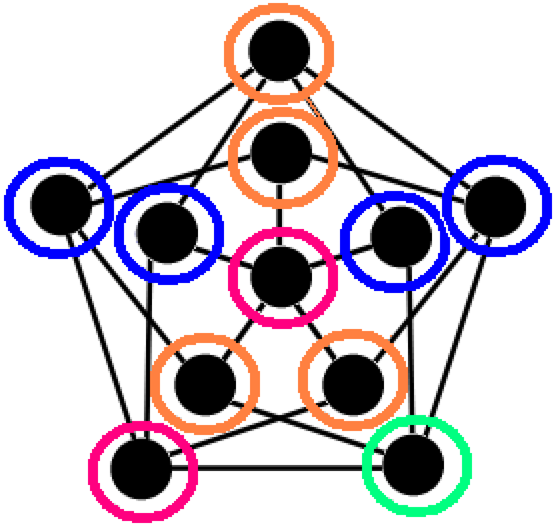
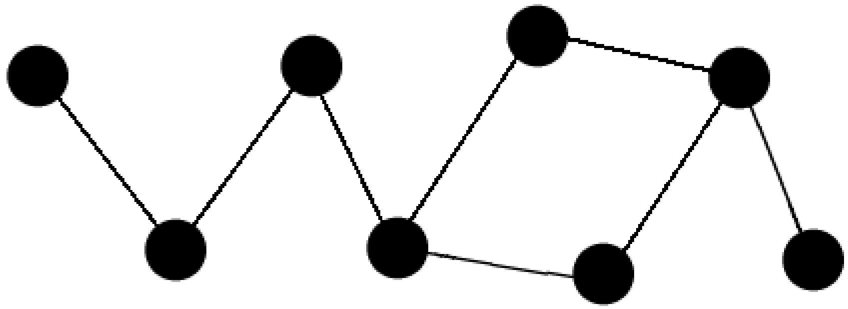
The Evolution of Grötzsch's Theorem



by Lauren DeDieu
Advisor: Dr. James Preen

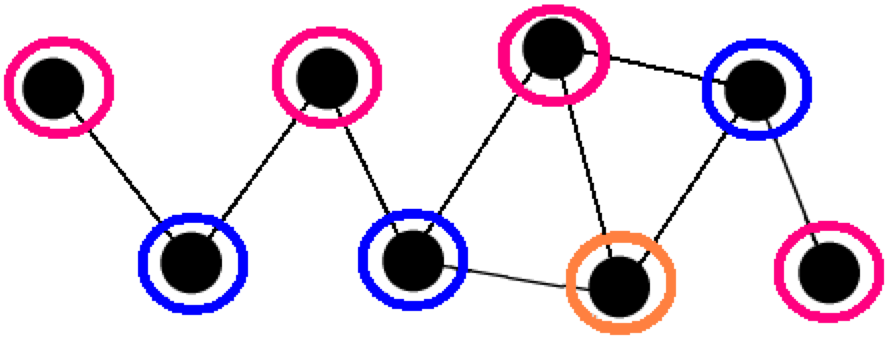
What is a graph?



vertices

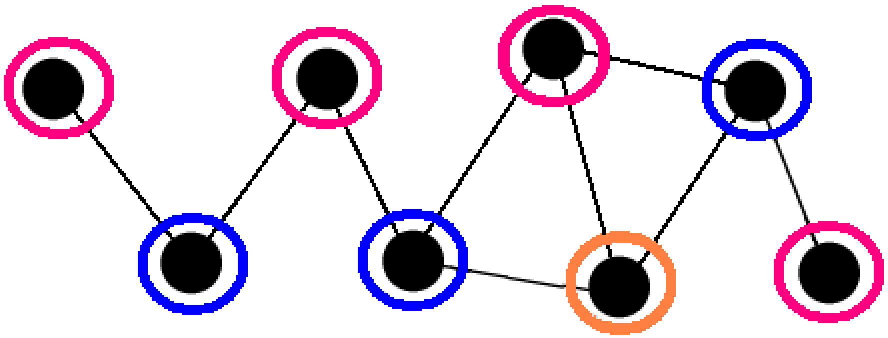
edges

What is graph colouring?

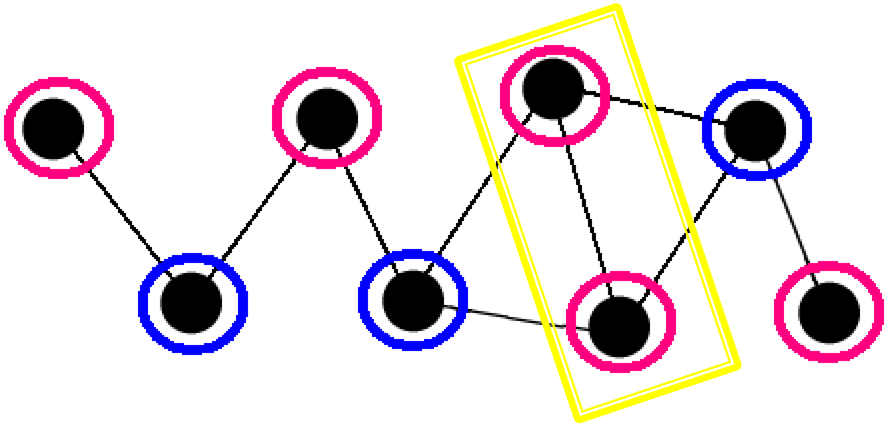


Proper Colouring

What is graph colouring?

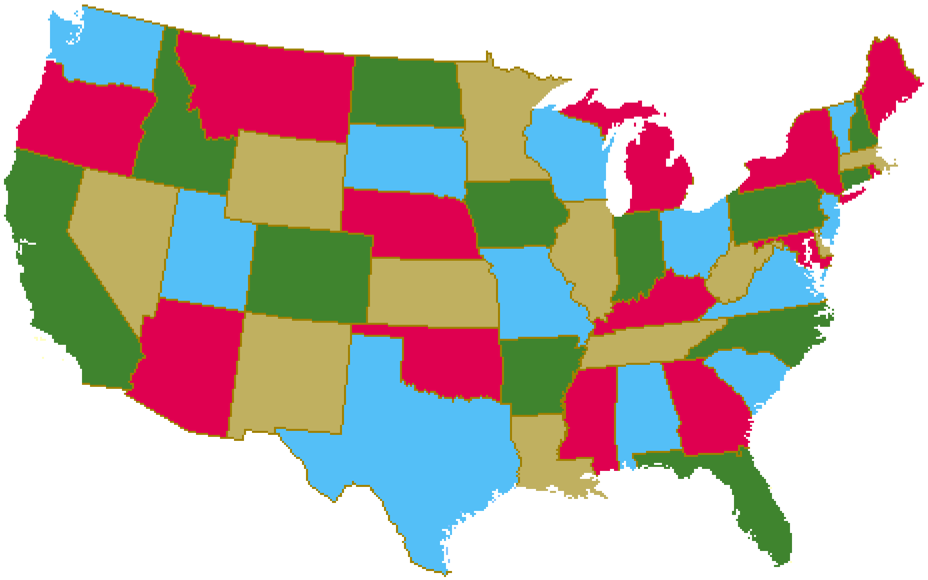


Proper Colouring



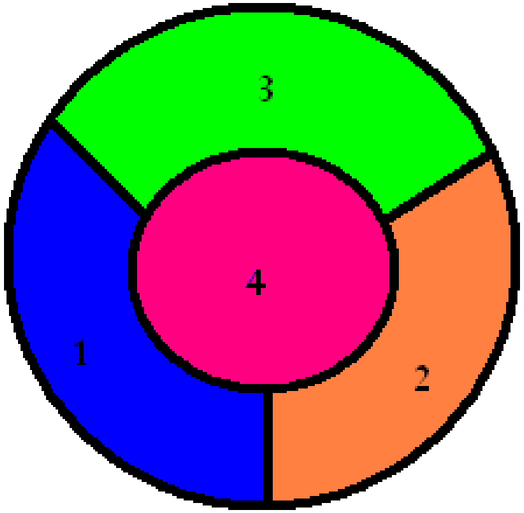
Improper Colouring

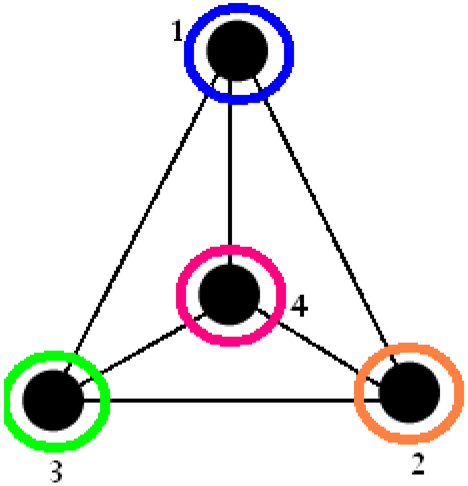
Origins of Graph Colouring



Can a geographical map be coloured using only 4 colours?

In 1878, A. Cayley represented the four colour problem using vertices and edges.





● **Country**

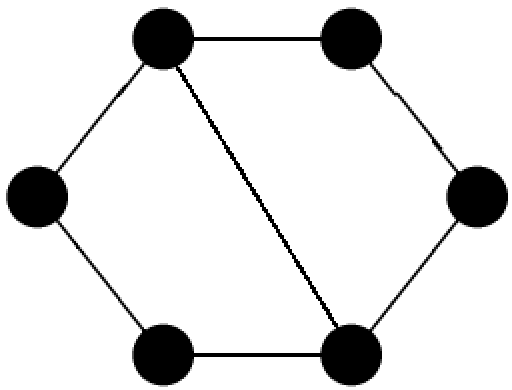
— **Border between countries**

Grötzsch's Theorem

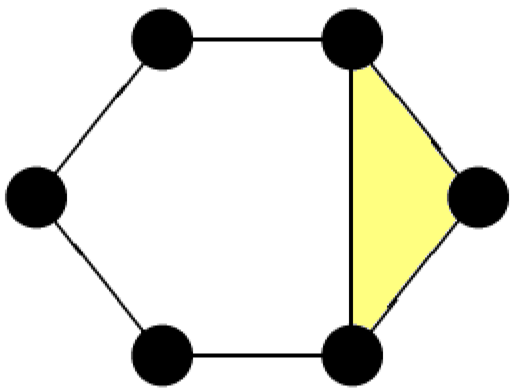
Any triangle-free planar graph can be properly coloured using at most 3 colours.

Grötzsch's Theorem

Any triangle-free planar graph can be properly coloured using at most 3 colours.



