the properties of asset price bubbles applying semimartingale theory and potential theory. We begin with a non standard definition:

Definition 5 (Strict Local Martingale). *A* **strict local martingale** *is a local martingale which is* not *a martingale.*

The term "strict local martingale" is not common in the literature, but it can be found in the recent book of Delbaen and Schachermayer [10], who in turn refer to a paper of Elworthy et al [13]. We hasten to remark that *their definition of a strict local martingale is different from our definition.* Indeed, Delbaen and Schachermayer refer to a strict local martingale as being a local martingale which is not a uniformly integrable martingale. They allow a strict local martingale to be actually a martingale, as long as the martingale itself is not uniformly integrable. Our definition is more appropriate for the study of bubbles, as will be made clear shortly.

1.3.1 Characterization of Bubbles

Theorem 2. *If there exists a non-trivial bubble $\beta_t \not\equiv 0$ in an asset's price, then we have three and only three possibilities:*

1. *β_t is a local martingale (which could be a uniformly integrable martingale) if $\mathbb{P}(\tau = \infty) > 0$.*
2. *β_t is a local martingale but not a uniformly integrable martingale if is unbounded, but with $\mathbb{P}(\tau < \infty) = 1$.*
3. *β_t is a strict $\mathbb{Q}$-local martingale, if τ is a bounded stopping time.*

Proof. Fix $\mathbb{Q} \in \mathcal{M}^e_{loc}(W)$. Since W_t is a closable supermartingale, (See proof of Lemma 2), there exists $W_\infty \in L^1(\mathbb{Q})$ such that $W_t \to W_\infty$ almost surely. Let

$$\beta'_t = W_t - E_\mathbb{Q}[W_\infty|\mathcal{F}_t] \tag{1.20}$$

Then β'_t is a (non-negative) local martingale since it is a difference of a local martingale and a uniformly integrable martingale. By Lemma 3,

$$E_\mathbb{Q}[W_\infty|\mathcal{F}_t] = E_\mathbb{Q}[W^*_\infty|\mathcal{F}_t] + E[S_\infty|\mathcal{F}_t] = W^*_t + E[S_\infty|\mathcal{F}_t] \tag{1.21}$$

By the definition of wealth processes and (1.20), (1.21):

$$\begin{aligned}\beta_t &= S_t - S^*_t \\ &= W_t - W^*_t \\ &= \left(E_\mathbb{Q}[W_\infty|\mathcal{F}_t] + \beta^1_t\right) - \left(E_\mathbb{Q}[W_\infty|\mathcal{F}_t] - E_\mathbb{Q}[S_\infty|\mathcal{F}_t]\right) \\ &= \beta'_t + E_\mathbb{Q}[S_\infty|\mathcal{F}_t].\end{aligned} \tag{1.22}$$

If $\tau < T$ for $T \in \mathbb{R}_+$, then $S_\infty = 0$. A bubble $\beta_t = \beta'_t = 0$ for $t \geq \tau$ and in particular $\beta_T = 0$. If β_t is a martingale,

$$\beta_t = E[\beta_T|\mathcal{F}_t] = 0 \qquad \forall t \leq T \tag{1.23}$$

It follows that $\beta.$ is a strict local martingale. This proves (1). For (2) assume that β_t is uniformly integrable martingale. Then by Doob's optional sampling theorem, for any stopping time $\tau_0 \leq \tau$,

$$\beta_{\tau_0} = E_{\mathbb{Q}}[\beta_\tau | \mathcal{F}_{\tau_0}] = 0 \tag{1.24}$$

and since β is optional, it follows from (for example) the section theorems of P.A. Meyer that $\beta = 0$ on $[0, \tau]$. Therefore the bubble does not exist. For (3), $E_{\mathbb{Q}}[S_\infty | \mathcal{F}_t]$ is a uniformly integrable martingale and claim holds .

As indicated, there are three types of bubbles that can be present in an asset's price. Type 1 bubbles occur when the asset has infinite life with a payoff at $\{\tau = \infty\}$. Type 2 bubbles occur when the asset's life is finite, but unbounded. Type 3 bubbles are for assets whose lives are bounded. In a subsequent section, we will provide an intuitive economic explanation for why these bubbles exist. Before that, however, we provide some examples.

1.3.2 Examples

This section presents simple examples of bubbles of types 1,2 and 3.

A Uniformly Integrable Martingale Bubble: Fiat Money (Type 1)

Example 2. Let $S_t = 1$ for all t be fiat money. Fiat money is money that the government declares to be legal tender although it cannot be converted into standard specie. Since money never matures, $\tau = \infty$ and $X_\infty = 1$. Money pays no dividend and hence $D_t \equiv 0$. Therefore $S_t^* \equiv 0$ and

$$\beta_t = S_t - S_t^* = 1. \tag{1.25}$$

The entire value of money comes from the bubble, its payoff $X_\infty = 1$, and it is a trivial uniformly integrable martingale. Note that in our setting, fiat money is equivalent to our money market account (paying zero interest for all times).

