MATH 1ZC3/1B03: Test 1 - Version 1

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Date: February 26, 2013 - Group A
Duration: 75 min.

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Instructions:

This test paper contains 20 multiple choice questions printed on both sides of the page. The questions are on pages 2 through 11. Pages 12 to 14 are available for rough work. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF THE INVIGILATOR.

Select the one correct answer to each question and ENTER THAT ANSWER INTO THE SCAN CARD PROVIDED USING AN HB PENCIL. Room for rough work has been provided in this question booklet. You are required to submit this booklet along with your answer sheet. HOWEVER, NO MARKS WILL BE GIVEN FOR THE WORK IN THIS BOOKLET. Only the answers on the scan card count for credit. Each question is worth 1 mark. The test is graded out of 20. There is no penalty for incorrect answers.

Computer Card Instructions:

IT IS YOUR RESPONSIBILITY TO ENSURE THAT THE ANSWER SHEET IS PROPERLY COMPLETED. YOUR TEST RESULTS DEPEND UPON PROPER ATTENTION TO THESE INSTRUCTIONS.

The scanner that will read the answer sheets senses areas by their non-reflection of light. A heavy mark must be made, completely filling the circular bubble, with an HB pencil. Marks made with a pen or felt-tip marker will **NOT** be sensed. Erasures must be thorough or the scanner may still sense a mark. Do **NOT** use correction fluid.

- Print your name, student number, course name, and the date in the space provided at the top of Side 1 (red side) of the form. Then the sheet **MUST** be signed in the space marked SIGNATURE.
- Mark your student number in the space provided on the sheet on Side 1 **and fill the corresponding bubbles underneath**.
- Mark only **ONE** choice (A, B, C, D, E) for each question.
- Begin answering questions using the first set of bubbles, marked "1".

1. Which of the following matrices are in reduced row echelon form?

i) $\begin{bmatrix} 0 & 1 & 9 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 3 & 6 & 5 \\ 2 & 6 & 12 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 3 & 6 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

iv) $\begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ v) $\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

• Things in green should be 0's... need 0's above & below leading 1's in r.r.e.f..

• Things in purple should be a "1".

A) ii), iii) and v) only
B) none of them
C) iii) and v) only
D) i) and ii) only
E) iv) only

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2. Let

$$A = \begin{bmatrix} -1 & -5 & -2 & 1 \\ 0 & 1 & 4 & -2 \\ 2 & 3 & 1 & -13 \end{bmatrix} \quad r_3 \leftarrow r_3 + 2r_1$$

$$\begin{bmatrix} -1 & -5 & -2 & 1 \\ 0 & 1 & 4 & -2 \\ 0 & -7 & -3 & -11 \end{bmatrix} \quad r_1 \leftarrow r_1 \times -1$$

What is the reduced row echelon form of A?

A) $\begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ B) $\begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ C) $\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

D) $\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ E) $\begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 5 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 0 & -7 & -3 & -11 \end{bmatrix} \begin{matrix} r_1 \leftarrow r_1 - 5r_2 \\ r_3 \leftarrow r_3 + 7r_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & -18 & 9 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 25 & -25 \end{bmatrix} r_3 \leftarrow r_3 \cdot \tfrac{1}{25} \rightarrow \begin{bmatrix} 1 & 0 & -18 & 9 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} r_1 \leftarrow r_1 + 18r_3 \\ r_2 \leftarrow r_2 - 4r_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

3. Suppose that the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 4x_2 + 3x_4 = 1 \Rightarrow x_1 = 1 - 4s - 3t$
$x_3 + 5x_4 = 6 \Rightarrow x_3 = 6 - 5t$
$x_5 = 0$
$x_2 = s$, $x_4 = t$ } no leading 1's in these spots, so free variables

is the augmented matrix of a linear system of four equations in the five unknowns x_1, x_2, x_3, x_4, x_5. Solve the linear system.

A) $x_1 = 3t - 4s + 1$, $x_2 = s$, $x_3 = 6 - 5t$, $x_4 = t$, $x_5 = u$

B) $x_1 = 3t - 4s + 1$, $x_2 = s$, $x_3 = 6 - 5t$, $x_4 = t$, $x_5 = 0$

C) $x_1 = -3t - 4s + 1$, $x_2 = s$, $x_3 = 6 - 5t + u$, $x_4 = t$, $x_5 = u$

D) $x_1 = -3t - 4s + 1$, $x_2 = s$, $x_3 = 6 - 5t$, $x_4 = t$, $x_5 = 0$

E) The system is inconsistent

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4. Consider the following linear system

$$\begin{aligned} x + y + z &= 1 \\ ax + ay - 3z &= -3 \\ y + 3az &= 0 \end{aligned}$$

Find all the possible values that a can take in order for the system to have exactly one solution.

A) $a > -3$ B) a can be any value C) $a \neq -3$ D) $a \neq 0, -3$ E) $a > 0$

Augmented matrix for this system: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ a & a & -3 & -3 \\ 0 & 1 & 3a & 0 \end{array}\right]$. Or, we can think about it as $\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ a & a & -3 \\ 0 & 1 & 3a \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$. We know this system has exactly one solution $\Leftrightarrow \det A \neq 0$.

$\det A = 1 * [3a^2 + 3] - a[3a - 1] = 3a^2 + 3 - 3a^2 + a = a + 3 \neq 0 \Leftrightarrow a \neq -3.$

5. Let A, B and C be three matrices such that $(A^T CB)^T$ can be formed and it has size 5×3. If C has size 2×4, what are the sizes of A and B?

$(A^TCB)^T = B^TC^TA$.

A) 2×3 and 4×5 B) 3×2 and 4×5 C) 2×3 and 5×4
D) 5×2 and 4×3 E) 2×5 and 4×3

C has size $2\times 4 \Rightarrow C^T$ has size 4×2. Multiplication well-defined $\Rightarrow$ A has 2 rows & B^T has 4 columns $\Rightarrow$ B has 4 rows. B^TC^TA $5\times 3 \Rightarrow B^T$ has 5 rows $\Rightarrow$ B has 5 columns & $\Rightarrow$ A has 3 columns. So, A is 2×3 & B is 4×5.

6. Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & b \\ 1 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det A = -1 \begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} = -1[1-0] = -1.$

$A^{-1} = \frac{1}{\det A}\operatorname{adj}(A)$.

Which of the following is the sum of the entries of A^{-1}?

A) $3 + a + b$ B) $3 + a + b - ab$ C) $3 + ab - a - b$
D) $3 + a^2 + b^2$ E) $3 + a^2 + b^2 - ab$

$$\operatorname{adj}(A) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \begin{pmatrix} a & -1 & 0 \\ -1 & 0 & 0 \\ -ab & b & -1 \end{pmatrix}^T = \begin{pmatrix} a & -1 & -ab \\ -1 & 0 & b \\ 0 & 0 & -1 \end{pmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{\det A}\operatorname{adj}(A) = -\operatorname{adj}(A) = \begin{bmatrix} -a & 1 & ab \\ 1 & 0 & -b \\ 0 & 0 & 1 \end{bmatrix}.$$

Sum of the entries: $-a + 1 + ab + 1 - b + 1 = ab - a - b + 3$

7. Let J be the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad J^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Calculate J^2. Which of the following are true?
i) $J^2 + I = 0$ ii) J is singular iii) $J + J^{-1} = 0$

A) i) and ii) only B) ii) and iii) only C) i) and iii) only D) i) only E) iii) only

(i) $J^2 + I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. ✓ (ii) $\det(J) = 0 - (-1) = 1 \neq 0 \Rightarrow J$ not singular. ✗

(iii) $J^{-1} = \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. $J + J^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. ✓

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8. Gauss-Jordan elimination applied to a matrix A yields the row reduced matrix

$$R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad EA = R.$$

Applying the same sequence of elementary row operations in the same order to the identity matrix yields a matrix E.
Which of the following must be true:

i) E is singular ii) A is singular iii) $EA\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 0$ iv) $A\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 0$

A) i), ii) and iii) only B) ii), iii) and iv) only C) ii) and iii) only
D) ii) and iv) only E) iii) and iv) only

(i) E is the product of elementary matrices. Elementary matrices are always invertible ⇒ the determinant of each is nonzero ⇒ the determinant of their product is nonzero ⇒ $\det E \neq 0$ ⇒ E is not singular. ✗

(ii) A invertible ⇔ the r.r.e.f. of A is I. In other words, A is singular ⇔ the r.r.e.f. of A is not I. We get R by performing row op.'s to A, so we can never get the identity ⇒ A is singular. ✓

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(iii) $\underset{R}{EA}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = R\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. ✓

(iv) We know E is invertible, b/c it's the product of elementary matrices. So, $EA\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow A\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = E^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow A\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. ✓

9. Find the values of x for which

$$\det\left(\begin{bmatrix} x & 0 \\ 0 & x+1 \end{bmatrix}\right) = \quad x(x+1)=0 \Leftrightarrow x=0 \text{ or } x=-1.$$

Sum of values: $0+-1=-1$.

is singular. What is the sum of these values?

A) -2 B) -1 C) 0 D) 1 E) 2

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10. Suppose that the following equations have more than one simultaneous solution for $(x_1, x_2, x_3, y_1, y_2, y_3)$:

$Ax=y$

$By=c.$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = y_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = y_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = y_3$$
$$b_{11}y_1 + b_{12}y_2 + b_{13}y_3 = c_1$$
$$b_{21}y_1 + b_{22}y_2 + b_{23}y_3 = c_2$$
$$b_{31}y_1 + b_{32}y_2 + b_{33}y_3 = c_3$$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \quad B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

or view as block matrix: $\begin{bmatrix} A & -I \\ 0 & B \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ c \end{bmatrix}$. simultaneous solution $\Rightarrow$ singular $\Rightarrow$ $\det(M)=0 \Rightarrow \det(AB)=0$.