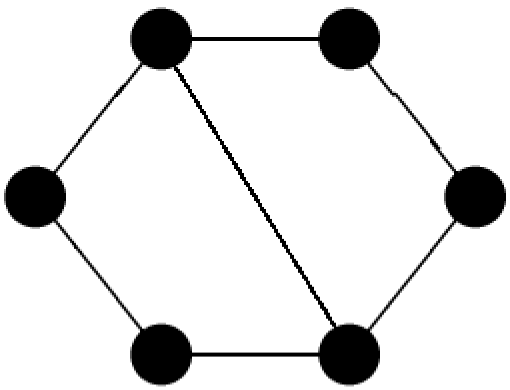
Triangle-free



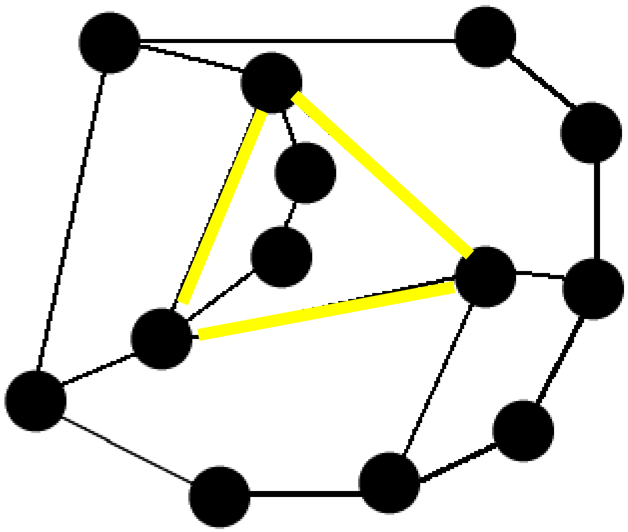
Not triangle-free

Grötzsch's Theorem

Any triangle-free planar graph can be properly coloured using at most 3 colours.

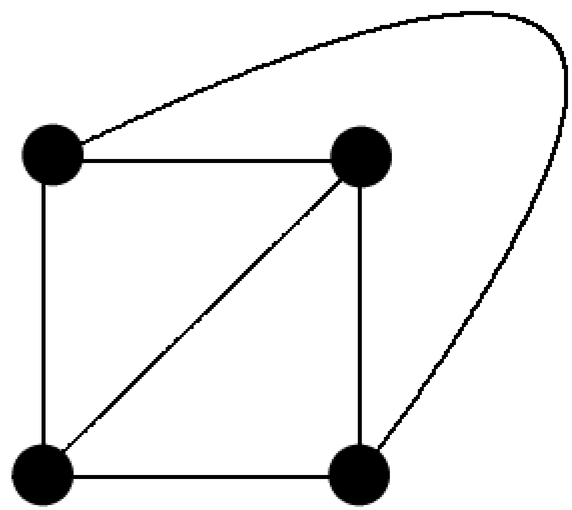


Triangle-free



Not triangle-free

Graphs with triangles are problematic because:



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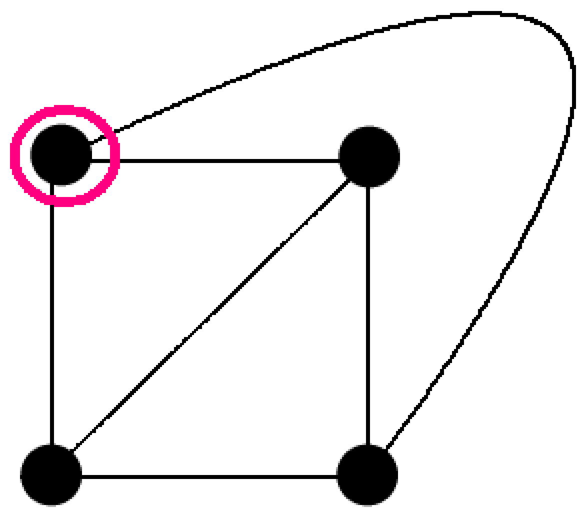

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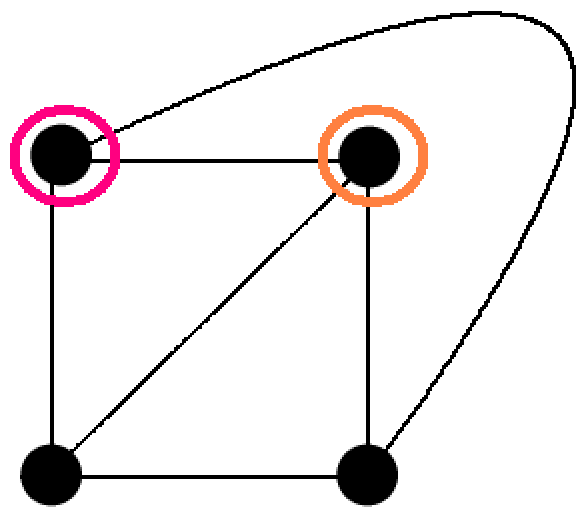

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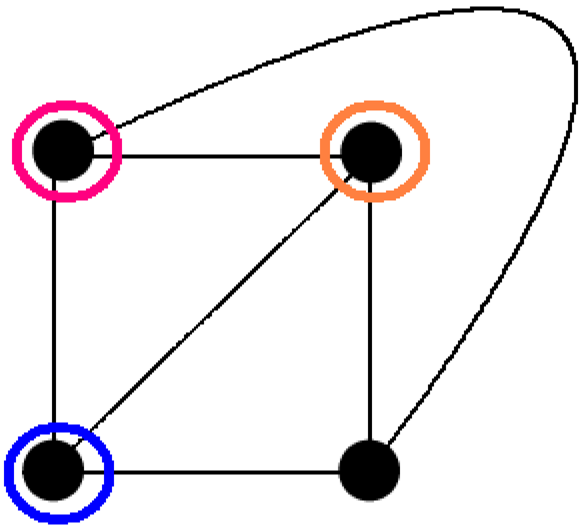

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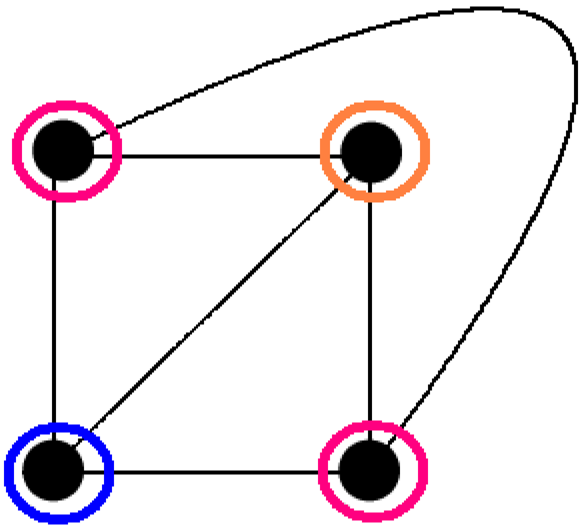
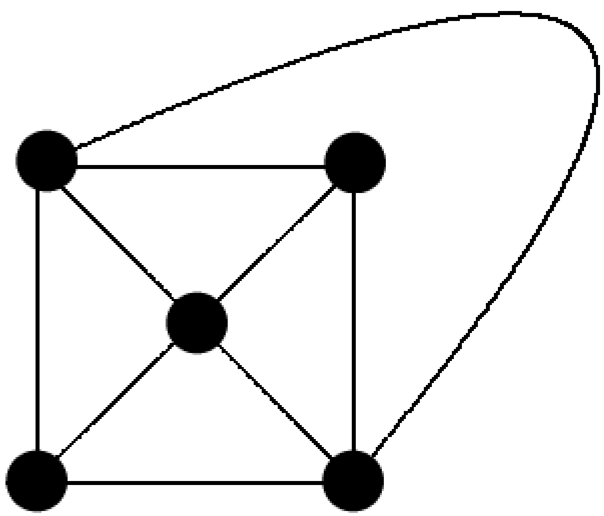


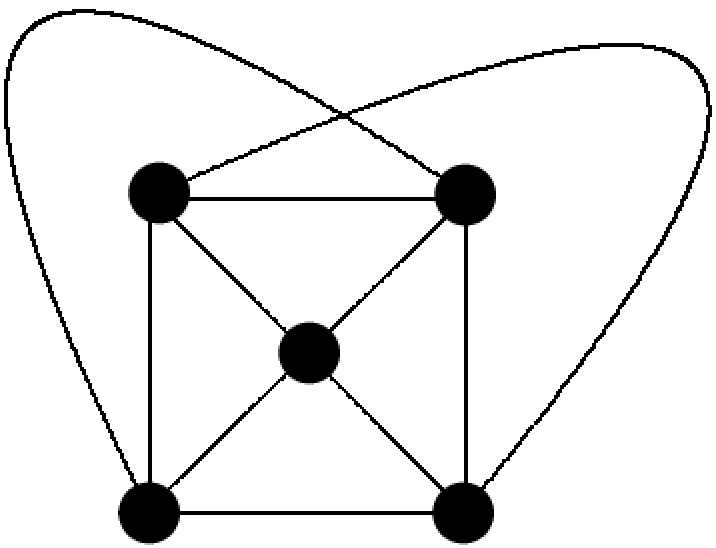
Not a proper colouring!

Grötzsch's Theorem

Any triangle-free planar graph can be properly coloured using at most 3 colours.

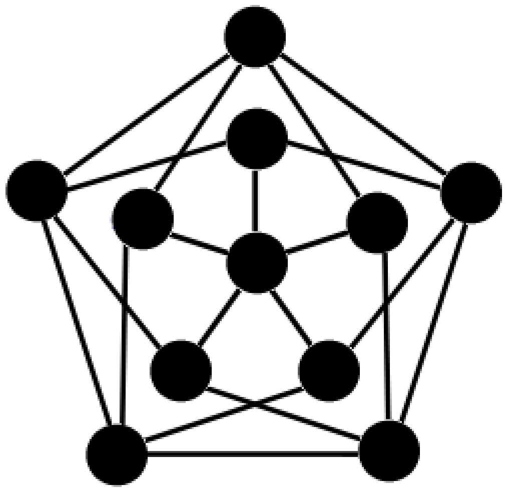


Planar



Non-Planar

Non-planar graphs are problematic because:



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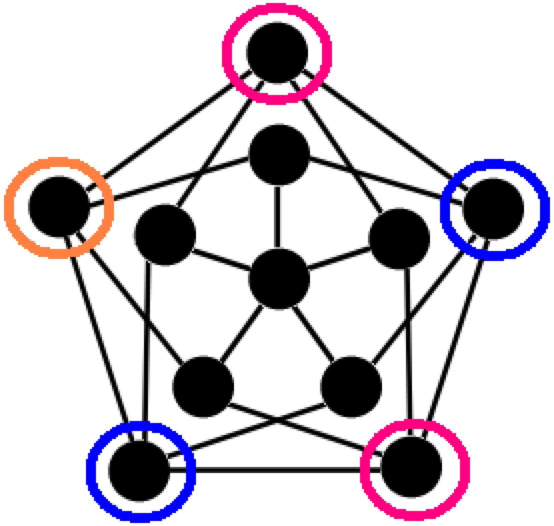