A Martingale Bubble (Type 2)

Example 3. Let the asset's maturity τ be a positive random time with $P(\tau > t) > 0$ for all t. Let the fundamental price process be $S_t^* = 1_{\{t<\tau\}}$ with payoff 1 at time τ. Set

$$\beta_t = \frac{1 - 1_{\{\tau \le t\}}}{P(\tau > t)}. \tag{1.26}$$

Lemma 4 shows that β_t is a martingale which is not a uniformly integrable martingale, with $\beta_\infty = 0$. Then

$$S_t = S_t^* + \beta_t \tag{1.27}$$

is a price process with a non-uniformly integrable bubble.

Lemma 4 *Let τ_0 be a positive finite random variable such that $P(\tau_0 > t) > 0$ for all t. Let $D_t = 1_{\{\tau_0 \le t\}}$, $\mathcal{D}_t$ be a filtration generated by D_t. Then*

$$N_t = \frac{1 - D_t}{P(\tau_0 > t)} \tag{1.28}$$

is a martingale which is not a uniformly integrable martingale, and $N_\infty = 0$.

Proof. By the structure of $\mathcal{D}_t$ (for example, see Protter [25, Lemma on page 121]) for $s < t$,

$$P(\tau_0 > t|\mathcal{D}_s) = 1_{\{\tau_0 > t\}} P(\tau_0 > t|\tau_0 > s) = 1_{\{\tau_0 > s\}} \frac{P(\tau_0 > t)}{P(\tau_0 > s)} \tag{1.29}$$

Therefore

$$E\left[1 - D_t|\mathcal{D}_s\right] = (1 - D_s)\frac{P(\tau_0 > t)}{P(\tau_0 > s)} \tag{1.30}$$

This shows that N_t is a martingale. Observe that $N_t = 0$ on $\{t > \tau_0\}$ and hence $N_t \to 0$ a.s. because $\tau_0 < \infty$. If $N = (N_t)_{t \ge 0}$ is a uniformly integrable martingale, then N is closable by N_∞ and $N_t = E[N_\infty|\mathcal{D}_t] \equiv 0$, which is not true. Therefore N is not uniformly integrable. □

This example has the asset's maturity τ having a positive probability of continuing past any given future time t. Although finite with probability one, the asset's life is unbounded.

A Strict Local Martingale Bubble (Type 3)

The following example is essentially that contained in Cox and Hobson [7], Example 3.5 in page 9 and 2.2.1 on page 4.

Example 4. Let the fundamental price process $S_t^* = 1_{[0,T)}$ with a payoff of 1 at time T. Define

$$\beta_t = \int_0^t \frac{\beta_u}{\sqrt{T-u}} dB_u \tag{1.31}$$

where B_t is a standard Brownian motion. Lemma 5 shows that β_t is a strict local martingale with $\beta_T = 0$. Then

$$S_t = S_t^* + \beta_t \tag{1.32}$$

is a price process with a strict local martingale bubble.

Lemma 5 *A process β_t defined by equation (1.31) is a continuous local martingale on $[0, T]$.*

Proof. The stochastic integral $\int_0^{\cdot} 1/\sqrt{T-s}dB_s$ is a local martingale but not a martingale on $[0,T)$ (because it is a stochastic integral of a predictable integrand w.r.t Brownian motion), such that

$$\left[\int_0^{\cdot} 1/\sqrt{T-s}dB_s, \int_0^{\cdot} 1/\sqrt{T-s}dB_s\right]_u = -ln\left[1 - \frac{u-t}{T}\right] := A_u \qquad (1.33)$$

and continuous on $[0,T)$. By Dubins-Schwartz theorem, there exists a Brownian motion $\tilde{B}$ such that

$$d\beta_u = \beta_u d\tilde{B}_{A_u} \qquad (1.34)$$

and

$$\beta_u = \beta_0 \mathcal{E}\left(\tilde{B}_{A_u}\right) = \beta_0 \exp\left(\tilde{B}_{A_u} - \frac{1}{2}A_u\right) \qquad (1.35)$$

for all $u < T$. By the Law of the Iterated Logarithm, we can show that $\lim_{t\to\infty} \mathcal{E}(B_t) = 0$. Since A_u is monotonic and $\lim_{u\to\infty} A_u = \infty$,

$$\lim_{u\to T} \beta_u = 0 \qquad \text{a.s.} \qquad (1.36)$$

Since we set$\beta_T = 0$, $\beta_{T-} = S_T$ and $\{\beta_t\}$ is continuous on $[0,T]$. $E[\beta_T] = 0 < E[\beta_0]$ implies that $\{\beta_t\}$ is not a martingale. □

In this example, although the asset has finite maturity T, a bubble still exists.