Let $A=[a_{ij}]$ and $B=[b_{ij}]$.

Let $x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $c=\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$.

Which of the following must be true:

i) A is singular ii) B is singular

iii) AB is singular iv) BA is singular

A) none B) i) and ii) only C) iii) only D) iii) and iv) only E) iv) only

(i) A invertible (nonsingular) $\Leftrightarrow$ $Ax=b$ has exactly one solution for each b. But this just tells us that there exist different x & y combonations which makes this true... $\Rightarrow$ not enough info. ✗

(ii) Again not enough info. $By=c$ may have only one solution b/c perhaps our simultaneous solutions for $(x_1,x_2,x_3,y_1,y_2,y_3)$ only changes the x's (i.e. $(1,2,3,y_1,y_2,y_3)$ & $(4,5,6,y_1,y_2,y_3)$ perhaps). ✗

(iv) $BAx=By=c$. If B is singular $\Rightarrow$ BA singular. If B is not singular $\Rightarrow$ only the x changes in our simultaneous solution $\Rightarrow$ $BAx=c$ has multiple solutions $\Rightarrow$ BA singular. ✓

(iii) BA singular $\Rightarrow$ $\det(BA)=0 \Rightarrow \det(B)\det(A)=0 \Rightarrow \det(A)\det(B)=0 \Rightarrow \det(AB)=0$
$\Rightarrow$ AB singular. ✓

11. Given the two matrices:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad B = \begin{bmatrix} 2a-3d & 2b-3e & 2c-3f \\ g & h & i \\ 4d & 4e & 4f \end{bmatrix} \; r_3 \leftarrow r_3 \cdot \tfrac{1}{4} \quad (4)$$

and that det$B$=8, find det$A$.

A) 4 B) −1 C) −8 D) −64 E) 24

$$\begin{bmatrix} 2a-3d & 2b-3e & 2c-3f \\ g & h & i \\ d & e & f \end{bmatrix} r_1 \leftarrow r_1 + 3r_3 \quad \begin{bmatrix} 2a & 2b & 2c \\ g & h & i \\ d & e & f \end{bmatrix} r_1 \leftarrow r_1 \cdot \tfrac{1}{2} \; (2) \quad \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} r_2 \leftrightarrow r_3 \; (-1) \quad \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{A}.$$

So, $\det(B) = 4 \cdot 2 \cdot (-1) \det(A) \Rightarrow 8 = -8 \det(A) \Rightarrow \det(A) = -1.$

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12. Which of the following statements must be true for all invertible 3×3 matrices A and B?

i) $\det(3A) = 27\det(A)$
ii) $\det(BAB^{-1}) = \det(A)$
iii) $\det(A + B) = \det(A) + \det(B)$

A) i) and iii) B) ii) only C) i) and ii) D) ii) and iii)
E) None of the statements

(i) $\det(3A) = 3^3 \det(A) = 27 \det(A)$. ✓
(ii) $\det(BAB^{-1}) = \det(B)\det(A)\det(B^{-1}) = \det(B)\det(A) \cdot \frac{1}{\det(B)} = \det(A)$. ✓
(iii) This is not true in general. e.g. let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.
$\det(A+B) = \det\left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right) = 2.$ $\det(A) + \det(B) = 1 + 0 = 1.$ ✗

13. Compute the determinant of

$$A = \begin{bmatrix} 2 & 3 & 17 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

A) 24 B) -12 C) 6 D) -18 E) 0

$$\det A = 2 \cdot \begin{vmatrix} 0 & -2 & 1 \\ 2 & 1 & 4 \\ 0 & 0 & 3 \end{vmatrix} = 2 \cdot (-2) \begin{vmatrix} -2 & 1 \\ 0 & 3 \end{vmatrix} = -4\,[-6] = 24.$$

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14. What is the second column of adj(M), the adjoint of the matrix M, if M is given by

$$M = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix}$$

A) $\begin{bmatrix} -4 \\ 6 \\ 0 \end{bmatrix}$ B) $\begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$ C) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ D) $\begin{bmatrix} 0 \\ -6 \\ -4 \end{bmatrix}$ E) $\begin{bmatrix} 0 \\ 12 \\ -12 \end{bmatrix}$

$$\mathrm{adj}(M) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}.$$

$C_{21} = -\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} = -[12-15] = 3.$ So, $\begin{bmatrix} C_{21} \\ C_{22} \\ C_{23} \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}.$

$C_{22} = \begin{vmatrix} 1 & 5 \\ 0 & 6 \end{vmatrix} = 6.$

$C_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = -3.$

15. Find the eigenvalues of

$$A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix},$$

$Ax = \lambda x$
$\Leftrightarrow Ax - \lambda x = 0$
$\Leftrightarrow (A - \lambda I)x = 0.$

x exists & is nonzero $\Leftrightarrow \det(A - \lambda I) = 0.$

A) 1 and 3 B) 2 and 5 C) 2 and 3 D) 1 and -2 E) 0 and 4

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda) + 8 = -5 - 5\lambda + \lambda + \lambda^2 + 8$$

$$= \lambda^2 - 4\lambda + 3 = 0 \Leftrightarrow (\lambda - 3)(\lambda - 1) = 0 \Leftrightarrow \lambda = 3 \text{ or } \lambda = 1.$$

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16. Which of the following is an eigenvector corresponding to an eigenvalue of $\lambda = 5$ for the matrix

$$A = \begin{bmatrix} 6 & 2 & -4 \\ 3 & 7 & -4 \\ -4 & 0 & 5 \end{bmatrix}$$

$\boxed{Ax = 5x.}$

A) $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ B) $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ C) $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ D) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ E) $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Method 2:

$Ax = 5x \Leftrightarrow (A - 5I)x = 0.$

Let's solve the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 3 & 2 & -4 & 0 \\ -4 & 0 & 0 & 0 \end{array}\right] \quad r_1 \leftarrow r_1 - r_2$$

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 3 & 2 & -4 & 0 \\ -4 & 0 & 0 & 0 \end{array}\right]$$

$-2x = 0 \Rightarrow x = 0$
$3x + 2y - 4z = 0$
$\Rightarrow 2y - 4z = 0$
$\Rightarrow 2y = 4z$
$\Rightarrow y = 2z = 2t.$
$z = t.$

$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} t$ solves the system.

So, is an eigenvector corr. $\lambda = 5$ for any value of t.

Method 1:

A) $[6 \ 2 \ -4]\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = 6 - 4 = 2 \neq 5(0)$. X

B) $[6 \ 2 \ -4]\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = 18 - 4 = 14 \neq 5(3)$. X

C) $[6 \ 2 \ -4]\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 4 - 4 = 0.$

might work!

D) $[6 \ 2 \ -4]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -2 \neq 5(1)$. X

E) $[6 \ 2 \ -4]\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 12 - 4 = 8 \neq 5(2)$ X

So it must be C.

$$\begin{bmatrix} 6 & 2 & -4 \\ 3 & 7 & -4 \\ -4 & 0 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$ ✓

17. Given a matrix in Matlab defined as:

$$A = [\,1\ 3\ 7;\ -2\ 5\ 2\,]$$

$A = \begin{bmatrix} 1 & 3 & 7 \\ -2 & 5 & 2 \end{bmatrix}.$

What is the output of the command $A(:,2)$?

A) ans =

$$\begin{matrix} 1 & 3 & 7 & 2 \\ -2 & 5 & 2 & 2 \end{matrix}$$

B) ans =

$$\begin{matrix} 3 \\ 5 \end{matrix}$$

C) ans =

$$\begin{matrix} 3 & 7 \\ 5 & 2 \end{matrix}$$

D) ans =

$$\begin{matrix} 1 & 3 \\ -2 & 5 \end{matrix}$$

E) ans =

$$\begin{matrix} -2 & 5 & 2 \end{matrix}$$

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18. Suppose that a matrix A has eigenvalues 2, 0, and -3, with corresponding eigenvectors $[2,1,0]^T$, $[2,2,1]^T$, and $[3,1,3]^T$. Then the matrix A^2 can be written in the form PDP^{-1} with

A) $P = \begin{bmatrix} 4 & 4 & 9 \\ 1 & 4 & 1 \\ 0 & 1 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

shouldn't do anything to P. ✗

B) $P = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} -9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

not a square of one of our eigenvalues. ✗

C) $P = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ ✓

D) $P = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

need to square D. ✗

E) $P = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

need to put eigenvectors along the columns... not the rows. ✗

$A = PDP^{-1}$

$\Rightarrow A^2 = PD^2P^{-1}.$

19. Only the first row of the 4x4 matrix A is given to you:

$$A = \begin{bmatrix} 2 & 3 & -4 & 1 \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

let $X = \begin{bmatrix} 2 \\ 0 \\ 3 \\ -4 \end{bmatrix}$.

From your knowledge of what it means for a vector to be an eigenvector of A, if $[2, 0, 3, -4]^T$ is an eigenvector, what is the corresponding eigenvalue?

X eigenvector

A) -6 B) -5 C) 8 D) -9 E) 7

$\Rightarrow$ $AX = \lambda X$, for some λ.

$$\begin{bmatrix} 2 & 3 & -4 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -12 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \lambda\begin{bmatrix} 2 \\ 0 \\ 3 \\ -4 \end{bmatrix} \Rightarrow -12 = 2\lambda \Rightarrow \lambda = -6.$$

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20. The 4x4 matrix A is not diagonalizable. Which one of the following statements could be true?