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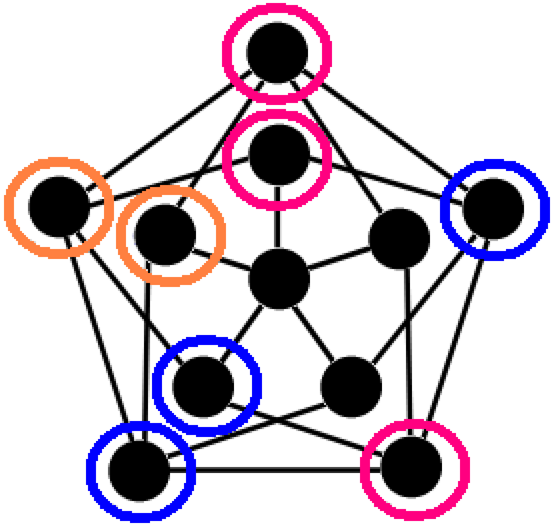

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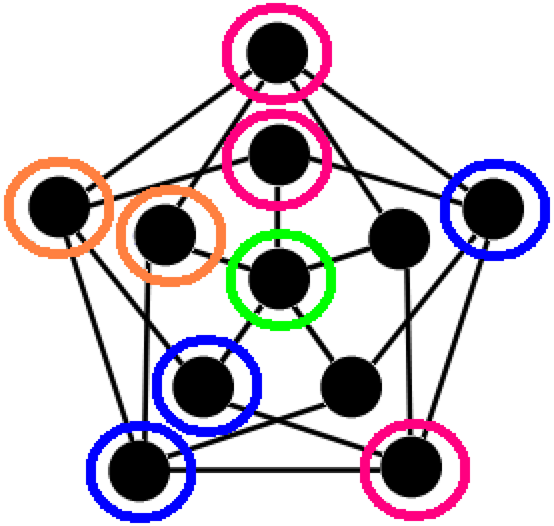


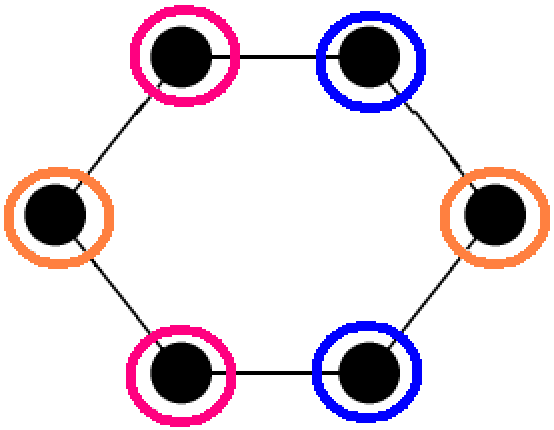
Triangle-free, but not 3-colourable!

Proof of Grötzsch's Theorem

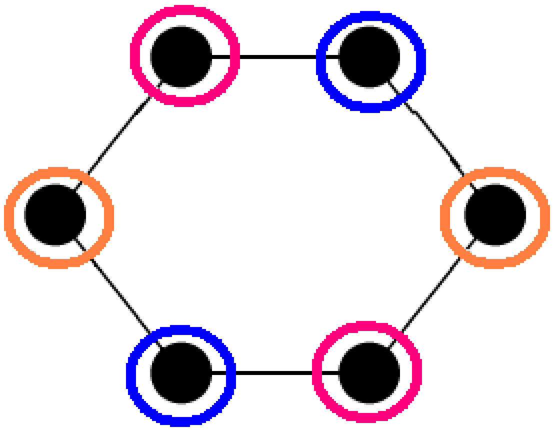
↳ **L. Kowalik**

- **Any <u>triangle-free</u> <u>planar</u> graph G is 3-colourable.**
- **Moreover, if the <u>boundary</u> of the <u>outer face</u> of G is a <u>cycle C</u> of length at most <u>6</u>, then any <u>safe</u> 3 -colouring of the boundary can be extended to a 3-colouring of G.**

Proof of Grötzsch's Theorem



Safe
(p,b,o,b,p,o)



Not Safe
(p,b,o,p,b,o)

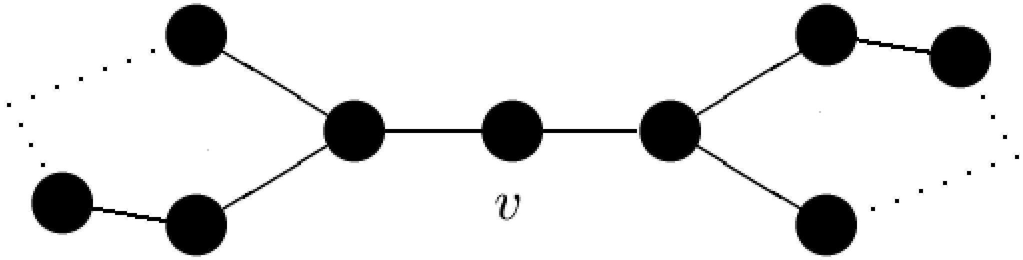
Proof of Grötzsch's Theorem

↳**L. Kowalik**

- **<u>Induction</u> on n, the number of vertices in G.**
- **Assume <u>true</u> for <u>n-1</u> or fewer <u>vertices</u>.**

Case 1

G has an uncoloured vertex v with degree at most 2.



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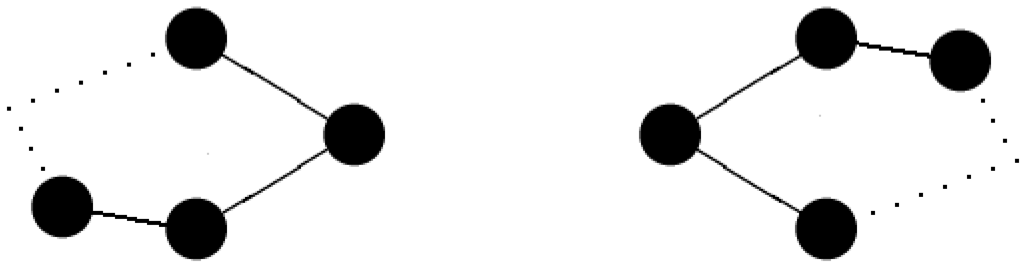
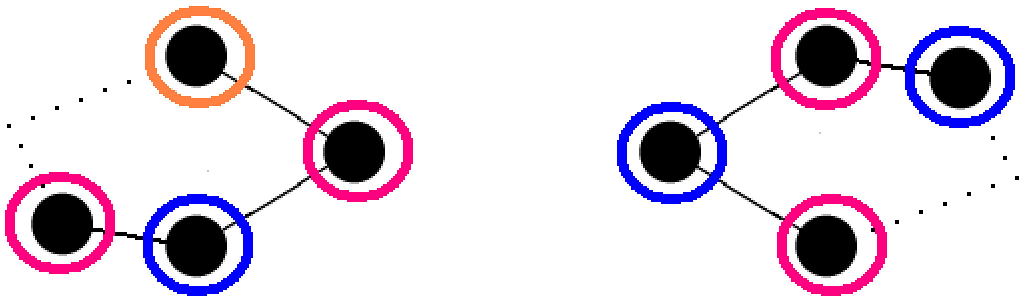


Remove v.

Case 1

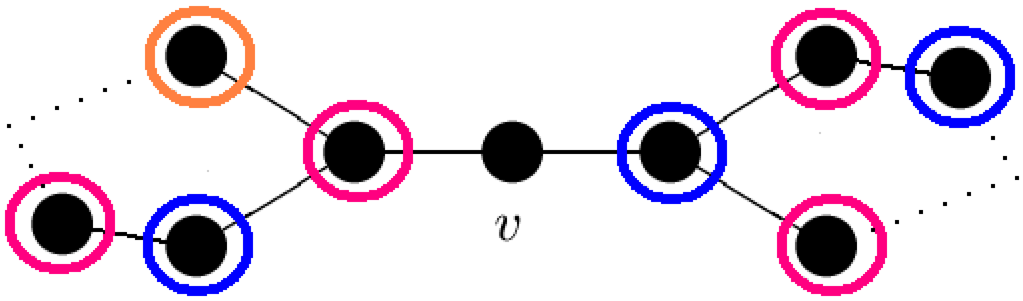
G has an uncoloured vertex v with degree at most 2.



Induction.

Case 1

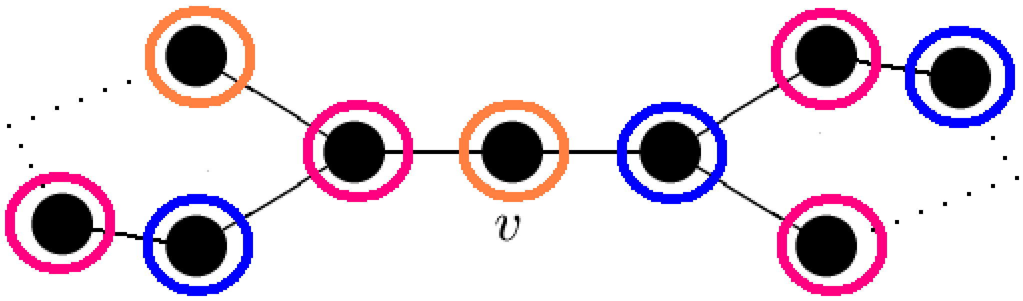
G has an uncoloured vertex v with degree at most 2.



Put v back in G.

Case 1

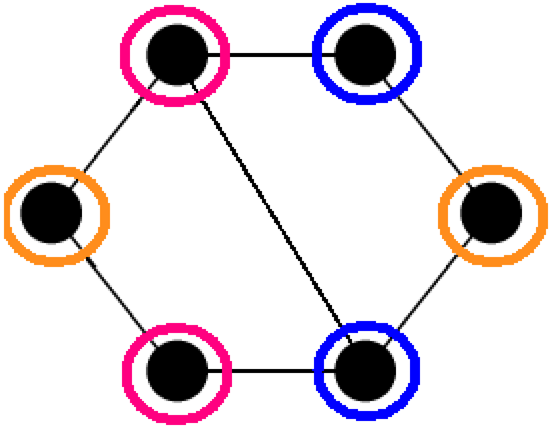
G has an uncoloured vertex v with degree at most 2.



Colour v with an available colour.

Case 2

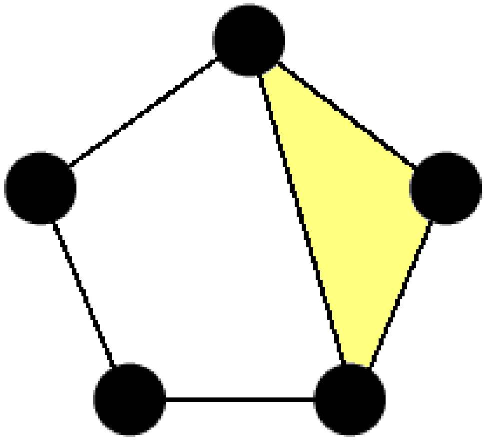
The boundary C is coloured and has a chord.



C has size 6 and is safely coloured.

Case 2

The boundary C is coloured and has a chord.



If C has a chord, it must have size 6, because otherwise there would be a triangle in G.