1.3.3 A Bubble Decomposition

In this section, we refine Theorem 2 to obtain a unique decomposition of an asset price bubble that yields some additional insights. The key tool is the decomposition of a positive supermartingale.

Theorem 3 (Riesz decomposition I). *Let X be a right continuous supermartingale such that $EX_t^- = \lim_{t\to\infty} EX_t^- < \infty$. Then there exists the limit $X_\infty = \lim_{t\to\infty} X_t$ a.s. exists and $E|X_\infty| < \infty$. X has the decomposition $X = U + V$, where $\{U_t\}$ is a right continuous version of the uniformly integrable martingale $E[X_\infty|\mathcal{F}_t]$ and $V_t = X_t - U_t$ is a right continuous supermartingale which is zero a.s. at infinity. V_t is positive if the .r.v X_t^- are uniformly integrable.*

Proof. See Dellacherie and Meyer [11, V.34, 35 and VI.8 in page 73] □

Definition 6 (Potential). *A positive right continuous supermartingale such that $\lim_{t\to\infty} EZ_t = 0$ is called a potential.*

Theorem 4 (Riesz decomposition II). *Every right continuous positive supermartingale X can be decomposed as a sum of $X = Y + Z$ where Y is a right continuous martingale and Z is a potential. This decomposition is unique except on an evanescent set and Y is the greatest right continuous martingale bounded above by X.*

Proof. See Dellacherie and Meyer [11, VI.9 in page 73] □

Theorem 5. *S_t admits a unique (up to an evanescent set) decomposition*

$$S_t = S_t^* + \beta_t = S_t^* + (\beta_t^1 + \beta_t^2 + \beta_t^3), \tag{1.37}$$

where $\beta = (\beta_t)_{t\geq 0}$ is a càdlàg local martingale and

- *β_t^1 is a càdlàg non-negative uniformly integrable martingale with $\beta_t^1 \to X_\infty$ almost surely,*
- *β_t^2 is a càdlàg non-negative non-uniformly integrable martingale with $\beta_t^2 \to 0$ almost surely,*
- *β_t^3 is a càdlàg non-negative supermartingale (and strict local martingale) such that $E\beta_t^3 \to 0$ and $\beta_t^3 \to 0$ almost surely. That is, β_t^3 is a potential.*

Furthermore, $(S_t^* + \beta_t^1 + \beta_t^2)$ is the greatest submartingale bounded above by W_t.

Proof. Let $\beta_t^1 = E_{\mathbb{Q}}[S_\infty|\mathcal{F}_t]$. Define

$$K_t = W_t - (W_t^* + \beta_t^1) = W_t - E_{\mathbb{Q}}[W_t|\mathcal{F}_t]. \tag{1.38}$$

By Theorem 3, $K = (K_t)_{t\geq 0}$ is a non-negative supermartingale and $K_t \to 0$ almost surely. Let M_t be a uniformly integrable martingale such that

$$0 \leq M_t \leq W_t - K_t \qquad \forall t \geq 0 \tag{1.39}$$

Since $W_t - K_t \to 0$ almost surely, $M_t \to 0$ almost surely. Then $M_t \equiv 0$. Therefore K is unique up to evanescent set. By Theorem 4, K has a unique decomposition:

$$K_t = \beta_t^2 + \beta_t^3, \tag{1.40}$$

where β^2 is a martingale, β^3 is a non negative supermartingale such that $E\beta_t^3 \to 0$, which implies $\beta_t^3 \to 0$ almost surely. Since $K_t \to 0$ almost surely, $\beta_t^2 = K_t - \beta_t^3 \to 0$ almost surely. Since β_t^2 is defined as

$$\beta_t^2 = \lim_{u\to\infty} E_{\mathbb{Q}}[K_{t+u}|\mathcal{F}_t] \tag{1.41}$$

and $K_s \geq 0$ for all $s \in [0,\infty)$, $\beta_t^2 \geq 0$. This complete the proof. □

As in the previous Theorem 2, β_t^1, β_t^2, β_t^3 give the type 1, 2 and 3 bubbles, respectively. First, for type 1 bubbles with infinite maturity, we see that the type 1 bubble component converges to the asset's value at time ∞, X_∞. This time ∞ value X_∞ can be thought of as analogous to fiat money, embedded as part of the asset's price process. Indeed, it is a residual value that pays zero dividends for all finite times. Second, this decomposition also shows that for finite maturity assets, $\tau < \infty$, the critical threshold is that of uniform integrability. This is due to the fact that when $\tau < \infty$, the type 2 and 3 bubble components of $\beta = (\beta_t)_{t\geq 0}$ have to converges to 0 almost surely, while they need not converge in L^1.

As a direct consequence of this theorem, we obtain the following corollary.

Corollary 1 *Any asset price bubble β_t has the following properties:*

1. $\beta_t \geq 0$,
2. $\beta_\tau 1_{\{\tau<\infty\}} = 0$, *and*
3. *if* $\beta_t = 0$ *then* $\beta_u = 0$ *for all* $u \geq t$.