A) A is not invertible, and has eigenvalues 1, 1, 2, 3
B) A is invertible, and has eigenvalues 1, 1, 2, 3
C) A is invertible, and has eigenvalues 0, 1, 2, 3
D) A is not invertible, and has eigenvalues 0, 1, 2, 3
E) A is invertible, and has eigenvalues 0, 1, 1, 3

Recall: A not invertible $\Leftrightarrow$ $\lambda = 0$ is an eigenvalue.

Recall: If A is $n \times n$ with n distinct eigenvalues $\Rightarrow$ A is diagonalizable

(A) A not invertible, but 0 not an eigenvalue. ✗
(B) May or may not be true. $\lambda = 1$ double root, so may get 2 distinct eigenvectors. ✓

END OF TEST QUESTIONS

(C) A invertible, but $\lambda = 0$ is an eigenvalue. ✗
(D) 4 distinct eigenvalues $\Rightarrow$ A diagonalizable. ✗
(E) A invertible, but $\lambda = 0$ eigenvalue. ✗

Extra page for rough work. DO NOT DETACH!

Math 1B03

1st Sample Test #1

Name: Dedieu Lauren

(Last Name) (First Name)

Student Number:________________ **Tutorial Number:**________________

This test consists of 20 multiple choice questions worth 1 mark each (no part marks), and 1 question worth 1 mark (no part marks) on proper computer card filling. All questions must be answered on the COMPUTER CARD with an HB PENCIL. Marks will not be deducted for wrong answers (i.e., there is no penalty for guessing). You are responsible for ensuring that your copy of the test is complete. Bring any discrepancy to the attention of the invigilator. Calculators are NOT allowed.

1. Which of the following matrices are in reduced row echelon form?

(i) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix}$ ← must swap r_2 & r_3 (iii) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$ ← $r_1 \leftarrow r_1 + 7r_2$

Recall: A matrix is in r.r.e.f. if:

(1) Rows of zeros are at bottom.

(2) The first non zero entry in a non-all-zeros-row is "1". (leading "1")

(3) Columns w/ a leading "1" have zeros everywhere else in it.

(4) Leading "1"'s in lower rows must occur farther to the right than the leading "1"'s above it.

(a) (i), (iii), and (iv) only
(b) (iii) only
(c) (i), (ii), and (iii) only
(d) (i) and (iii) only
(e) none of them

2. Let $A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 3 & 5 & 1 \\ 3 & -1 & -7 & 2 \end{bmatrix}$. Find the reduced row echelon form of A.

(a) $\begin{bmatrix} 1 & 0 & -\frac{8}{5} & 0 \\ 0 & 1 & \frac{9}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & \frac{8}{5} & 0 \\ 0 & 1 & \frac{11}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 0 & -\frac{8}{5} & 0 \\ 0 & 1 & \frac{11}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & -\frac{8}{5} & 0 \\ 0 & 1 & -\frac{11}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **(e)** $\begin{bmatrix} 1 & 0 & -\frac{7}{5} & 0 \\ 0 & 1 & \frac{11}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(see paper)

3. Suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row echelon form. Solve the system.

$$\left[\begin{array}{ccccc|c} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4 = -5x_5 + 8$
$x_3 = -4x_5 + 7$
$x_1 = 6x_2 - 3x_5 - 2$

$x_2 = r$, $x_5 = w$ $\rightarrow$ $x_4 = -5w + 8$, $x_3 = -4w + 7$, $x_1 = 6r - 3w - 2$

(a) $x_1 = -2s - 3t + 6u$, $x_2 = u$, $x_3 = 7s - 4t$, $x_4 = 8s - 5t$, $x_5 = t$

(b) $x_1 = -2 - 3s + 6t$, $x_2 = t$, $x_3 = 7 - 4t$, $x_4 = 8 - 5t$, $x_5 = s$

(c) $x_1 = 2 - 3t + 6s$, $x_2 = s$, $x_3 = -7 - 4t$, $x_4 = -8 - 5t$, $x_5 = t$

$w = t$
$r = s$

(d) $x_1 = -2 - 3t + 6$, $x_2 = 0$, $x_3 = 7 - 4t$, $x_4 = 8 - 5t$, $x_5 = t$

(e) $x_1 = -2 - 3t + 6s$, $x_2 = s$, $x_3 = 7 - 4t$, $x_4 = 8 - 5t$, $x_5 = t$

4. Solve the following system of equations

$$\begin{aligned} 2x_1 - x_2 + x_3 + x_4 - 2x_5 &= 1 \\ 3x_1 - 3x_2 + 2x_3 + 3x_5 &= 0 \\ 3x_2 - x_3 + 3x_4 - 12x_5 &= -1 \end{aligned}$$

(a) no solution

(b) $x_1 = 3 - 2s - 4t$, $x_2 = -1 + 2t$, $x_3 = -5 + 3s - t$, $x_4 = s$, $x_5 = t$

(c) $x_1 = 3 - 2s + 8t$, $x_2 = s$, $x_3 = -6 + 3s - 15t$, $x_4 = 1 + 2s - 12t$, $x_5 = t$

(See handout)

(d) $x_1 = 3 - 2s + 8t$, $x_2 = -1 - t$, $x_3 = -6 + 3s - 15t$, $x_4 = s$, $x_5 = t$

(e) $x_1 = -1 + s + 4t$, $x_2 = -1 - 3s + t$, $x_3 = s$, $x_4 = 2s + 6t$, $x_5 = t$

5. If ABC^T can be formed, A is 3×2, and C is 4×5, what size is B?
(a) 2×2 **(b)** 2×5 **(c)** 3×4 **(d)** 2×4 **(e)** 3×5

${}_3A_2\,B\,{}_5C^T_4 \rightarrow B$ is a 2×5.

6. Find conditions on a and b such that the following system has exactly one solution

$$x + by = -1$$
$$2ax + 2y = 5$$

(a) $ab = 1$ and $a \neq -\frac{5}{2}$
(b) $ab \neq 1$
(c) $a = -\frac{5}{2}$, $b = -\frac{2}{5}$
(d) $a = 3b$, $b \neq -\frac{2}{5}$
(e) $ab = 2$, $a \neq -\frac{5}{2}$

$$\left[\begin{array}{cc|c} 1 & b & -1 \\ 2a & 2 & 5 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 & b \\ 2a & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 5 \end{bmatrix}}_{B}$$

We know A invertible $\Leftrightarrow A\vec{x} = b$ has exactly one solution for every $n \times 1$ matrix b. And A invertible $\Leftrightarrow \det(A) \neq 0$.
$\det(A) = 2 - 2ab = 2(1-ab) \neq 0 \Leftrightarrow 1 - ab \neq 0 \Leftrightarrow ab \neq 1$.

7. Consider the following system.

$$2x - y + 2z = 5$$
$$x - y + 3z = 1$$
$$x + 2y + 4z = 6$$

Given that the inverse of $\overbrace{\begin{bmatrix} 2 & -1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}}^{A}$ is equal to $\begin{bmatrix} \frac{10}{13} & -\frac{8}{13} & \frac{1}{13} \\ \frac{1}{13} & -\frac{6}{13} & \frac{4}{13} \\ -\frac{3}{13} & \frac{5}{13} & \frac{1}{13} \end{bmatrix}$, which of the following gives a solution to the above system?

(a) $\begin{bmatrix} \frac{10}{13} & \frac{1}{13} & -\frac{3}{13} \\ -\frac{8}{13} & -\frac{6}{13} & \frac{5}{13} \\ \frac{1}{13} & \frac{4}{13} & \frac{1}{13} \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{10}{13} & -\frac{8}{13} & \frac{1}{13} \\ \frac{1}{13} & -\frac{6}{13} & \frac{4}{13} \\ -\frac{3}{13} & \frac{5}{13} & \frac{1}{13} \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{10}{13} & \frac{1}{13} & -\frac{3}{13} \\ -\frac{8}{13} & -\frac{6}{13} & \frac{5}{13} \\ \frac{1}{13} & \frac{4}{13} & \frac{1}{13} \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{10}{13} & -\frac{8}{13} & \frac{1}{13} \\ \frac{1}{13} & -\frac{6}{13} & \frac{4}{13} \\ -\frac{3}{13} & \frac{5}{13} & \frac{1}{13} \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix}$

(e) none of the above

$A\vec{x} = \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix}$

$\Rightarrow \vec{x} = A^{-1} \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix}$

8. Find the matrix A if

$$(A^T - 2I)^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

(a) $A = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{11}{4} \end{bmatrix}$ (c) $A = \frac{1}{4}\begin{bmatrix} 1 & -2 \\ -1 & 5 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & -2 \\ -1 & 5 \end{bmatrix}$ (e) $A = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{5}{4} \end{bmatrix}$

$A^T - 2I = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}^{-1}$

$A^T - 2I = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} \frac{1}{4}$

$A^T = \begin{bmatrix} 1/2 & -1/4 \\ -1/2 & 3/4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$A^T = \begin{bmatrix} 5/2 & -1/4 \\ -1/2 & 11/4 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 5/2 & -1/2 \\ -1/4 & 11/4 \end{bmatrix}$

9. Find an elementary matrix E such that $B = EA$.

$3 + x2 = -1$
$2x = -4$
$x = -2$

$$A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}_{r_1 \leftarrow r_1 - 2r_2}, \quad B = \begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} r_1 \leftarrow r_1 - 2r_2$ $\quad E = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

(a) $\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix}$ **(c)** $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **(e)** $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

10. Which of the following matrices are *always* symmetric.
(i) $A + A^T$ (ii) AA^T (iii) kA for any scalar k (iv) $A - A^T$

A symmetric if $A^T = A$.