Case 2

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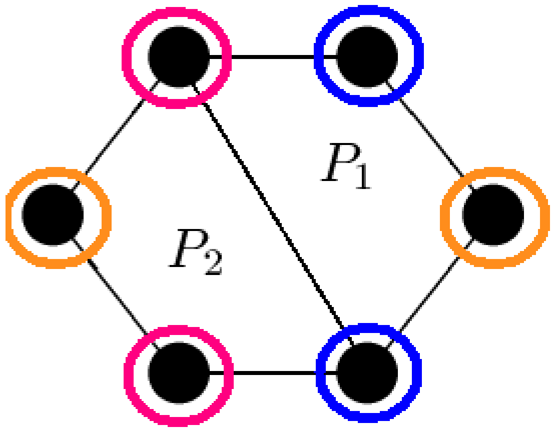

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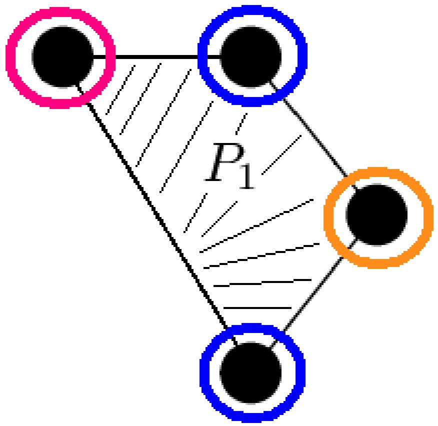
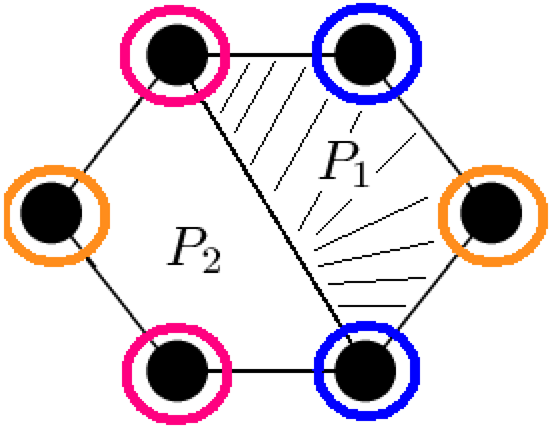


Colour P_1 by induction.

Case 2

The boundary C is coloured and has a chord.



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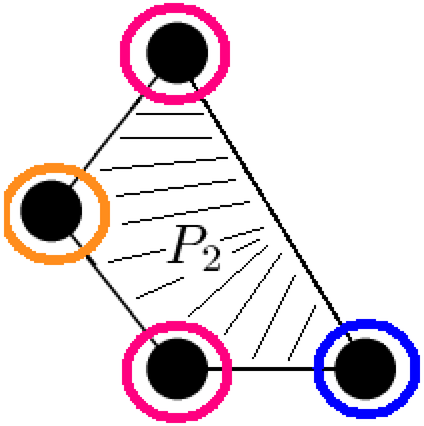
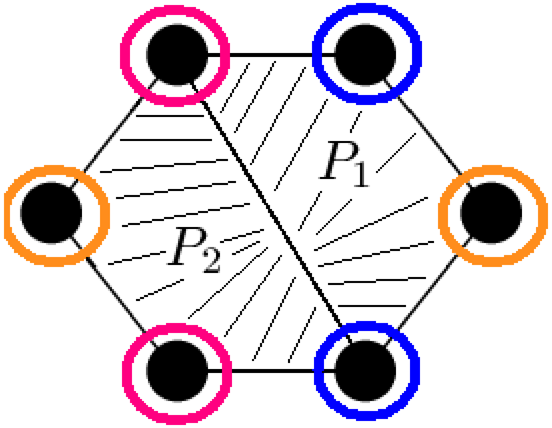


Colour P_2 by induction.

Case 2

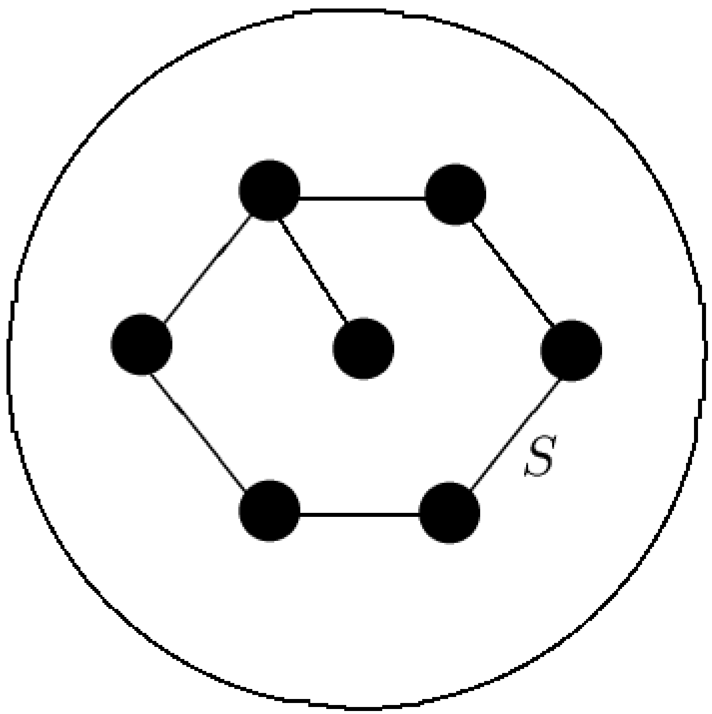
The boundary C is coloured and has a chord.



G is properly coloured.

Claim 1

If G has a separating cycle S, where S has size at most 6, then we can complete the proof by induction.



S is a separating cycle.

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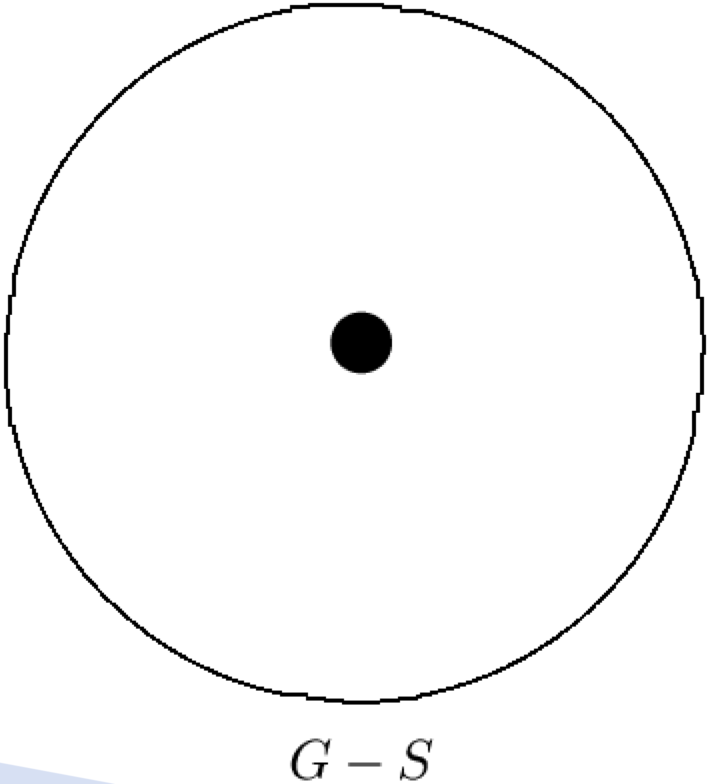

Removing S disconnects the graph.

Claim 1

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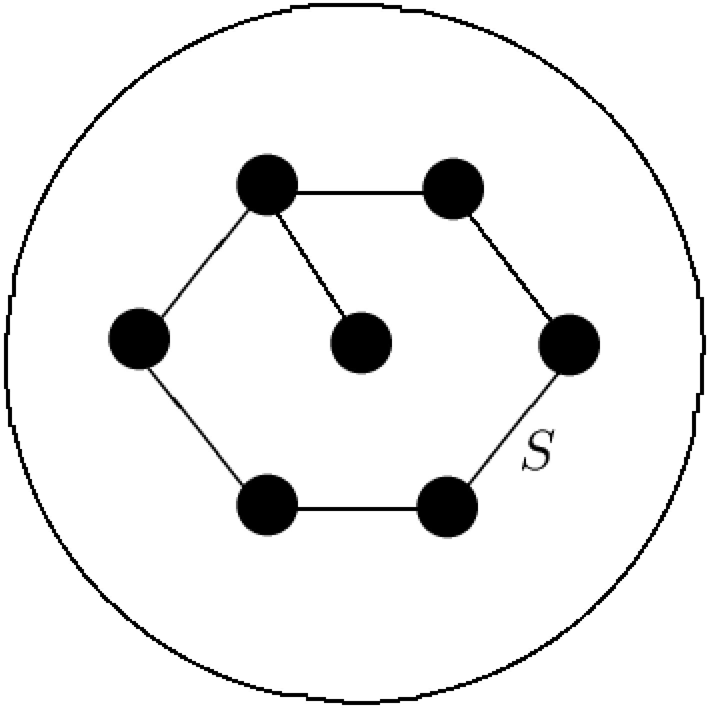

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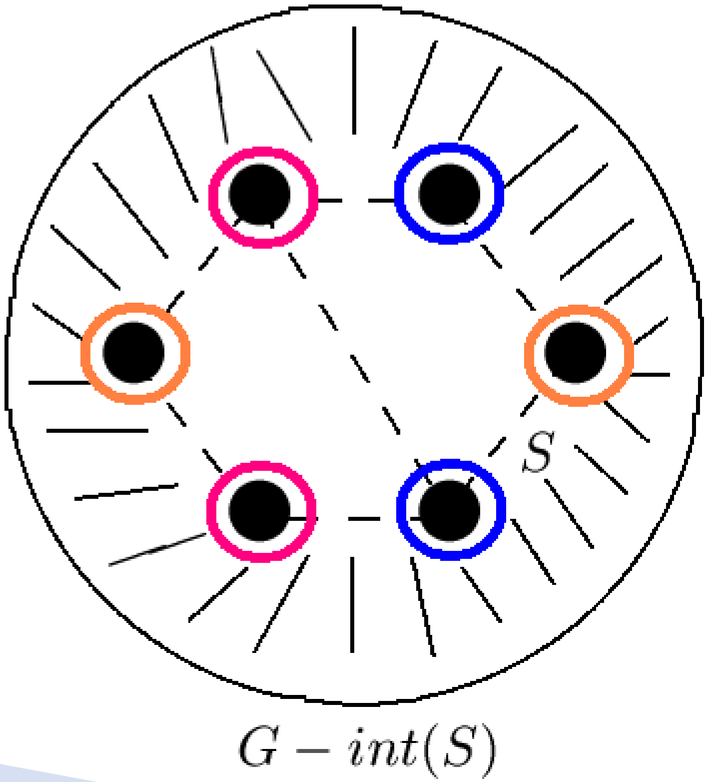
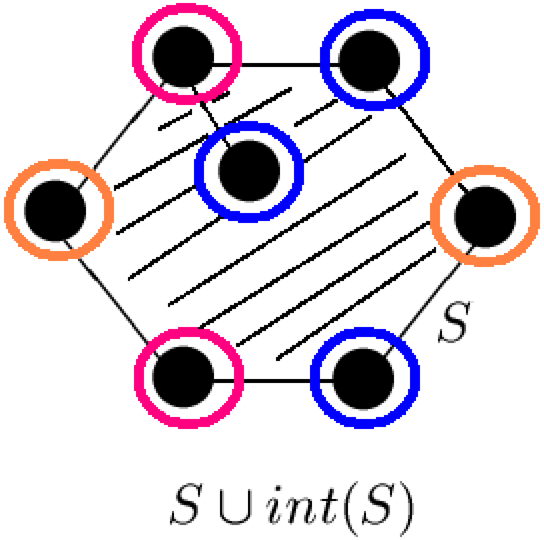


First colour everything except the interior of S by induction.