Proof. (1), (2) hold by Theorem 5. A non-negative supermartingale stays at 0 once it hits 0, which implies (3). □

This is a key result. Condition (1) states that bubbles are always non-negative, i.e. the market price can never be less than the fundamental value. Condition (2) states that if the bubble's maturity is finite $\tau < \infty$, then the bubble must burst on or before τ. Finally, Condition (3) states that if the bubble ever bursts before the asset's maturity, then it can never start again. Alternatively stated, Condition (3) states that in the context of our model, bubbles must either exist at the start of the model, or they never will exist. And, if they exist and burst, then they can not start again. The fact that this model does not include bubble birth is a weakness of the theory, due in part to the fact that the markets are complete and there is a unique martingale measure.

1.3.4 No Dominance

In this section, we add Assumption 3 (the assumption of No Dominance) to the previous structure to see what additional insights can be obtained. We only consider assets whose maturities are finite, i.e. $\tau < \infty$ a.s. *This means that we only consider bubbles of type 2 and 3.*

Let W_t be the wealth process generated by the asset with price S_t. Now, by our complete markets Assumption 2, we know that there exists a local martingale representation

$$W_t^* = W_0^* + \int_0^t \pi_u^1 dW_u, \qquad \beta_t = \beta_0 + \int_0^t \pi_u^2 dW_u \tag{1.42}$$

where W^* is the fundamental wealth process and β_t is the asset price bubble. Let $\{\eta_t^i\}_{t\geq 0}$ be holdings in money market account given by equation (1.3) so that the trading strategies (π^i, η^i) are self-financing. Since $\beta_\infty = 0$, W_t and W_t^* have the same cash flows. Since $W_\infty^* \geq 0$, $\{\pi_t^i\}$ represents an admissible trading strategy. This observation implies that there are two alternative ways of obtaining the asset's cash flows. The first is to buy and hold the asset, obtaining the wealth process W. The second is to hold the admissible trading strategy $\{\pi_t^1\}$, obtaining the wealth process W^* instead. The cost of obtaining the first position is $W_0 \geq W_0^*$, with strict inequality if a bubble exists. This implies that if there is a bubble, then the second method for buying the asset dominates the first, yielding the following proposition.

Proposition 1 *Under Assumption 3 , type 2 and type 3 bubbles do not exist.*

Proof. For any admissible payoff function, there is an admissible trading strategy to replicate S_t^*. Under Assumption 3, $V_0(S_0^*) \geq V_0(S_0)$ and $V_0(S_0) \geq V_0(S_0^*)$. Hence $V_0(S_0) = V_0(S_0^*)$, since the cash flow of a synthetic asset S^* and an asset S_t are the same. It follows that $\beta \equiv 0$ and type 2 or type 3 bubbles do not exist. □

This proposition implies that given both the NFLVR assumption 1 and the no dominance assumption 3, the only possible asset price bubbles are those of type 1. Essentially, under these two weak no arbitrage assumptions, only infinite horizon assets can have bubbles in complete markets.

1.4 The Economic Intuition

This section provides the economic intuition underlying the existence of asset price bubbles of types 1, 2, and 3.

1.4.1 Type 1 Bubbles

Type 1 bubbles are for assets with infinite lives, with positive probability. As argued after Theorem 5, type 1 bubbles are due to a component of an asset's price process X_∞ that is obtained at time ∞. This component of the asset's price is analogous to fiat money, a residual value received at time ∞. As such, bubbles of type 1 are uninteresting from an economic perspective because they represent a permanent (but stochastic) wedge between an asset's fundamental value and its market price, generated by an exogenously given value at time ∞.

1.4.2 Type 2 Bubbles

Type 2 bubbles are for assets with finite, but unbounded lives. In a type 2 bubble, the market price of the asset exceeds its fundamental value. To take advantage of this discrepancy, one would form a trading strategy that is long the fundamental value, and short the asset's price. This is possible because the market is complete. If held until the asset's maturity, when $\beta_\tau^2 1_{\{\tau<\infty\}} = 0$, this would (if possible) create an arbitrage opportunity (FLVR). Unfortunately, for any sample path of the asset price process, the trading strategy must terminate at some finite time. And, there is a positive probability that the bubble exceeds this termination time, ruining the trading strategy, and making it "risky" and not an arbitrage. This situation enables asset price bubbles of type 2 to exist in our economy.

1.4.3 Type 3 Bubbles

Type 3 bubbles are for assets with finite and bounded lives. In a type 3 bubble, the market price of the asset exceeds it fundamental value. Just as for a type 2 bubble, to take advantage of this discrepancy, one would form a trading strategy that is long the fundamental value, and short the asset's price. This is possible because the market is complete. If held until the asset's maturity, when $\beta^3_\tau 1_{\{\tau<\infty\}} = 0$, this would (if possible) create an arbitrage opportunity (FLVR). Unfortunately, to be a FLVR trading strategy, the trading strategy must be admissible. Shorting the asset is an inadmissible trading strategy, because if the price of the asset becomes large enough, the value of the trading strategy will fall below any given lower bound. Hence, there are NFLVR with type 3 bubbles.