(a) (i), (ii), and (iii) only
(b) (i), (ii), and (iv) only
(c) (ii) and (iv) only
(d) (i) and (ii) only
(e) (i), (ii), (iii), and (iv)

$(A + A^T)^T = A^T + A$ ✓
$(AA^T)^T = A^{TT}A^T = AA^T$ ✓
$(KA)^T = KA^T \neq KA$
$(A - A^T)^T = A^T - A \neq A - A^T$

11. Given that $\det \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = 4$, compute $\det \begin{bmatrix} r & s & t \\ x - 8r & y - 8s & z - 8t \\ 8u & 8v & 8w \end{bmatrix}$.

(see paper)

(a) 32 **(b)** -32 **(c)** 256 **(d)** -256 **(e)** 0

12. If $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -3$, calculate $\det \begin{bmatrix} 2 & -2 & 0 \\ c+1 & -1 & 2a \\ d-2 & 2 & 2b \end{bmatrix}$.

(see paper)

(a) 4 **(b)** -12 **(c)** 12 **(d)** -4 **(e)** -3

13. If A is 3×3 and $\det(2A^{-1}) = -3 = \det(A^3(B^{-1})^T)$, find $\det B$.
(a) $\frac{3^2}{8^3}$ **(b)** $\frac{8^3}{3^4}$ **(c)** $\frac{8^3}{3^2}$ **(d)** $\frac{2^3}{3^4}$ **(e)** $\frac{2^3}{3^2}$

$\det(A^3(B^{-1})^T) = -3$
$\det(A^3)\det((B^{-1})^T) = -3$
$\Rightarrow \det(A^3)\det(B^{-1}) = -3$
$\Rightarrow [\det(A)]^3 \det(B^{-1}) = -3$
$\Rightarrow (\det A)^3 \frac{1}{\det B} = -3$
$\Rightarrow -\frac{1}{3}(\det(A))^3 = \det B.$
And $\det(2A^{-1}) = -3$
$\Rightarrow 2^3 \det(A^{-1}) = -3$
$\Rightarrow 8 \frac{1}{\det A} = -3$
$\Rightarrow -8/3 = \det A$

So, $\det B = -\frac{1}{3} * \left(-\frac{8}{3}\right)^3 = \frac{8^3}{3^4}$

14. Compute the determinant of the following matrix,

$$\begin{bmatrix} 3 & 1 & -5 & 2 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 5 & 2 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

(a) -31 **(b)** -132 **(c)** -131 **(d)** -130 **(e)** 0

15. Find the adjoint of the following matrix $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

(see page)

(a) $\begin{bmatrix} 1 & -1 & -4 \\ 9 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 1 & -4 \\ -9 & 1 & 0 \\ 0 & -1 & 4 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 1 & -2 \\ 3 & 1 & -6 \\ -3 & -1 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -1 & -2 \\ -3 & 1 & 6 \\ -3 & 1 & 4 \end{bmatrix}$ **(e)** $\begin{bmatrix} 1 & 3 & 0 \\ -1 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

16. Find the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$.

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & 3 & -2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)[(2-\lambda)(-2-\lambda)+3]=0$$

$$= (2-\lambda)[-4+\lambda^2+3] = (2-\lambda)(\lambda^2-1)$$

$$\to \lambda=2,\ \lambda=\pm 1$$

(a) $2, 1, -1$ **(b)** $1, -1$ **(c)** $2, 1$ **(d)** $2, -1$ **(e)** $2, 1, 0$

17. Suppose that a matrix A (not given) has eigenvalues $\lambda = 1, -2, 3$ with eigenvectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, respectively. Find P and D so that $P^{-1}AP = D$.

(a) $P = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) $P = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(c) $P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(d) $P = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $P = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

18. Suppose $p(\lambda) = (\lambda - 1)^3$ for some diagonalizable 3×3 matrix A (not given). Calculate A^{25}.

We know A has

(a) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} -25 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & -25 \end{bmatrix}$ **(c)** $\begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **(e)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

19. Suppose that λ_1 is an eigenvalue of A with eigenvector $\mathbf{x}$, and λ_2 is an eigenvalue of B with the same eigenvector $\mathbf{x}$. Consider the following statements.
(i) $\lambda_1 + \lambda_2$ is an eigenvalue of the matrix $(A + B)$
(ii) $\lambda_1 \lambda_2$ is an eigenvalue of the matrix BA
(iii) λ_1^3 is an eigenvalue of the matrix A^3

(i) $\left.\begin{array}{l} Ax = \lambda_1 x \\ Bx = \lambda_2 x \end{array}\right\} \Rightarrow Ax + Bx = \lambda_1 x + \lambda_2 x \Rightarrow (A+B)x = (\lambda_1 + \lambda_2)x$

$\Rightarrow \lambda_1 + \lambda_2$ eigenvalue of $A+B$.

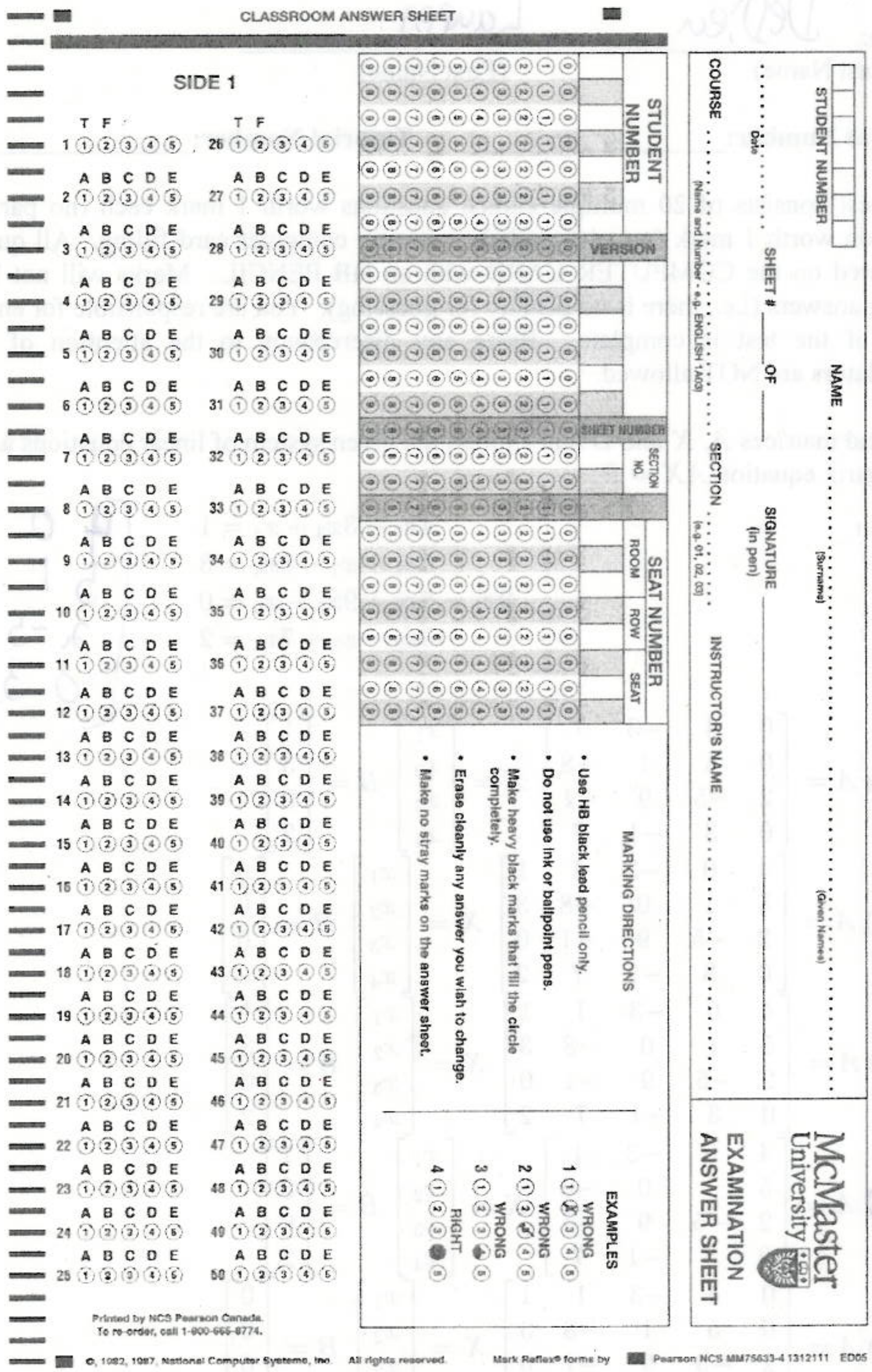
Which of the above statements are always true?
(a) (i), (ii), and (iii)
(b) (i) and (ii) only
(c) (i) and (iii) only
(d) (ii) only
(e) (i) only

(iii) $A^3x = A^2Ax = A^2\lambda_1 x = A\lambda_1 Ax = A\lambda_1^2 x = \lambda_1^2 Ax = \lambda_1^3.$

(ii) $BAx = B\lambda_1 x = \lambda_1 Bx = \lambda_1\lambda_2 x$

20. In Matlab what command could be used to create the row vector $(3, 5, 7, 9, 11, 13, 15, 17, 19)$?
(a) `>>[3 by 2 to 19]` **(b)** `>>3:2:19` **(c)** `>>[3 to 19 by 2]`
(d) `>>for (i = 3 to 19 by 2) x[i] = i end`
(e) `>>[3;5;7;9;11;13;15;17;19]`

21. Correctly fill out the bubbles corresponding to your student number and the version number of your test in the correct places on the computer card. (Use the below computer card for this sample test.)