Claim 1

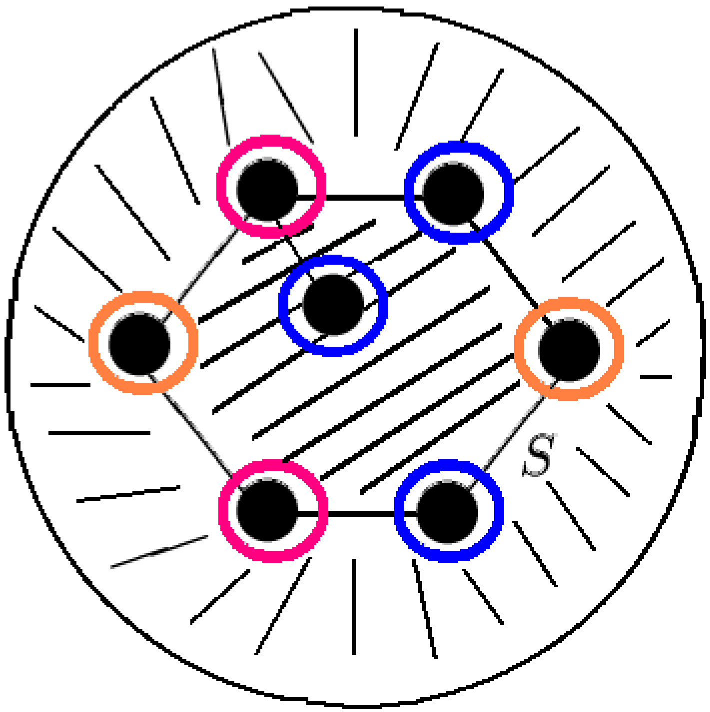
If G has a separating cycle S, where S has size at most 6, then we can complete the proof by induction.



Now colour S and its interior by induction.

Claim 1

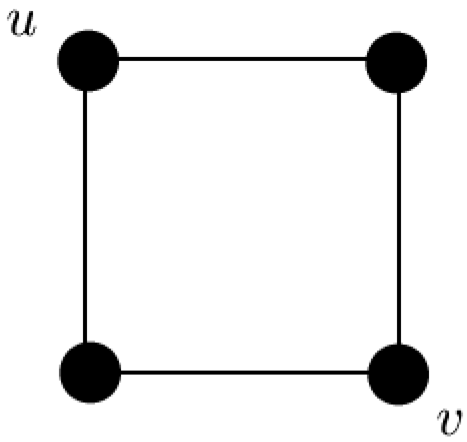
If G has a separating cycle S, where S has size at most 6, then we can complete the proof by induction.



G is now properly coloured.

Identifying Vertices

Identifying vertices makes the graph G smaller, and allows us to use induction.



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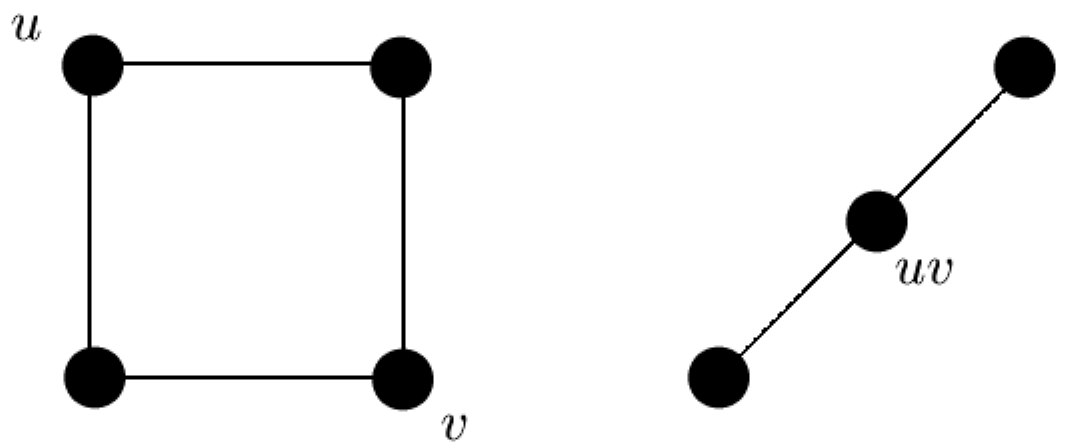

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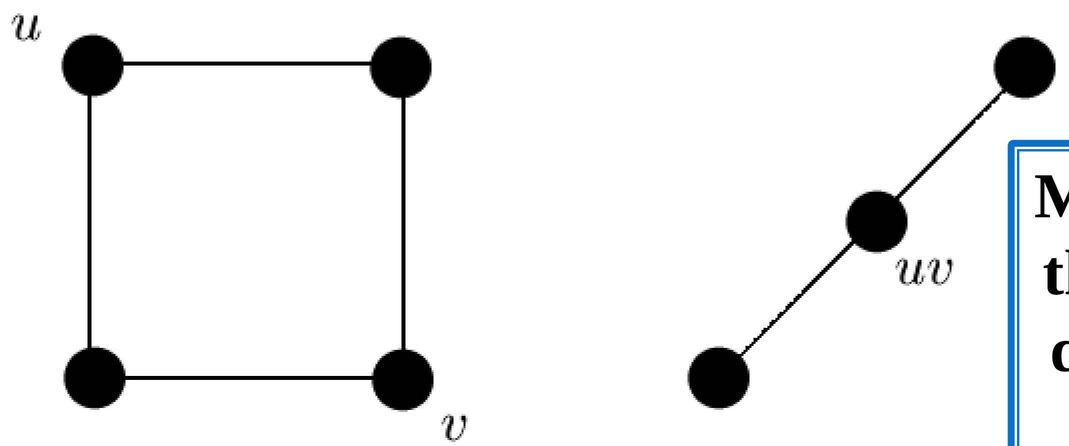


Must make sure that identifying does not create a <u>chord</u> through the boundary C, or a <u>triangle</u>.

Proof of Grötzsch's Theorem

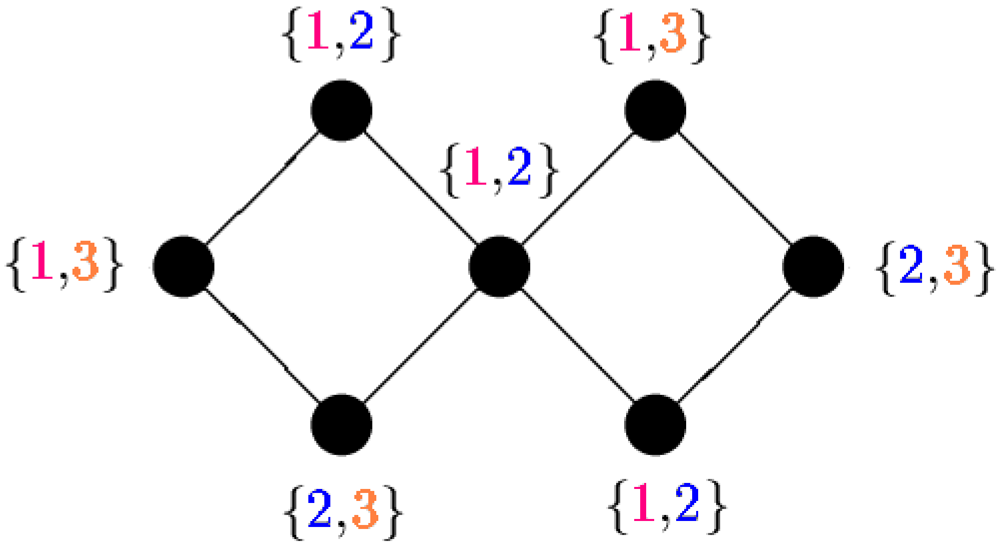
Kowalik uses these techniques when considering each of the following cases:

- **G has a face of size 6 or greater.**
- **G has a face of size 4.**
- **G has a face of size 5.**

Once all of these cases are considered, the proof is complete.

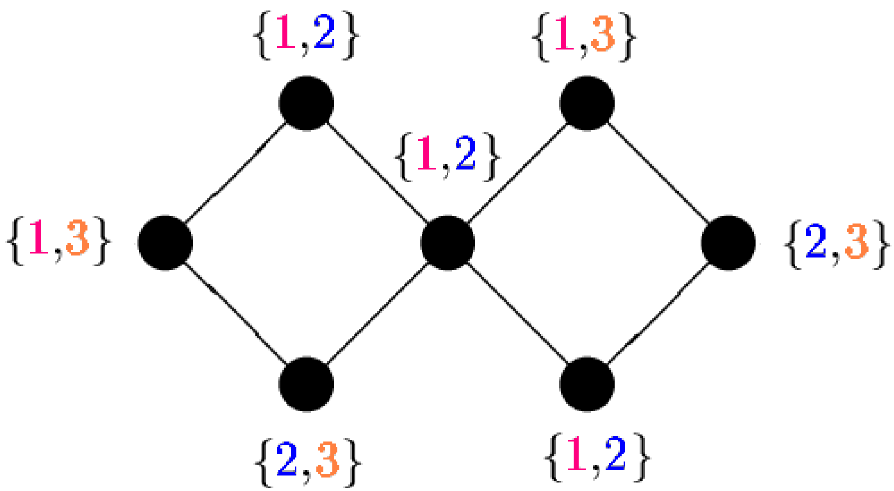
What is list colouring?

- A type of graph colouring in which each vertex is <u>assigned</u> a <u>list</u> of <u>potential colours</u>.