Alternatively stated, the first fundamental theorem of asset pricing is formulated for admissible trading strategies. And, admissible trading strategies are used to exclude doubling strategies, which would be possible otherwise. Restricting the class of trading strategies to be admissible (to exclude doubling strategies) implies that it also excludes shorting the asset for a fixed time horizon (short and hold) as an admissible trading strategy. This removes downward selling pressure on the asset price process, allowing bubbles to exist in an arbitrage free setting.

The question naturally arises, therefore, whether the class of admissible trading strategies can be relaxed further, to exclude both doubling strategies, but still allow shorting the stock over a fixed investment horizon. Unfortunately, the answer is no. To justify this statement, we briefly explore the concept of admissibility. The standard definition of admissibility, the one we adopted, yield the following set of possible trading strategy values:

$$\mathcal{W} = \cup_a \{W_u : W_u \geq -a, \forall u \in [0,T]\} \tag{1.43}$$

As usual, W is the wealth process generated by a risky asset with price process S. The weakest notion of admissibility consistent with NFLVR (see Strasser [29]) yields the following set of trading strategy values:

$$\mathcal{W}^* = \left\{ X = H \cdot W : H \in L(W) \wedge \lim_{n\to\infty} E_{\mathbb{Q}}[(H \cdot W)^-_{\sigma_n} 1_{\{\sigma_n<\infty\}}] = 0 \right\} \tag{1.44}$$

where the notation Z^- for a random variable Z means $Z^- = -(Z \wedge 0)$, and

$$\sigma_n = \inf\{t \in [0,T] : X_t \leq -n\}. \tag{1.45}$$

Clearly $\mathcal{W} \subset \mathcal{W}^*$. Replacing our definition by this weaker notion of admissibility does not affect our analysis for type 3 bubbles. Short selling an asset with type 3 bubble is not admissible even in the sense of (1.44) as follows from Lemma 6:

Lemma 6 *Assume that S_t has a type 3 bubble β_t. Then A trading strategy $H_t = -1_{(0,T]}$ is not an admissible strategy and $W_0 - W_T \notin \mathcal{W}^*$.*

Proof. It suffices to show that if $W_0 - W_T \in \mathcal{W}^*$ then type 3 bubble does not exist. Let $H = -1_{(0,T]} \in L(W)$. Then

$$(H \cdot W)^-_{\sigma_n} = (W_0 - W_{\sigma_n \wedge T})^- = (W_{\sigma_n \wedge T} - W_0)^+ \geq W_{\sigma_n \wedge T} - W_0 \quad (1.46)$$

By definition σ_n takes a value in $[0,T] \cup \{\infty\}$ and $(\sigma_n \wedge T)1_{\{\sigma_n < \infty\}} = \sigma_n$. By (1.46) and hypothesis,

$$\begin{aligned} \lim_{n\to\infty} E_{\mathbb{Q}}[W_{\sigma_n} 1_{\{\sigma_n<\infty\}}] &= \lim_{n\to\infty} E_{\mathbb{Q}}[(W_{\sigma_n} - W_0)1_{\{\sigma_n<\infty\}}] \\ &\leq \lim_{n\to\infty} E_{\mathbb{Q}}[(H\cdot W)^-_{\sigma_n} 1_{\{\sigma_n<\infty\}}] \\ &= 0. \end{aligned} \quad (1.47)$$

Since $W_t \geq 0$, $\lim_{n\to\infty} E_{\mathbb{Q}}[W_{\sigma_n} 1_{\{\sigma_n<\infty\}}] = 0$. Since W_t is a supermartingale and $W_0 \geq 0$

$$E[(H\cdot W)^-_T] = E[(W_T - W_0)^+] \leq EW_T \leq EW_0 < \infty \quad (1.48)$$

By [29, Theorem 1.4], $(H \cdot W)_t = W_0 - W_t$ is a supermartingale. Then by [1, Theorem 3.3], there exists a martingale M such that $(W_0 - W_t)^- \leq M_t$ for $0 \leq t \leq T$. Then $W_t \leq M_t + W_0$ for $0 \leq t \leq T$. Since W_t is a local martingale and $M_t + W_0$ is a martingale, W_t is a martingale. Since $0 \leq \beta_t \leq W_t$, β_t is also a martingale and type 3 bubble does not exist. □

This motivation for the existence of stock price bubbles is consistent with the rich literature on the question *"If stocks are overpriced, why aren't prices corrected by short sales?"*. To answer this question, two types of short-sales constraints were used. The first constraint is a structural limitation in the economy caused by a limited ability and/or costs to borrow an asset for a short-sale (see, for example, Oftek and Richardson [23], Duffie Gârleanu and Pedersen [12], Chen, Hong and Stein [6], D'Avolio [8]). The second constraint is indirect and is caused by the risk associated with short sales (see, for example, DeLong et al. [3] and Shleifer and Vishny [28]). Using Internet stock data from the alleged bubble period (1999 to 2000), Battalio and Schultz [2] argue that put-call parity holds and the constraint on short-sales was not the reason for the alleged Internet stock bubble.