Math 1B03

2nd Sample Test #1

Name: DeDien Lauren
(Last Name) (First Name)

Student Number:__________ **Tutorial Number:**__________

This test consists of 20 multiple choice questions worth 1 mark each (no part marks), and 1 question worth 1 mark (no part marks) on proper computer card filling. All questions must be answered on the COMPUTER CARD with an HB PENCIL. Marks will not be deducted for wrong answers (i.e., there is no penalty for guessing). You are responsible for ensuring that your copy of the test is complete. Bring any discrepancy to the attention of the invigilator. Calculators are NOT allowed.

1. Find matrices A, X and B that express the given system of linear equations as a single matrix equation $AX = B$.

$$\begin{aligned} 4x_1 - 3x_3 + x_4 &= 1 \\ 5x_1 + x_2 - 8x_4 &= 3 \\ 2x_1 - 5x_2 + 9x_3 - x_4 &= 0 \\ 3x_2 - x_3 + 7x_4 &= 2 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -3 & 1 & 1 \\ 5 & 1 & 0 & -8 & 3 \\ 2 & -5 & 9 & -1 & 0 \\ 0 & 3 & -1 & 7 & 2 \end{array}\right]$$

(a) $A = \begin{bmatrix} 0 & 4 & -3 & 1 \\ 0 & 5 & 1 & -8 \\ 2 & -5 & 9 & -1 \\ 0 & 3 & -1 & 7 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 4 & 0 & -3 & 1 & 1 \\ 5 & 1 & 0 & -8 & 3 \\ 2 & -5 & 9 & -1 & 0 \\ 0 & 3 & -1 & 7 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 4 & 0 & -3 & 1 & 1 \\ 5 & 1 & 0 & -8 & 3 \\ 2 & -5 & 9 & -1 & 0 \\ 0 & 3 & -1 & 7 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 4 & 0 & -3 & 1 \\ 5 & 1 & 0 & -8 \\ 2 & -5 & 9 & -1 \\ 0 & 3 & -1 & 7 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$

(e) $A = \begin{bmatrix} 0 & 4 & -3 & 1 & 1 \\ 0 & 5 & 1 & -8 & 3 \\ 2 & -5 & 9 & -1 & 0 \\ 0 & 3 & -1 & 7 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

2. Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute $C^T A^T + 2E^T$, if possible.

(see paper)

(a) $\begin{bmatrix} 15 & 7 & 10 \\ 10 & 0 & 9 \\ 14 & 10 & 13 \end{bmatrix}$ **(b)** $\begin{bmatrix} 15 & 3 & 12 \\ 14 & 0 & 7 \\ 12 & 12 & 13 \end{bmatrix}$ **(c)** undefined **(d)** $\begin{bmatrix} 15 & 14 & 12 \\ 3 & 0 & 12 \\ 12 & 7 & 13 \end{bmatrix}$

(e) $\begin{bmatrix} 15 & 10 & 14 \\ 7 & 0 & 10 \\ 10 & 9 & 13 \end{bmatrix}$

3. Solve the following system of equations

$$\begin{aligned} 2x_1 - x_2 + x_3 + x_4 - 2x_5 &= 1 \\ 3x_1 - 3x_2 + 2x_3 + 3x_5 &= 0 \\ 2x_1 + x_2 + x_3 + x_4 &= -1 \end{aligned}$$

(a) no solution

(b) $x_1 = 3 - 2s - 4t$, $x_2 = -1 + 2t$, $x_3 = -5 + 3s - t$, $x_4 = s$, $x_5 = t$

(c) $x_1 = 3 - 2s + 8t$, $x_2 = s$, $x_3 = -6 + 3s - 15t$, $x_4 = 1 + 2s - 12t$, $x_5 = t$

(d) $x_1 = 3 - 2s + 8t$, $x_2 = -1 - t$, $x_3 = -6 + 3s - 15t$, $x_4 = s$, $x_5 = t$

(e) $x_1 = -1 + s + 4t$, $x_2 = -1 - 3s + t$, $x_3 = s$, $x_4 = 2s + 6t$, $x_5 = t$

4. Use determinants to find all of the possible real values of a which make the following matrix *not* invertible.

$$A = \begin{bmatrix} 1 & 1 & a \\ -a & 1 & -a \\ a & -1 & 2 \end{bmatrix} \quad r_2 \leftarrow r_2 + r_3$$

A not invertible $\Leftrightarrow \det A = 0$

(a) 2 and -1 **(b)** ± 1 **(c)** -1 **(d)** ± 2 **(e)** 0

$$\begin{bmatrix} 1 & 1 & a \\ 0 & 0 & -a+2 \\ a & -1 & 2 \end{bmatrix}$$

$$\det(A) = -(-a+2)\begin{vmatrix} 1 & 1 \\ a & -1 \end{vmatrix} = (a-2)[-1-a] = 0$$

$\Leftrightarrow$ $a = 2$ or $-1 - a = 0$

$\Rightarrow a = -1$

5. Find conditions on a, b, and c such that the system has infinitely many solutions

$$\begin{aligned} -cx + 3y + 2z &= -8 \\ x + z &= 2 \\ 3x + 3y + az &= b \end{aligned}$$

(see paper)

(a) $a - c - 5 \neq 0$
(b) $a - c = 0$ and $b - 2c + 2 = 5$
(c) $a - c - 5 = 0$ and $b - 2c + 2 = 0$
(d) $a - c = 0$ and $b - 2c + 2 \neq 5$
(e) $a - c - 5 = 0$ and $b - 2c + 2 \neq 0$

6. Find the diagonal entries of the inverse of $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$.

(see paper)

(a) $\begin{bmatrix} \frac{2}{5} & * & * \\ * & -\frac{2}{5} & * \\ * & * & \frac{4}{25} \end{bmatrix}$ **(b)** $\begin{bmatrix} \frac{2}{5} & * & * \\ * & \frac{2}{5} & * \\ * & * & \frac{4}{25} \end{bmatrix}$ **(c)** $\begin{bmatrix} -\frac{2}{5} & * & * \\ * & -\frac{2}{5} & * \\ * & * & \frac{4}{25} \end{bmatrix}$

(b) $\begin{bmatrix} \frac{2}{5} & * & * \\ * & -\frac{2}{5} & * \\ * & * & -\frac{4}{25} \end{bmatrix}$ **(b)** $\begin{bmatrix} -\frac{2}{5} & * & * \\ * & \frac{2}{5} & * \\ * & * & -\frac{4}{25} \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} r_2 \leftarrow r_2 \times \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} r_1 \leftarrow r_1 - 2r_2 \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

7. Consider the following matrix,

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

Note that A can be reduced to I using the following row operations:

So. $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(i) $r_2 \rightarrow \frac{1}{3} r_2$
(ii) $r_1 \rightarrow r_1 - 2r_2$

Using the above two row operations in the above order, find elementary matrices E_1 and E_2 such that $A = E_1^{-1} E_2^{-1}$.

(a) $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, E_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ **(b)** $E_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

(c) $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **(d)** $E_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

(e) $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

So, $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{-1}$

$A = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}^{-1}}_{E_1^{-1}} \underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{-1}}_{E_2^{-1}}$

A symmetric if $A = A^T$.

$(A^{-1})^T = (A^T)^{-1} = A^{-1}$ ✓

$(AB)^T = B^T A^T = BA \neq AB$ ✗

$(AB - BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB$ ✗

8. Suppose that A and B are symmetric matrices. Which of the following matrices are *always* symmetric?
(i) A^{-1} (ii) AB (iii) $AB - BA$

(a) (i) only **(b)** (i) and (ii) only **(c)** (i) and (iii) only **(d)** (ii) and (iii) only
(e) none of them

9. If $A^3 = 0$, which of the following is equal to $(I - A)^{-1}$?

(a) $I + A$ **(b)** $I + A + A^2$ **(c)** $I - A$ **(d)** $I - A - A^2$ **(e)** $I - A + A^2$

$(I-A)(I+A+A^2)$
$= I(I+A+A^2) - A(I+A+A^2)$
$= I + A + A^2 - A - A^2 - \underbrace{A^3}_{0}$
$= I \Rightarrow I + A + A^2 = (I-A)^{-1}$.

10. A matrix A is **skew-symmetric** if $A^T = -A$. Suppose that A and B are both skew-symmetric. Which of the following matrices are *always* skew-symmetric?
(i) $A + B$ (ii) AB (iii) kA

(a) (i) only
(b) (i) and (iii) only
(c) (iii) only
(d) (i), (ii), and (iii)
(e) (ii) only

$(A+B)^T = A^T + B^T = -A - B = -(A+B)$ ✓

$(AB)^T = B^T A^T = (-B)(-A) = BA \neq AB$ ✗

$(kA)^T = kA^T = k(-A) = -kA$ ✓

11. Consider the following statements,
(i) $(A - B)^2 = (B - A)^2$ for all $n \times n$ matrices A and B.
(ii) $\det(A + B^T) = \det(A^T + B)$
(iii) If $AB = 0$ then $A = 0$ or $B = 0$.

Which of the above statements are always true?
(a) (i) only
(b) (i) and (ii) only
(c) (i) and (iii) only
(d) (ii) and (iii) only
(e) all of them

$(A-B)(A-B) = A^2 - AB - BA + B^2$ same
$(B-A)(B-A) = B^2 - BA - AB = A^2$

$\det(A + B^T) = \det(A^T + B^T)$
$= \det((A^T + B)^T) = \det(A^T + B)$ ✓

(iii) is false:

$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

12. Let $A = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 4 & 0 & -1 & 2 \\ 3 & 3 & 7 & 0 \\ 3 & 5 & 6 & -4 \end{bmatrix}$. Given that $\det A = -4$, use the adjoint to find the entry in row 1 column 2 of A^{-1}.