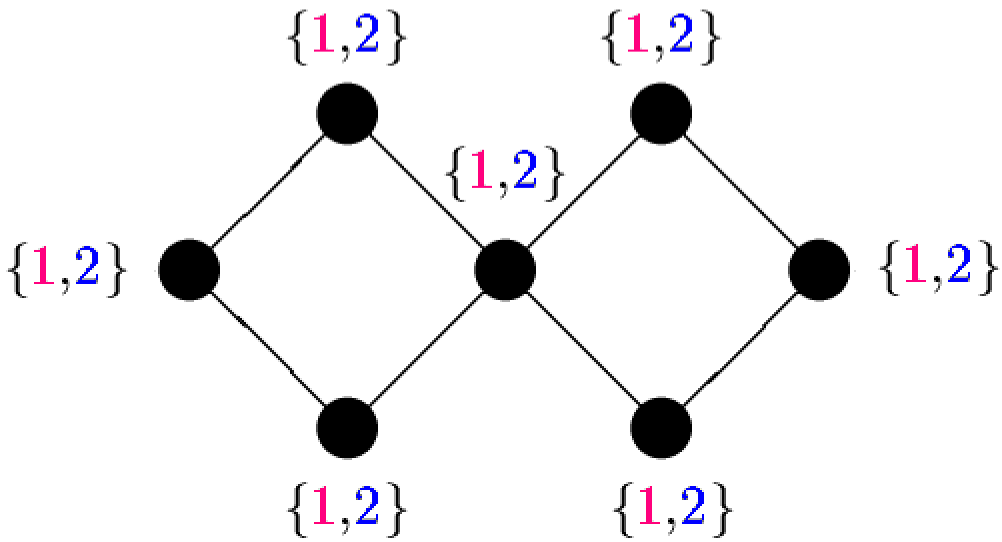
A graph is k-list colourable if:

- each vertex has a list size of at most k
- G can be properly coloured regardless of which colours are assigned to each vertex's k-sized list.



Relationship between list colouring and regular colouring:

Regular colouring is a special case of list colouring where each vertex is assigned the same list of colours.



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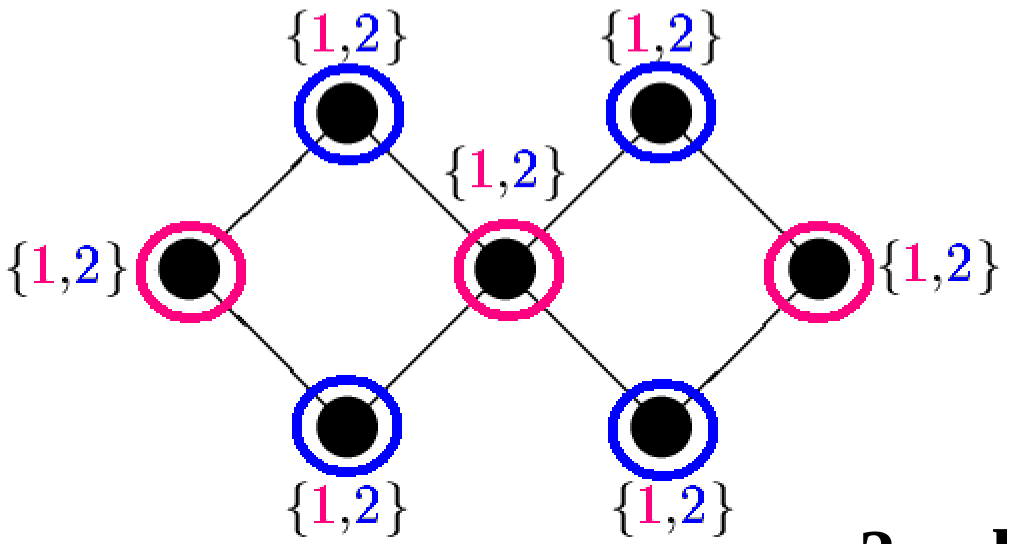
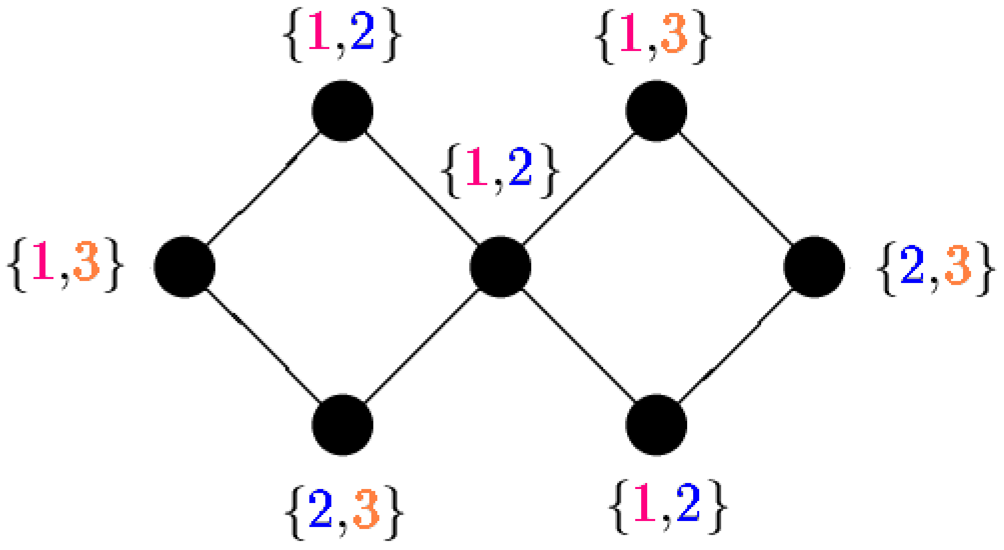


2-colourable!

Relationship between list colouring and regular colouring:



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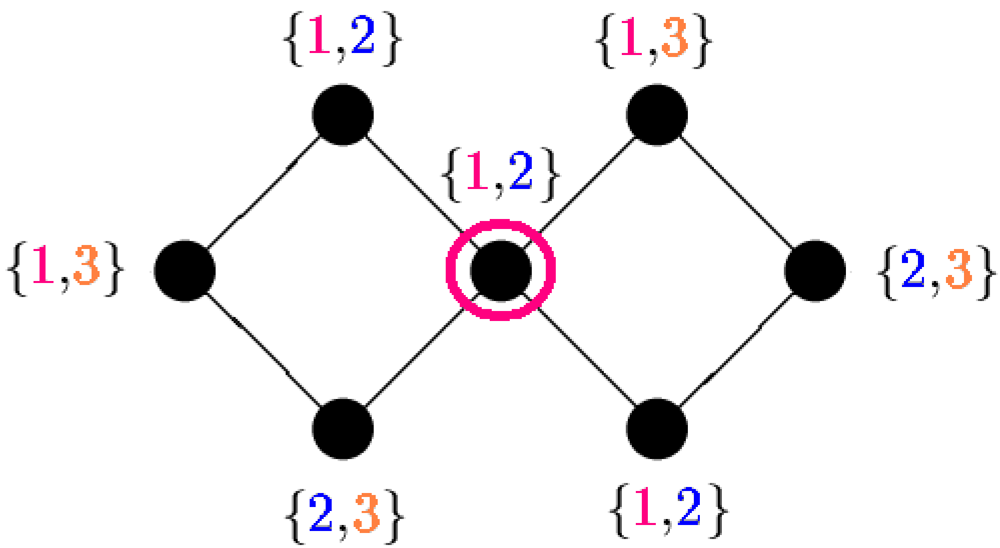

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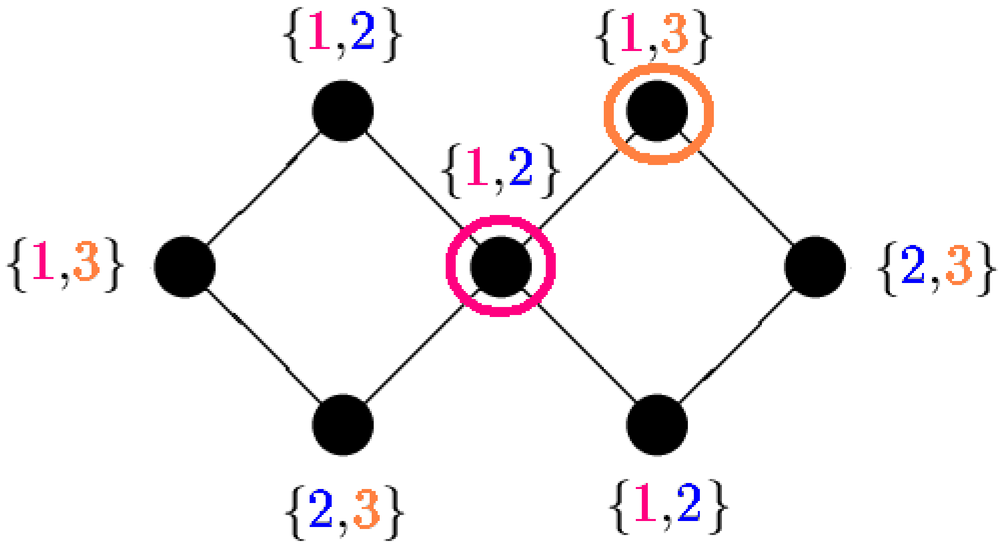

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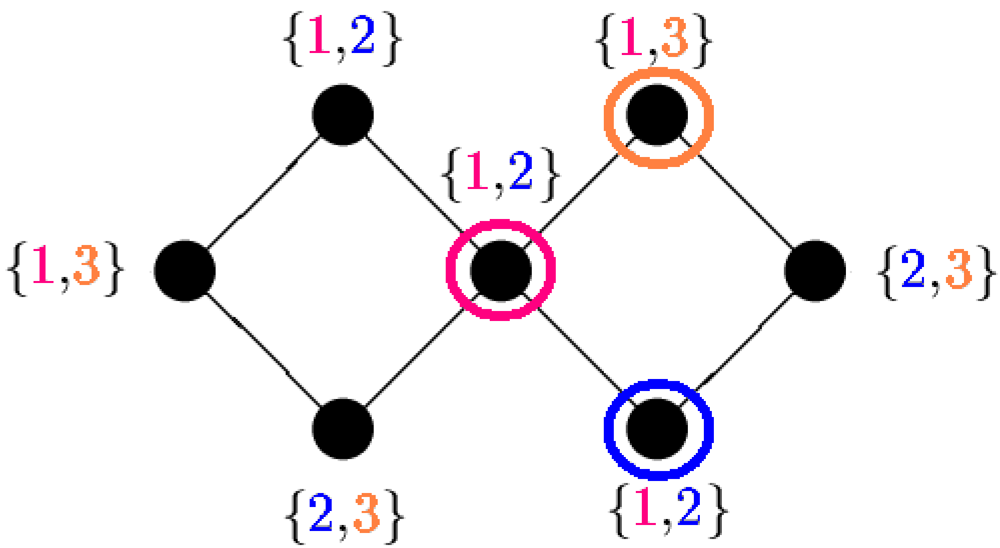

