$\boxed{A}$ Are the following vector spaces?

(a) Set of 2×2 invertible matrices, w/ standard matrix addition & scalar multiplication.

No! Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible (i.e. $\det A \neq 0$).

$$0 * A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad \det(0*A) = 0$$

$\Rightarrow$ $0*A$ <u>not</u> invertible $\Rightarrow$ this set is <u>not</u> closed under scalar $*$ $\Rightarrow$ <u>not</u> a vector space.

(b) Pairs of real numbers of the form $(1,x)$ s.t. $(1,y)+(1,y') = (1, y+y')$, $K(1,y) = (1, Ky)$.

$\boxed{1}$ $(1,y)+(1,y') = (1, y+y')$ ✓

$\boxed{2}$ $(1,y)+(1,y') = (1,y+y') = (1,y'+y) = (1,y')+(1,y)$. ✓

$\boxed{3}$ $(1,z)+(1,y'+y) = (1, y+y'+z) = (1, z+y')+(1,y)$. ✓

$\boxed{4}$ $(1,0)+(1,y) = (1,0+y) = (1,y) \Rightarrow \bar{0} = (1,0)$.

$\boxed{5}$ $(1,y)+(1,-y) = (1,0) = \bar{0}$. ✓

$\boxed{6}$ $(1,Ky)$ ✓

$\therefore$ A vector space!

$\boxed{7}$ $K(u+v) = Ku + Kv$

$\boxed{8}$ $(K+m)u = Ku + mu$

$\boxed{9}$ $K(mu) = (Km)u$.

$\boxed{10}$ $1u = 1(1,y) = (1, 1\cdot y) = (1,y)$. ✓

$\boxed{B}$ Find a basis for the following subspaces of $\mathbb{R}^3$:

4.5 #7

ⓐ The plane $3x - 2y + 5z = 0$.

$3x = 2y - 5z$

$x = \frac{2}{3}y - \frac{5}{3}z = \frac{2}{3}t - \frac{5}{3}s$.

$y = t$.

$z = s$.

$\begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix} t, \begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix} s.$

$\left\{ \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix} \right\}.$

ⓑ The plane $x - y = 0$.

$x = y = t$.

$y = t$

$z = s$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} s.$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

ⓒ The line $x = 2t$

$y = -t$

$z = 4t$.

$\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} t.$ $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right\}.$

ⓓ $\{(a,b,c) \in \mathbb{R}^3 \mid b = a+c\}$.

$a = t$

$b = a + c = t + s$.

$c = s$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} s.$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$

4.5 #20

$\boxed{c}$ Find a basis for the subspace of P_2 spanned by the vectors

$1+x,\ x^2,\ -2+2x^2,\ -3x.$ $\therefore W$

$w \in \text{span}\{1+x,\ x^2,\ -2+2x^2,\ -3x\} \Rightarrow \exists$ scalars k_1, k_2, k_3 s.t.

$$k_1(1+x) + k_2(x^2) + k_3(-2+2x^2) + k_4(-3x) = w.$$

$$(k_1 - 2k_3) + (k_1 - 3k_4)x + (k_2 + 2k_3)x^2 = w_1 + w_2 x + w_3 x^2.$$

$$\underbrace{\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 \end{bmatrix}}_{\text{Want to find a basis for the column space of this matrix.}} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 \end{bmatrix} r_2 \leftarrow r_2 - r_1$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 1 & 2 & 0 \end{bmatrix} \begin{matrix} r_1 \leftarrow r_1 + r_2 \\ r_3 \leftarrow r_3 - r_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 0 & 2 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix} r_2 \leftrightarrow r_3$$

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & -3 \end{bmatrix} r_3 \leftarrow r_3 \cdot \tfrac{1}{2}$$

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -\tfrac{3}{2} \end{bmatrix}.$$

So, a basis for W is

$$\{1+x,\ x^2,\ -2+2x^2\}.$$

$$-3x = -\tfrac{3}{2}(-2+2x^2) + 3(x^2) - 3(1+x).$$

$\dim W = 3 = \dim(P_2) \Rightarrow W = P_2.$