(a) $\frac{9}{4}$ **(b)** $-\frac{9}{4}$ **(c)** 9 **(d)** $-\frac{65}{2}$ **(e)** -9

13. A square matrix P is called **idempotent** if $P^2 = P$. If P is idempotent, which of the following matrices are also idempotent?
(i) $I-P$ (ii) $I+P$ (iii) $I-2P$

$(I-P)^2 = (I-P)(I-P) = I-P-P(I-P)$
$= I-P-P+P^2 = I-P-\cancel{P}+\cancel{P} = I-P$ ✓
$(I+P)(I+P) = I+P+P+P^2 = I+P+P+P \neq I+P$
$(I-2P)(I-2P) = I-2P-2P+4P^2 = I-4P+4P$
$= I \neq I-2P$ ✗

(a) (i) only
(b) (i) and (ii)
(c) (i) and (iii)
(d) (ii) only
(e) (i), (ii), and (iii)

14. If A is 3×3 and $\det A = 2$, find $\det(A^{-1} + 4\,\mathrm{adj}\,A)$.

$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}\,A$
$\mathrm{adj}\,A = \det A\, A^{-1}$
$= 2A^{-1}$

$\det(A^{-1} + 4\,\mathrm{adj}\,A) = \det[A^{-1} + 4(2A^{-1})]$
$= \det(9A^{-1}) = 9^3 \det(A^{-1}) = 729 \frac{1}{\det A}$
$= \frac{729}{2}$.

(a) 364 **(b)** $\frac{729}{2}$ **(c)** 365 **(d)** 729 **(e)** $\frac{365}{2}$

15. Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$ and assume that $\det A = 2$. Compute $\det(2B^{-1})$ where

$$B = \begin{bmatrix} 4u & 2a & -p \\ 4v & 2b & -q \\ 4w & 2c & -r \end{bmatrix}.$$

(a) -1 **(b)** $-\frac{1}{2}$ **(c)** -16 **(d)** -2 **(e)** $-\frac{1}{4}$

16. Let A and B be $n \times n$ matrices. Consider the following statements.
(i) $\det(AB) = \det(BA)$
(ii) $\det(A+B) = \det A + \det B$
(iii) $\det(-A) = -\det(A)$

$\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA)$ ✓
$\det(A+B) \neq \det A + \det B$.
$\det(-A) = (-1)^n \det(A)$

Which of the above statements are always true?
(a) (i) only
(b) (i) and (ii) only
(c) (i) and (iii) only
(d) (i), (ii), and (iii)
(e) (iii) only

$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$ $\begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = 0$

$\begin{vmatrix} 4 & 1 \\ 6 & 5 \end{vmatrix} = 14 \neq 4.$

17. Given that the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ has $\lambda = -1$ as one of its eigenvalues, find the corresponding eigenvector(s).

(a) $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

18. Find a matrix P such that $P^{-1}AP$ is diagonal. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}$

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ **(e)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

19. Consider the following statements.
(i) If $P^{-1}AP$ is diagonal, and $P^{-1}BP$ is diagonal, then AB diagonalizable.
(ii) If A is diagonalizable then $\det(A) = \lambda_1\lambda_2\cdots\lambda_n$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A.
(iii) If A is diagonalizable then A must be invertible.

False. $\det(A)=0$, but A diagonalizable.

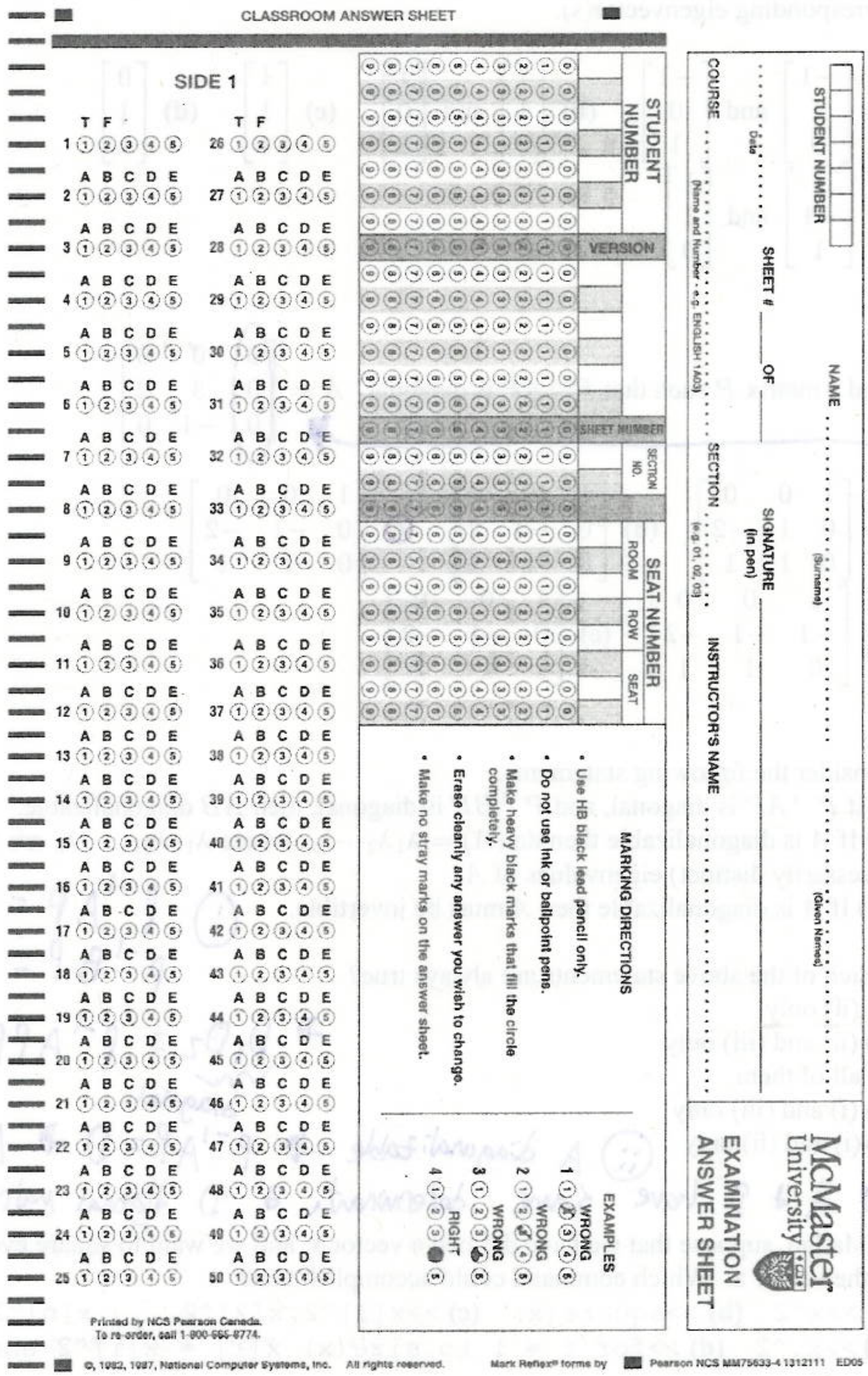
(i) $P^{-1}AP = D_1$, $P^{-1}BP = D_2$
$\Rightarrow \underbrace{D_1D_2}_{\text{diagonal}} = P^{-1}APP^{-1}BP = P^{-1}ABP$ ✓

Which of the above statements are always true?
(a) (ii) only
(b) (ii) and (iii) only
(c) all of them
(d) (i) and (iii) only
(e) (i) and (ii) only

(ii) A diagonalizable $\Rightarrow P^{-1}AP = D \Rightarrow A$ & D similar matrices $\Rightarrow A$ & D have same determinant. & D diagonal matrix w/ eigenvalues on diagonal $\Rightarrow \det(D) = \det(A) = \lambda_1\lambda_2\cdots\lambda_n$ ✓

20. In Matlab, suppose that we have defined a vector **x**, and we want to square every component of the vector **x**. Which command could accomplish this?
(a) `>>x^2` **(b)** `>>square(x)` **(c)** `>>x[1]^2,x[2]^2,...,x[n]^2`
(d) `>>x.^2` **(d)** `>>for i = 1 to size(x) x[i] = x[i]^2 endfor`

21. Correctly fill out the bubbles corresponding to your student number and the version number of your test in the correct places on the computer card. (Use the below computer card for this sample test.)


CLASSROOM ANSWER SHEET

SIDE 1

STUDENT NUMBER

VERSION

SHEET NUMBER

SECTION NO.

SEAT NUMBER ROOM ROW SEAT

MARKING DIRECTIONS

• Use HB black lead pencil only.

• Do not use ink or ballpoint pens.

• Make heavy black marks that fill the circle completely.

• Erase cleanly any answer you wish to change.

• Make no stray marks on the answer sheet.