1.5 Bubbles and Contingent Claims Pricing

This section studies the pricing of contingent claims in markets where the underlying asset price process has a bubble. Bubbles can have two impacts on a contingent claims value. The first is that a bubble in the underlying asset price process influences the contingent claims' price. The second is that the contingent claim itself can have a bubble. This section explores these possibilities in our market setting. For the remainder of this section we assume that the risky asset S does not pay dividends, so that $W_t = S_t$. We restrict our attention to European contingent claims in this paper because under the NFLVR and no dominance assumptions, American contingent claims provide no additional insights. However, this is not true for the incomplete market setting, see Jarrow, Protter, Shimbo [21]. Following our analysis for the underlying asset price process, the first topic to discuss is the fundamental price for a contingent claim.

1.5.1 The Fundamental Price of a Contingent Claim

Definition 7. *The fundamental price $V(H)_t$ of a European contingent claim with payoff function H at maturity T is given by*

$$V_t^*(H) = E_{\mathbb{Q}}[H(S)_T|\mathcal{F}_t], \tag{1.49}$$

where $H(S)_T$ denotes a functional of the path of S on the time interval $[0,T]$. That is, $H(S)_T = H(S_r; 0 \leq r \leq T)$.

Note that in this definition, the market price of the asset $S = (S_t)_{0 \leq t \leq T}$, and not its fundamental value, is used in the payoff function. This makes sense since the contingent claim is written on the market price of the asset, and not it fundamental value. As seen in Theorem 6 below, this definition is equivalent to the *fair price* as defined by Cox and Hobson [7]. We believe Definition 7 is more natural since it is valid in an incomplete market setting as well.

Theorem 6 (Cox and Hobson Theorem 3.3). *If the market is complete, the fundamental price is equivalent to the smallest initial cost to finance a replicating portfolio of a contingent claim.*

Proof. Let $\{\theta_u\}_{u\in[0,T]}$ be an admissible trading strategy and v_t be a wealth process associated with θ:

$$v_t = v_0 + \int_0^t \theta_u dS_u \tag{1.50}$$

Let $\mathcal{V}$ be a set of wealth process of admissible trading strategies θ_t:

$$\mathcal{V} = \{v_t = v_0 : v_T \geq H(S)_T \text{ , admissible, self-financing}\}. \tag{1.51}$$

Fix $v_t \in \mathcal{V}$. By the definition of risk neutral measure, v_t is a local martingale. Since $H(S)_T \geq 0$, v_t is non negative and hence v_t is a supermartingale. Then there exists a decomposition on $[0,T]$:

$$v_t = M_t + C_t, \tag{1.52}$$

where M_t is a uniformly integrable martingale and C_t is a potential (a non-negative supermartingale converging to 0). This decomposition is unique (up to an evanescent set). In addition, M_t is the greatest martingale dominated by v_t. (See Dellacherie and Meyer V.34, 35 and VI.8, 9 in Page 73 for the discussion of this decomposition.) At option maturity date T,

$$v_T = H(S)_T = M_T + C_T \tag{1.53}$$

and $v_t = M_t = C_t = 0$ on $t > T$, $C_T = 0$ and hence $M_T = H(S)_T$. Recall that $\{M_t\}$ is a uniformly integrable martingale, whence:

$$M_t = E[H(S)_T|\mathcal{F}_t] \qquad \text{a.s.} \tag{1.54}$$

Since we assume a complete market, there exists a predictable process θ_t such that

$$M_t = M_0 + \int_0^T \theta_u dS_u \tag{1.55}$$

Since H is positive, $M_t \geq 0$ and hence this strategy is admissible. Therefore for any potential C_t, $v_t = M_t + C_t$ is a super-replicating portfolio. (M_t is a replicating portfolio. Adding C makes it super-replicating except for the case $C \equiv 0$). The fair price is the infimum of such v_t's:

$$\begin{aligned} V_t^*(H) &= \inf_{v \in \mathcal{V}} v_t = M_t + \inf_{C_t:\text{potential with } C_T=0} C_t = M_t + 0 \\ &= E[H(S)_T|\mathcal{F}_t], \end{aligned} \tag{1.56}$$

which complete the proof. □

Since contingent claims discussed here have a fixed maturity T, by Theorem 2, contingent claims can not have type 1 or type 2 bubbles. The only possible bubbles in the contingent claims' price are of type 3. We explore these bubbles below. However this does not imply that the existence of type 1 or 2 bubbles in the underlying asset's price does not affect the price of the contingent claim. Indeed, it appears within the payoff function H as a component of the asset price S_T.