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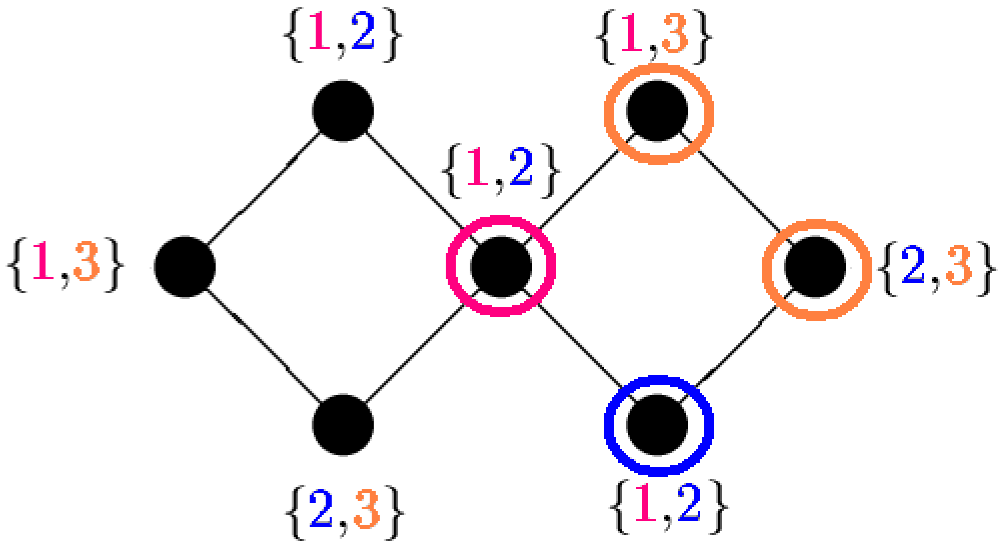
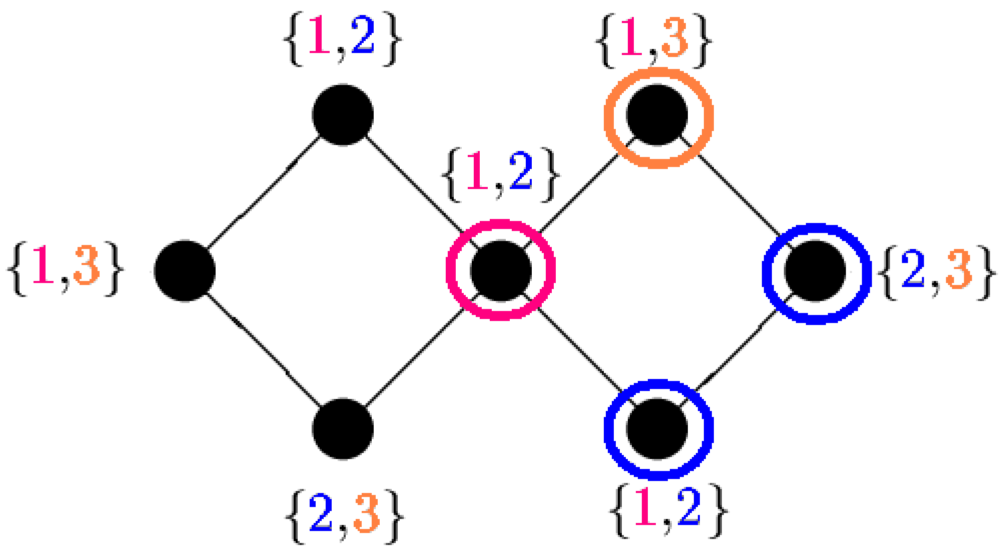


Improper colouring!

Relationship between list colouring and regular colouring:



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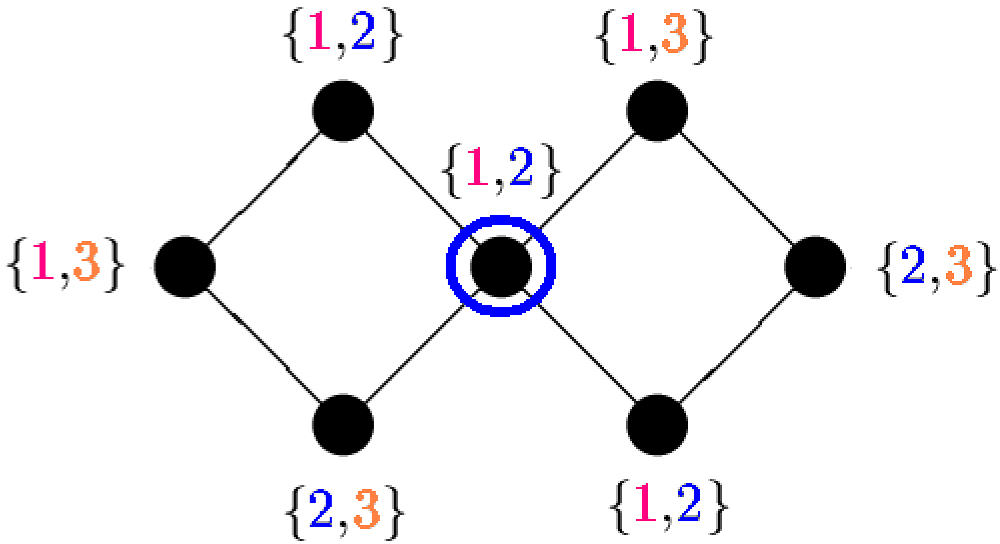

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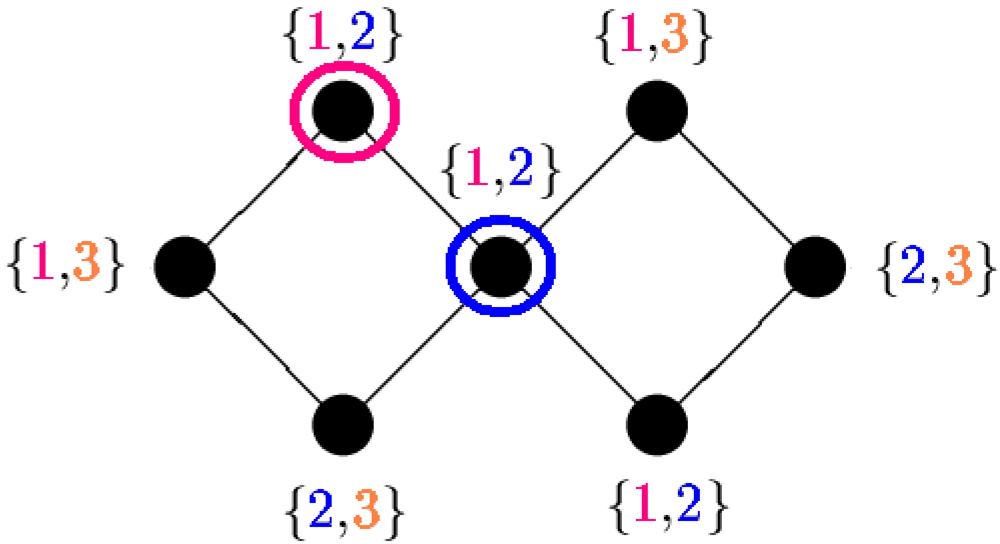

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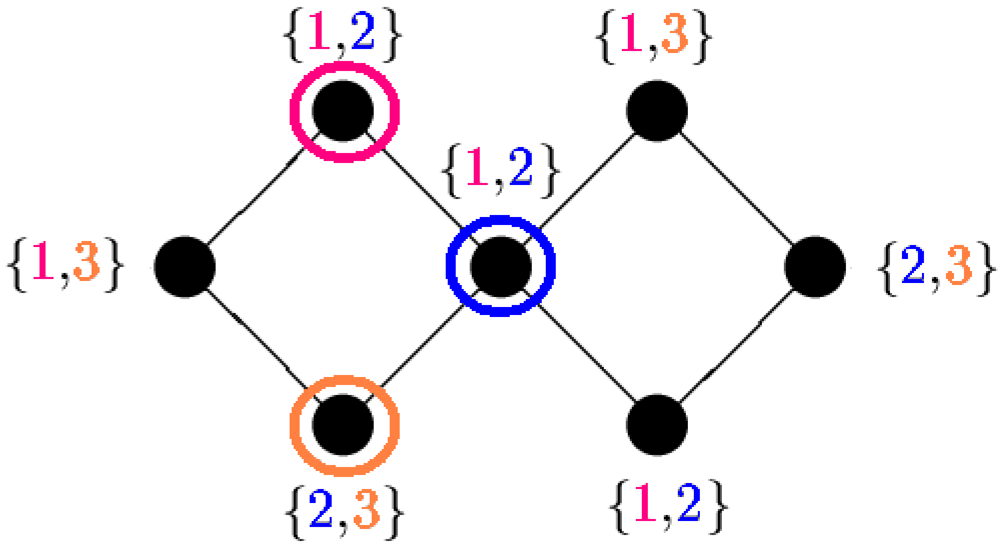

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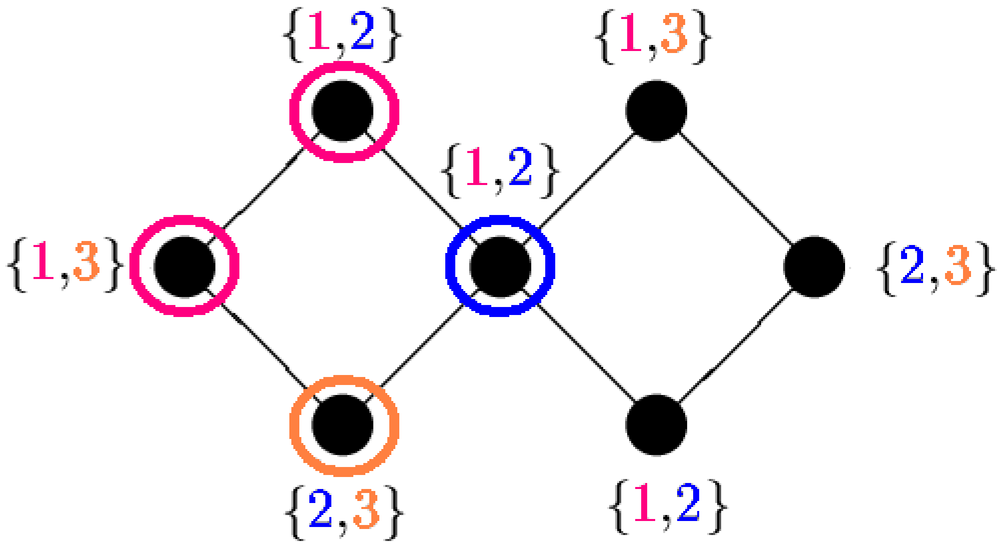