EXAMPLES

1 WRONG

2 WRONG

3 WRONG

4 RIGHT

McMaster University

EXAMINATION ANSWER SHEET

STUDENT NUMBER

NAME (Surname) (Given Names)

COURSE (Name and Number - e.g. ENGLISH 1A03)

Date

SHEET # ____ OF ____

SIGNATURE (in pen)

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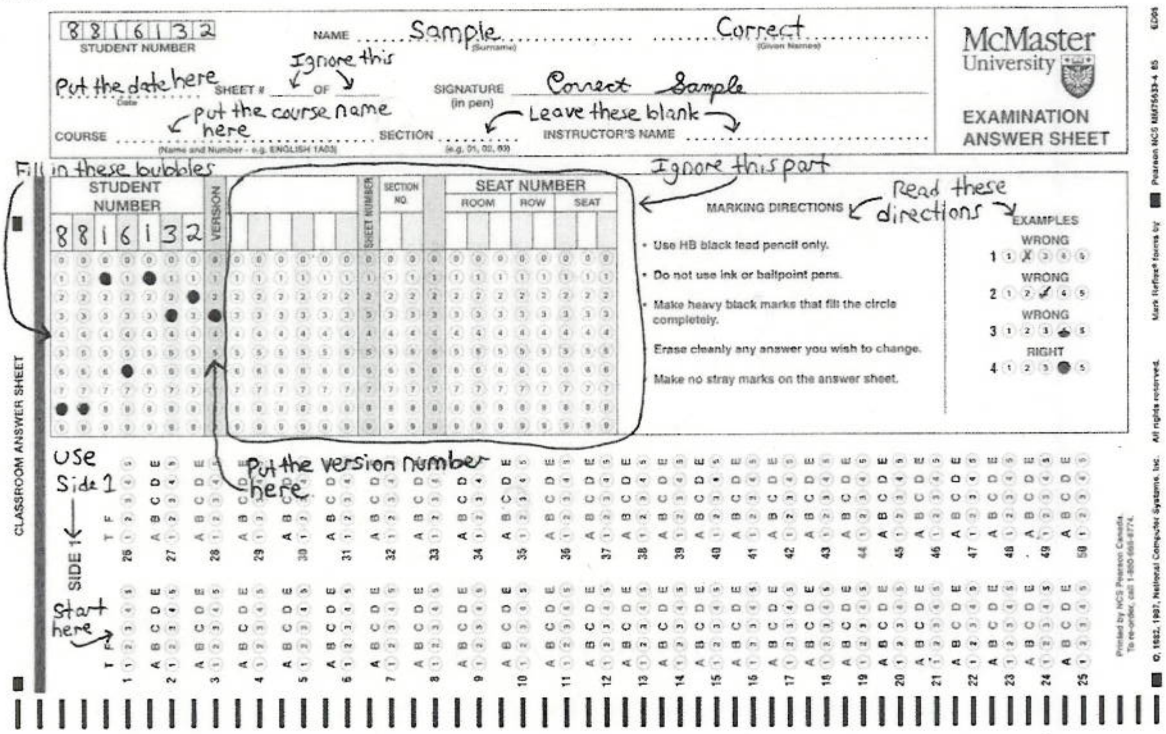
Answers for 1st Sample Test #1

1. d **2.** c **3.** e **4.** a **5.** b **6.** b **7.** b **8.** b **9.** e **10.** d
11. b **12.** c **13.** b **14.** b **15.** d **16.** a **17.** b **18.** e **19.** a **20.** b
21.



NOTE: On the sample tests, a version number is not given. On the actual tests, it will say "Version X" at the top, where X is the version number that you will have to fill in on the computer card. The sample answer above assumes that the test says "Version 3" at the top. On the actual test you will have to fill in the bubble corresponding to the version number of YOUR test (which may or may not be Version 3). The sample above also assumes that your student number is 8816132. On the actual test, you will have to fill in the bubbles corresponding to YOUR student number (not 8816132).

Answers for 2nd Sample Test #1

1. d **2.** b **3.** d **4.** a **5.** c **6.** a **7.** a **8.** a **9.** b **10.** b
11. b **12.** a **13.** a **14.** b **15.** b **16.** a **17.** a **18.** c **19.** e **20.** d
21. see the answer to #21 on the first sample test above.

Sample Test #1

2. $A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ 1 & 3 & 5 & 1 \\ 3 & -1 & -7 & 2 \end{bmatrix}$ $r_1 \leftarrow r_1 + r_3$ $\begin{bmatrix} 5 & 0 & -8 & 2 \\ 1 & 3 & 5 & 1 \\ 3 & -1 & -7 & 2 \end{bmatrix}$

$r_1 \leftarrow r_1 * \frac{1}{5}$

$\begin{bmatrix} 1 & 0 & -8/5 & 2/5 \\ 1 & 3 & 5 & 1 \\ 3 & -1 & -7 & 2 \end{bmatrix}$ $r_2 \leftarrow r_2 - r_1$, $r_3 \leftarrow r_3 - 3r_1$ $\begin{bmatrix} 1 & 0 & -8/5 & 2/5 \\ 0 & 3 & 8/5 + 25/5 & 5/5 - 2/5 \\ 0 & -1 & -7 + 24/5 & -6/5 + 10/5 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -8/5 & 2/5 \\ 0 & 3 & 33/5 & 3/5 \\ 0 & -1 & -11/5 & 4/5 \end{bmatrix}$ $r_2 \leftarrow r_2 * \frac{1}{3}$ $\begin{bmatrix} 1 & 0 & -8/5 & 2/5 \\ 0 & 1 & 33/15 & 3/15 \\ 0 & -1 & -11/5 & 4/5 \end{bmatrix}$

$r_3 \leftarrow r_3 + r_2$

$\begin{bmatrix} 1 & 0 & -8/5 & 2/5 \\ 0 & 1 & 33/15 & 3/15 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $r_1 \leftarrow r_1 - \frac{2}{5}r_3$, $r_2 \leftarrow r_2 - \frac{3}{15}r_3$ $\begin{bmatrix} 1 & 0 & -8/5 & 0 \\ 0 & 1 & 11/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

↑ reduced row echelon form

4. $\left[\begin{array}{ccccc|c} 2 & -1 & 1 & 1 & -2 & 1 \\ 3 & -3 & 2 & 0 & 3 & 0 \\ 0 & 3 & -1 & 3 & -12 & -1 \end{array}\right]$ $r_1 \leftarrow r_1 + r_3$

$\left[\begin{array}{ccccc|c} 2 & 2 & 0 & 4 & -14 & 0 \\ 3 & -3 & 2 & 0 & 3 & 0 \\ 0 & 3 & -1 & 3 & -12 & -1 \end{array}\right]$ $r_1 \leftarrow r_1 * \frac{1}{2}$ $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & -7 & 0 \\ 3 & -3 & 2 & 0 & 3 & 0 \\ 0 & 3 & -1 & 3 & -12 & -1 \end{array}\right]$

$r_2 \leftarrow r_2 - 3r_1$

$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & -7 & 0 \\ 0 & -6 & 2 & -6 & 24 & 0 \\ 0 & 3 & -1 & 3 & -12 & -1 \end{array}\right]$ $r_2 \leftarrow r_2 + 2r_3$ $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 3 & -1 & 3 & -12 & -1 \end{array}\right]$

So no solution

11. $B = \begin{bmatrix} r & s & t \\ x-8r & y-8s & z-8t \\ 8u & 8v & 8w \end{bmatrix}$ $\quad A = \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix}$ swap r_2 & r_3 $(*-1)$

$\begin{bmatrix} r & s & t \\ x & y & z \\ u & v & w \end{bmatrix}$ $r_2 \leftarrow r_2 - 8r_1$, $r_3 \leftarrow r_3 * 8$ $(*8)$ $\quad \begin{bmatrix} r & s & t \\ x-8r & y-8s & z-8t \\ 8u & 8v & 8w \end{bmatrix}$

So, since $\det(A) = 4 \rightarrow \det(B) = 4 * -1 * 8 = -32$.

12. $\det\left(\underbrace{\begin{bmatrix} 2 & -2 & 0 \\ c+1 & -1 & 2a \\ d-2 & 2 & 2b \end{bmatrix}}_{B}\right)$ $r_3 \leftarrow r_3 + r_1$ $= \det\left(\begin{bmatrix} 2 & -2 & 0 \\ c+1 & -1 & 2a \\ d & 0 & 2b \end{bmatrix}\right)$ $r_1 \leftarrow r_1 - 2r_2$

$= \det\left(\begin{bmatrix} -2c & 0 & -4a \\ c+1 & -1 & 2a \\ d & 0 & 2b \end{bmatrix}\right)$

$= -1 * \det\left(\begin{bmatrix} -2c & -4a \\ d & 2b \end{bmatrix}\right)$

And, we know $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -3 \rightarrow \begin{vmatrix} a & c \\ b & d \end{vmatrix} = -3$

$\begin{bmatrix} c & a \\ d & b \end{bmatrix}$ swap c_1 & c_2 $(*-1)$ $\quad \begin{bmatrix} -2c & -2a \\ d & b \end{bmatrix} (*-2) \quad \begin{bmatrix} -2c & -4a \\ d & 2b \end{bmatrix} (*2)$

So, $\det(B) = -1 * -3 * -1 * -2 * 2 = \boxed{12}$.