1.5.2 A Contingent Claims' Price Bubble

Analogous to the underlying asset, a contingent claims' price bubble is defined by

$$\delta_t = V_t(H) - V_t^*(H)$$

where $V_t(H)$ is the market price of the contingent claim at time t.

1.5.3 Bubbles under NFLVR

This section studies contingent claims' price bubbles under NFLVR. Assume that S_t is a non-dividend paying asset with $\tau > T$ almost surely for some $T \in \mathbb{R}_+$. Let $C_t(K)^*$, $P_t(K)^*$, $F_t(K)^*$ be the fundamental prices of a call option, put option and forward contract on S.

Lemma 7 (Put-Call parity for fundamental prices) *Fundamental prices satisfy put-call parity:*

$$C^*(K) - P^*(K) = F^*(K). \tag{1.57}$$

Proof. At maturity of an option with terminal time T,

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K \qquad \forall K \geq 0 \tag{1.58}$$

Since a fundamental price of contingent claims with payoff function H is $E_{\mathbb{Q}}[H(S)_T|\mathcal{F}_t]$,

$$\begin{aligned} C_t^*(K) - P_t^*(K) &= E_{\mathbb{Q}}[(S_T - K)^+|\mathcal{F}_t] - E_{\mathbb{Q}}[(K - S_T)^+|\mathcal{F}_t] \\ &= E_{\mathbb{Q}}[S_T - K|\mathcal{F}_t] \\ &= F_t^*(K) \end{aligned} \tag{1.59}$$

□

However, the market prices of the call, put, and forward need not satisfy put-call parity.

Example 5. Let $B_t^i, i = \{1, 2, 3, 4, 5\}$ be independent Brownian motions. Define M_t^i by

$$M_t^1 = \exp\left(B_t^1 - \frac{t}{2}\right), \qquad M_t^i = 1 + \int_0^t \frac{M_s^i}{\sqrt{T-s}} dB_s^i \qquad 2 \leq i \leq 5. \tag{1.60}$$

Consider a market with a finite time horizon $[0, T]$. The market is complete with respect to the filtration generated by $\{(M_t^i)_{t \geq 0}\}_{i=1}^5$. M_t^1 is a uniformly integrable martingale on $[0, T]$. By Lemma 5, $\{M_t^i\}_{i=2}^5$ are non-negative strict local martingales that converge to 0 almost surely as $t \to T$. Let $S_t^* = \sup_{s \leq t} M_s^1$. Suppose the market prices in this model are given by

- $S_t = S_t^* + M_t^2$
- $C_t(K) = C_t^*(K) + M_t^3$
- $P_t(K) = P_t^*(K) + M_t^4$
- $F_t(K) = F_t^*(K) + M_t^5$

All of the traded securities in this example have bubbles. To take advantage of any of these bubbles $\{M_t^i\}_{i=2}^4$ based on the time T convergence, an agent must short sell at least one asset. However, as shown in Lemma 6 shorting an asset with a type 3 bubble is not admissible. Therefore such strategies are not a free lunch with vanishing risk.

In summary, this example shows that Assumption 1 is not strong enough to exclude bubbles in contingent claims. And, given the existence of bubbles in calls and puts, we get various possibilities for put-call parity in market prices.

- $C_t(K) - P_t(K) = F_t(K)$ if and only if $\delta_t^F = \delta_t^c - \delta_t^p$.
- $C_t(K) - P_t(K) = S_t - K$ if and only if $\delta_t^S = \delta_t^c - \delta_t^p$.

This example validates the following important observation. In the well studied Black Scholes economy (a complete market under the standard NFLVR structure), contrary to common belief, the Black-Scholes formula need not hold! Indeed, if there is a bubble in the market price of the option (M_t^3), then the market price ($C_t(K)$) can differ from the option's fundamental price ($C_t^*(K)$) - the Black-Scholes formula. This insight has numerous ramifications, for example, it implies that the implied volatility (from the Black-Scholes formula) does not have to equal the historical volatility. In fact, if there is a bubble, then the implied volatility should exceed the historical volatility, and there exist no arbitrage opportunities! (Note that this is with the market still being complete.) This possibility, at present, is not commonly understood. However, all is not lost. *One additional assumption* returns the Black-Scholes economy to normalcy, but an additional assumption is required! This is the assumption of *no dominance*, which we discuss in the next section.

1.5.4 Bubbles under No Dominance

This section analyzes the behavior of the market prices of call and put options under Assumption 3. We start with a useful lemma.

Lemma 8 *Let H' be a payoff function of a contingent claim such that $V_t(H') = V_t^*(H')$. Then for every contingent claim with payoff H such that $H(S)_T \leq H'(S_T)$, $V_t(H) = V_t^*(H)$.*

Proof. Since contingent claims have bounded maturity, we only need to consider type 3 bubbles. Let $\mathcal{L}$ be a collection of stopping times on $[0, T]$. Then for all $L \in \mathcal{L}$, $V_L(H) \leq V_L(H')$ by Assumption 3. Since $\{V_t(H')\}_{t \in [0,T]}$ is a martingale it is uniformly integrable martingale and in class (D) on $[0, T]$. Then $\{V_t(H)\}$ is also in class (D) and it is a uniformly integrable martingale on $[0, T]$. (See Jacod and Shiryaev [19, Definition 1.46, Proposition 1.47 in page 11]). Therefore type 3 bubbles do not exist for this contingent claim. □

This lemma states that if we have a contingent claim with no bubbles, and this contingent claim dominates another contingent claims' payoff, then the dominated contingent claim will not have a bubble as well. Immediately, we get the following corollary.

Corollary 2 *If $H(S)_T$ is bounded, then $V_t(H) = V_t(H^*)$. In particular a put option does not have a bubble.*

Proof. Assume that $H(S)_T < \alpha$ for some $\alpha \in \mathbb{R}_+$. Then applying Lemma 8 for $H(x) = \alpha$, we have desired result. □

Theorem 7. $C_t(K) - C_t^*(K) = S_t - E[S_T|\mathcal{F}_t]$ *for all* $K \geq 0$*. This implies calls and forwards (with* $K = 0$*) can only have type 3 bubbles and that they must be equal to the asset price type 3 bubble.*

Proof. Let $C_t(K)$, P_t and $F_t(K)$ denote market prices of call, put option with strike K and a forward contract with delivery price K. Then

$$F_t^*(K) = E[S_T|\mathcal{F}_t] - K \leq S_t - K \tag{1.61}$$

By Assumption 3, the price of two admissible portfolios with the same cash flow are the same. Thus

$$F_t = S_t - K = F_t^*(K) + (S_t - E[S_T|\mathcal{F}_t]) \tag{1.62}$$

This implies a forward contract has a Type 3 bubble of size $\beta_t^3 = S_t - E[S_T|\mathcal{F}_t]$. To investigate put call parity, take the conditional expectation on the identity: $(S_T - K)^+ - (K - S_T)^+ = S_T - K$.

$$C_t^*(K) - P_t^*(K) = F_t^*(K) \leq S_t - K. \tag{1.63}$$

By Assumption 3 and (1.62),

$$C_t(K) - P_t(K) = F_t(K) = S_t - K \tag{1.64}$$

By subtracting (1.63) from (1.62)

$$[C_t(K) - C_t^*(K)] - [P_t(K) - P_t(K)^*] = \beta_t^3 \tag{1.65}$$

By Corollary 2, $P_t(K) - P_t(K)^* = 0$. The claim follows since $C_t(K) - C_t^*(K) = \beta_t^3$. □

This theorem states that a call option's bubble, if it exists, must equal the stock price's type 3 bubble. But, we know from proposition 1 that under the no dominance assumption, asset prices have no type 3 bubbles. Thus, call options have no bubbles under the no dominance assumption as well.

Since both European calls and puts have no bubbles under the no dominance assumption, put call parity (as in Merton [24]) holds as well.

1.6 Conclusion

This paper reviews and extends the mathematical finance literature on bubbles in complete markets. We provide a new characterization theorem for bubbles under the standard no arbitrage (NFLVR) framework, showing that bubbles can be of three types. Type 1 bubbles are uniformly integrable martingales, and these can exist for assets with infinite lifetimes. Type 2 bubbles are non-uniformly integrable martingales, and these can exist for assets with finite, but unbounded lives. Last, type 3 bubbles are strict local martingales, and these can exist for assets with finite lives. In addition, we show that bubbles can only be non-negative, and must exist at the start of the model. Bubble birth cannot occur in the standard NFLVR, complete markets structure.

When one adds a no dominance assumption (from Merton (1973)), we show that only type 1 bubbles are possible. In addition, under Merton's no dominance hypothesis, put-call parity holds and there are no bubbles in standard call and put options. Our analysis implies that if one believes asset price bubbles exist and are an important economic phenomena, then asset markets must be incomplete. Incomplete market bubbles are studied in a companion paper, which is in preparation.

Acknowledgments

It is a privilege to be included in this volume honoring Dilip Madan on his sixtieth birthday. Dilip is one of those pioneers who carried the torch through the dark years when only a few people in the United States and – for that matter – the world, considered mathematical finance a subject worthy of study. Now that he is 60, and a leader in the field, he can look back with satisfaction on a job well done. More than just an academic researcher, Dilip is also a entertaining communicator. He presentations are always lucid, often sarcastic, and humorous. He is also one of the few prominent researchers with intimate ties to industry. He is loved by his students and his colleagues. We are proud to be considered his friend.

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