14. $\underbrace{\begin{bmatrix} 3 & 1 & -5 & 2 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 5 & 2 \\ 1 & 1 & 2 & -1 \end{bmatrix}}_{B}$ $r_1 \leftarrow r_1 + r_3$ $\begin{bmatrix} 4 & 1 & 0 & 4 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 5 & 2 \\ 1 & 1 & 2 & -1 \end{bmatrix}$

$c_4 \leftarrow c_4 - c_1$

$\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 5 & 1 \\ 1 & 1 & 2 & -2 \end{bmatrix}$ $r_3 \leftarrow r_3 + \frac{1}{2} r_4$ $\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 3/2 & 1/2 & 6 & 0 \\ 1 & 1 & 2 & -2 \end{bmatrix}$

$c_2 \leftarrow c_2 - \frac{1}{4} c_1$

$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 11/4 & 0 & 0 \\ 3/2 & 1/8 & 6 & 0 \\ 1 & 3/4 & 2 & -2 \end{bmatrix}$ $\rightarrow \det(B) = 4 * \frac{11}{4} * 6 * -2$

$= 66 * -2 = \boxed{-132}$.

lower triangular

$-\frac{1}{4} + \frac{12}{4}$

$-\frac{3}{8} + \frac{4}{8}$

$66 \times 2 = 132$

15. $\begin{bmatrix} \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} \end{bmatrix}^T$

$= \begin{bmatrix} 1 & -3 & -3 \\ -1 & 1 & 1 \\ -2 & 6 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & -2 \\ -3 & 1 & 6 \\ -3 & 1 & 4 \end{bmatrix}$

Sample Test #2

2. $C^T A^T + 2E^T$

$$= \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + 2 \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 4 \\ 12 & -2 & 5 \\ 6 & 8 & 7 \end{bmatrix} + \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 3 & 12 \\ 14 & 0 & 7 \\ 12 & 12 & 13 \end{bmatrix}$$

3. $\left[\begin{array}{ccccc|c} 2 & -1 & 1 & 1 & -2 & 1 \\ 3 & -3 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & 1 & 0 & -1 \end{array}\right] r_1 \leftarrow r_1 - r_3$

$\left[\begin{array}{ccccc|c} 0 & -2 & 0 & 0 & -2 & 2 \\ 3 & -3 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & 1 & 0 & -1 \end{array}\right] r_1 \leftarrow r_1 \cdot \frac{1}{2}$ $\left[\begin{array}{ccccc|c} 0 & -1 & 0 & 0 & -1 & 1 \\ 3 & -3 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & 1 & 0 & -1 \end{array}\right]$

$r_2 \leftarrow r_2 + 3r_1$

$\left[\begin{array}{ccccc|c} 0 & -1 & 0 & 0 & -1 & 1 \\ 3 & -6 & 2 & 0 & 0 & 3 \\ 2 & 1 & 1 & 1 & 0 & -1 \end{array}\right] r_3 \leftarrow r_3 - \frac{1}{2} r_2$ $\left[\begin{array}{ccccc|c} 0 & -1 & 0 & 0 & -1 & 1 \\ 3 & -6 & 2 & 0 & 0 & 3 \\ \frac{1}{2} & 4 & 0 & 1 & 0 & -\frac{5}{2} \end{array}\right]$

$-\frac{3}{2} + 2$

$= -\frac{3}{2} + \frac{4}{2}$

$-\frac{3}{2} - \frac{3}{2}$

$-x_2 - x_5 = 1 \rightarrow x_5 = -w - 1$

$3x_1 - 6x_2 + 2x_3 = 3 \rightarrow 2x_3 = 3 - 3r + 6w$

$\frac{1}{2}x_1 + 4x_2 + x_4 = -\frac{5}{2} \rightarrow x_4 = -\frac{5}{2} - \frac{1}{2}r - 4w$

$x_2 = w$

$x_1 = r$

Argh sometimes it's hard to do the same row ops, & easy to make a mistake. Maybe easier to just plug em in & check each!

$$\begin{bmatrix} 2 & -1 & 1 & 4 & -2 \\ 3 & -3 & 2 & 0 & 3 \\ 2 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Sometimes it's hard to get the same free variables as in the multiple choice. Let's try out the solutions to see which one works.

(b) $\begin{pmatrix} 3 \\ -1 \\ -5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -4 \\ 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} t$

Let's try each.

(c) $\begin{pmatrix} 3 \\ 0 \\ -6 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ -3 \\ 2 \\ 0 \end{pmatrix} s + \begin{pmatrix} 8 \\ 0 \\ -15 \\ -12 \\ 1 \end{pmatrix} t$

$$\begin{bmatrix} 2 & -1 & 1 & 4 & -2 \\ 3 & -3 & 2 & 0 & 3 \\ 2 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} 3 \\ -1 \\ -5 \\ 0 \\ 0 \end{pmatrix}$$

(b) ✗

$= 2 \neq \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

(d) $\begin{pmatrix} 3 \\ -1 \\ -6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 6 \\ 3 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 8 \\ -1 \\ -15 \\ 0 \\ 1 \end{pmatrix} t$

(c) ✗

$$\begin{bmatrix} 2 & -1 & 1 & 4 & -2 \\ & & \vdots & & \\ & & & & \end{bmatrix} \begin{pmatrix} 3 \\ 0 \\ -6 \\ 1 \\ 0 \end{pmatrix}$$

(e) $\begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 1 \\ 2 \\ 0 \end{pmatrix} s + \begin{pmatrix} 4 \\ 1 \\ 0 \\ 6 \\ 1 \end{pmatrix} t$

$= 4 \neq \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

(d) $\begin{bmatrix} 2 & -1 & 1 & 1 & -2 \\ 3 & -3 & 2 & 0 & 3 \\ 2 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} 3 \\ -1 \\ -6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ ✓

So (d) works!

$\begin{bmatrix} 2 & -1 & 1 & 1 & -2 \\ 3 & -3 & 2 & 0 & 3 \\ 2 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ✓

$\begin{bmatrix} 2 & -1 & 1 & 1 & -2 \\ 3 & -3 & 2 & 0 & 3 \\ 2 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} 8 \\ -1 \\ -15 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ✓

---

5. $\left[\begin{array}{ccc:c} -c & 3 & 2 & -8 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & a & b \end{array}\right]$ $r_1 \leftarrow r_1 - r_3$

If we have a row of zeros, we know we'll have a parameter t so infinitely many solutions.

$\left[\begin{array}{ccc:c} -c-3 & 0 & 2-a & -8-b \\ 1 & 0 & 1 & 2 \\ 3 & 3 & a & b \end{array}\right]$ $a = 2$, $c = -3$, $b = -8$ works

Check $\left[\begin{array}{ccc:c} 3 & 3 & 2 & -8 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & 2 & -8 \end{array}\right]$ $r_3 \leftarrow r_3 - r_1$ $\left[\begin{array}{ccc:c} 3 & 3 & 2 & -8 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $r_1 \leftarrow r_1 - 3r_2$

$$\left[\begin{array}{ccc:c} 0 & 3 & -1 & -14 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$3y = -14 + z \Rightarrow y = \frac{-14}{3} + \frac{1}{3}t$

$x = 2 - z \Rightarrow x = 2 - t$

$z = t \qquad z = t$

And $a - c - 5 = 2 + 3 - 5 = 0$ ✓ So (c).

$b - 2c + 2 = -8 - 2(-3) + 2 = 0$ ✓

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6. We know $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{array}{l} r_1 \leftarrow r_1 - 3r_3 \\ r_2 \leftarrow r_2 - r_3 \end{array} \begin{bmatrix} 0 & -5 & -10 \\ 0 & -3 & -1 \\ 1 & 2 & 4 \end{bmatrix}$$

$\det(A) = 1 \times \begin{vmatrix} -5 & -10 \\ -3 & -1 \end{vmatrix} = 5 - 30 = -25$

$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$C_{11} = \begin{vmatrix} -1 & 3 \\ 2 & 4 \end{vmatrix} = -4 - 6 = -10$

$C_{22} = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 12 - 2 = 10$

$C_{33} = \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = -3 - 1 = -4$

So, $A^{-1} = \begin{bmatrix} 2/5 & * & * \\ * & -2/5 & * \\ * & * & 4/25 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 4 & 0 & -1 & 2 \\ 3 & 3 & 7 & 0 \\ 3 & 5 & 6 & -4 \end{bmatrix}$. We know $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

Want the a_{12} entry of A^{-1}.

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$C_{21} = -\begin{vmatrix} 2 & 4 & -1 \\ 3 & 7 & 0 \\ 5 & 6 & -4 \end{vmatrix} \xrightarrow{r_3 \leftarrow r_3 - 4r_1} -\begin{vmatrix} 2 & 4 & -1 \\ 3 & 7 & 0 \\ -3 & -10 & 0 \end{vmatrix}$$

$$= -1 * -1 * \begin{vmatrix} 3 & 7 \\ -3 & -10 \end{vmatrix} = -30 + 21 = -9$$

So, $a_{12} = \boxed{9/4}$.

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15. $\det A = 2 = \det \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix} = \begin{vmatrix} a & p & u \\ b & q & v \\ c & r & w \end{vmatrix} = -\begin{vmatrix} u & p & a \\ v & q & b \\ w & r & c \end{vmatrix}$

$$= \begin{vmatrix} u & a & p \\ v & b & q \\ w & c & r \end{vmatrix} = \frac{1}{4} * \frac{1}{2} * -1 \begin{vmatrix} 4u & 2a & -p \\ 4v & 2b & -q \\ 4w & 2c & -r \end{vmatrix} = \frac{1}{-8} \det(B)$$

$\boxed{-1/2}$

$\det(B) = -16$.

So, $\det(2B^{-1}) = 2^3 \det(B^{-1}) = \frac{8}{\det B} = \frac{8}{-16}$

17. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$Ax = \lambda x$

$(A - \lambda I)x = 0$

$\underline{\lambda = -1}$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} r_2 \leftarrow r_2 - r_1 \\ r_3 \leftarrow r_3 - r_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x = -y - z = -t - s$

$y = t$

$z = s$

$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} s$

eigenvectors

$\underline{\text{Check}}$ $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -1 * \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ✓

$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = -1 * \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ ✓

18. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}$ $\begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 3-\lambda & 2 \\ 0 & -1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix}$

$= -\lambda\left[(3-\lambda)(-\lambda) + 2\right] = -\lambda\left[-3\lambda + \lambda^2 + 2\right] = -\lambda(\lambda-2)(\lambda-1)$

$\lambda = 0 \quad \lambda = 2 \quad \lambda = 1$

$\underline{\lambda = 0}$: $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$

$y = 0$

$3y = -2z$

$-2z = 0 \rightarrow z = 0$

$x = t$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t$

$\lambda=1$: $\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2+2r_3} \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$

$x=0$
$-y = z \Rightarrow y=-t$
$z=t$

$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} t$

$\lambda=2$: $\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2+r_3} \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$

$x=0$
$-y=2z \Rightarrow y=-2t$
$z=t$

$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} t$

So, $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{pmatrix}$ is s.t. $P^{-1}AP$ is diagonal.