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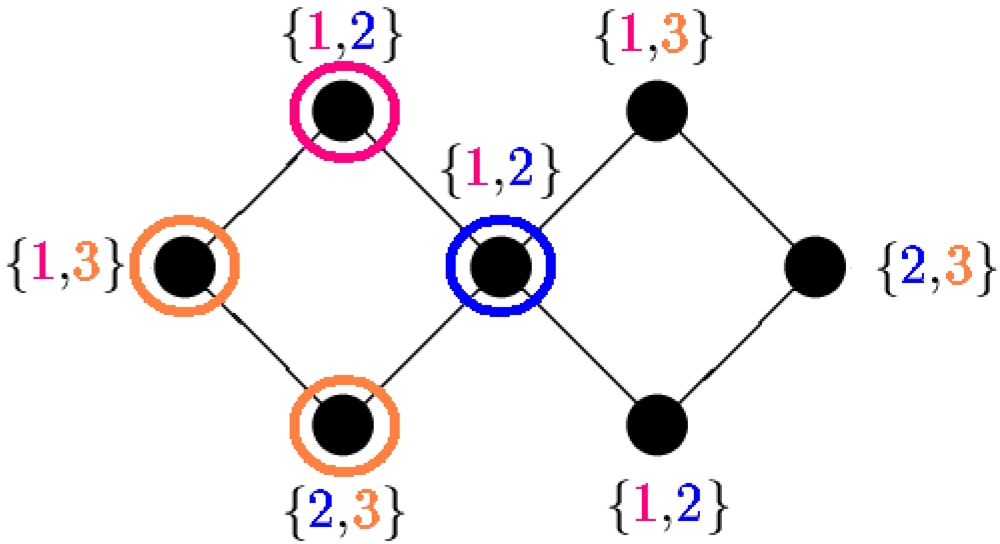

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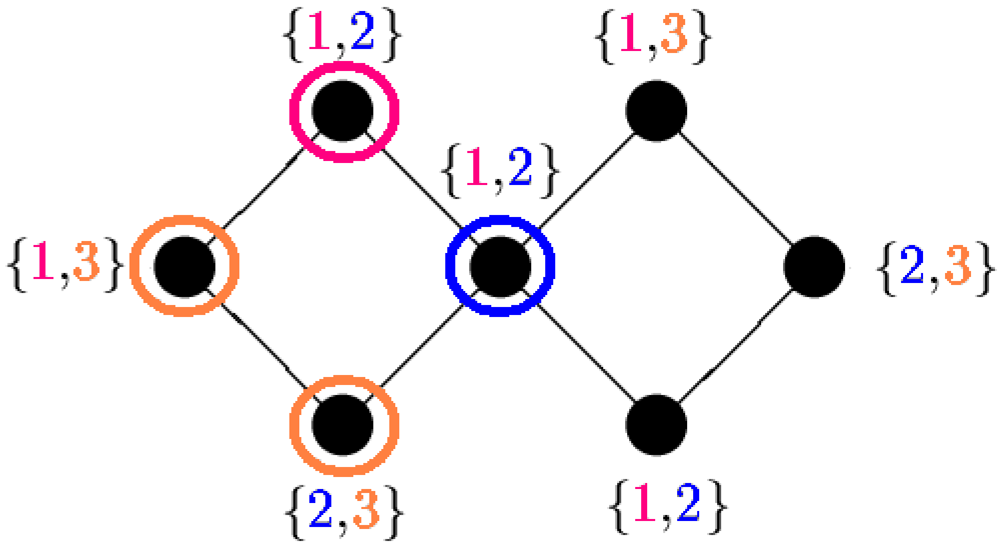
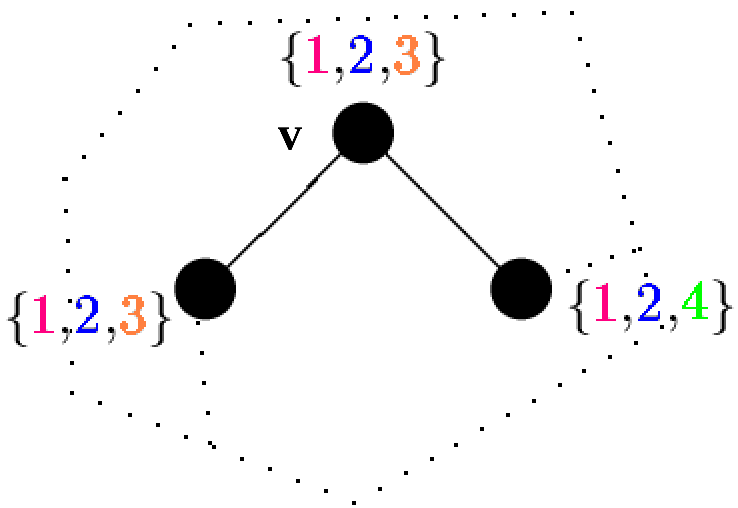


Not 2-list colourable!

List Colouring Proof Techniques

- <u>Non-list</u> colouring forces you to <u>rely</u> on the known <u>properties</u> of <u>planar graphs</u>.
- <u>List</u> colouring allows you to <u>manipulate</u> the <u>size</u> of a <u>list</u>:



List Colouring Proof Techniques

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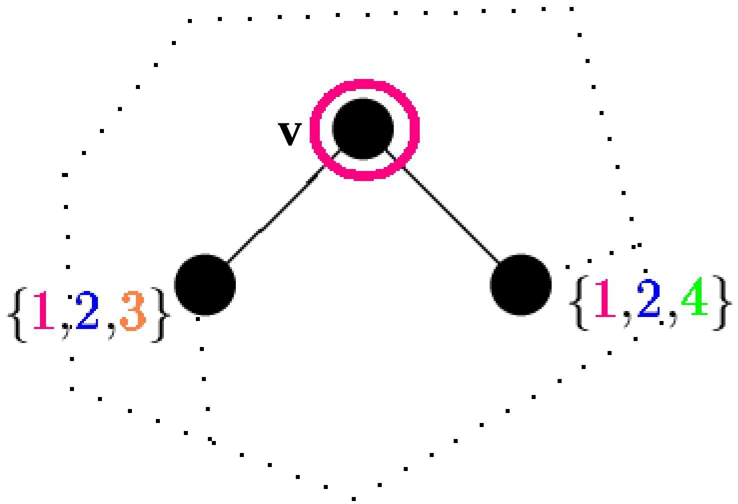

Colour v.

List Colouring Proof Techniques

- **Non-list colouring forces you to rely on the known properties of planar graphs.**
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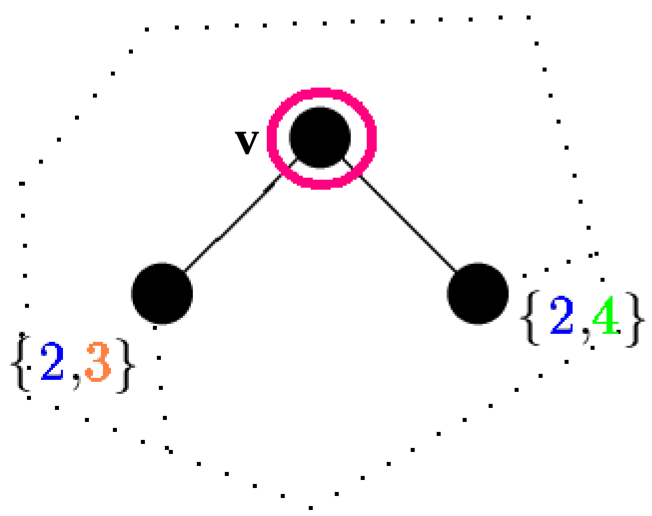

Delete v's colour from its neighbours' lists.

List Colouring Proof Techniques

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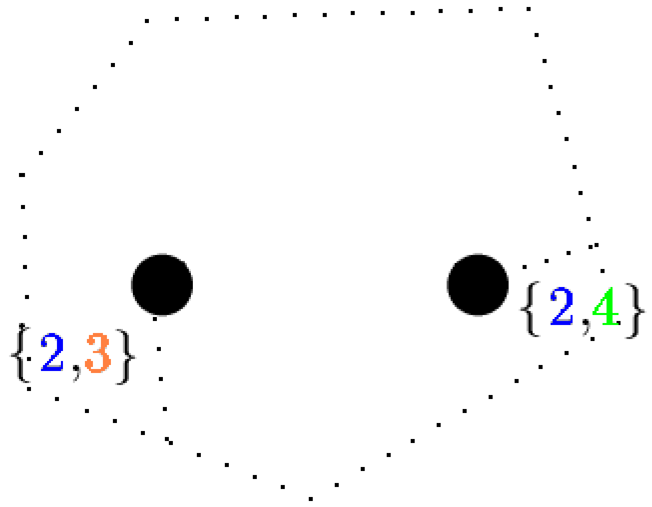

Delete v.

List Colouring Proof Techniques

- **<u>Non-list</u> colouring forces you to <u>rely</u> on the known <u>properties</u> of <u>planar graphs</u>.**
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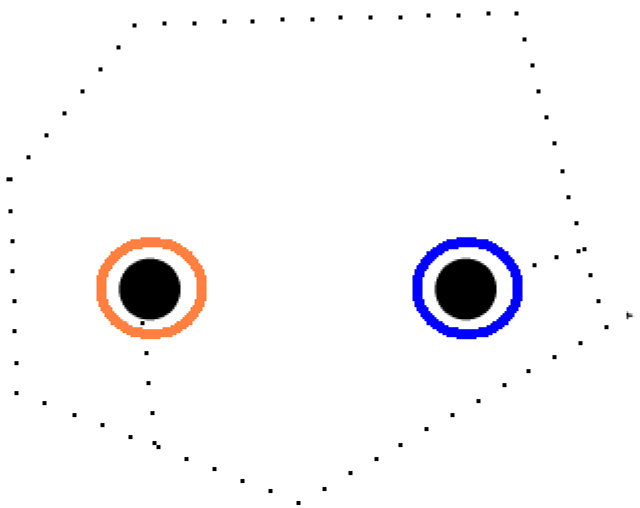

Colour by induction.

List Colouring Proof Techniques

- **<u>Non-list</u> colouring forces you to <u>rely</u> on the known <u>properties</u> of <u>planar graphs</u>.**
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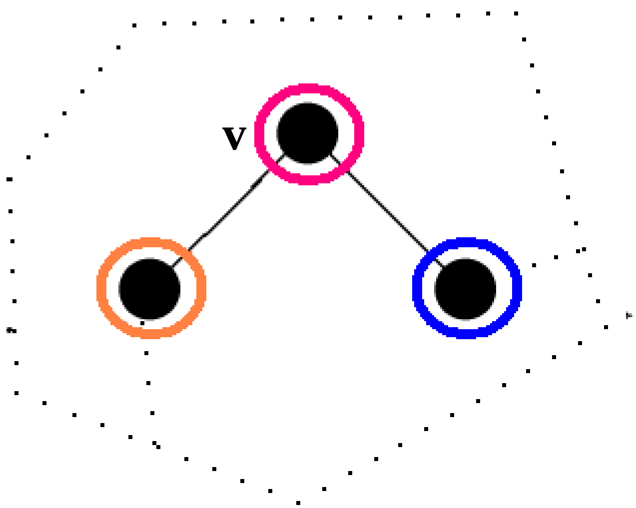


Add v back in.

Proof of Grötzsch's Theorem

↳ **C. Thomassen**

Any triangle-free planar graph G is 3-colourable.

Theorem 1

Any planar graph without triangles and without 4-cycles is 3-list-colourable.

Theorem 1

Any planar graph without triangles and without 4-cycles is 3-list-colourable.

and is therefore 3 colourable!

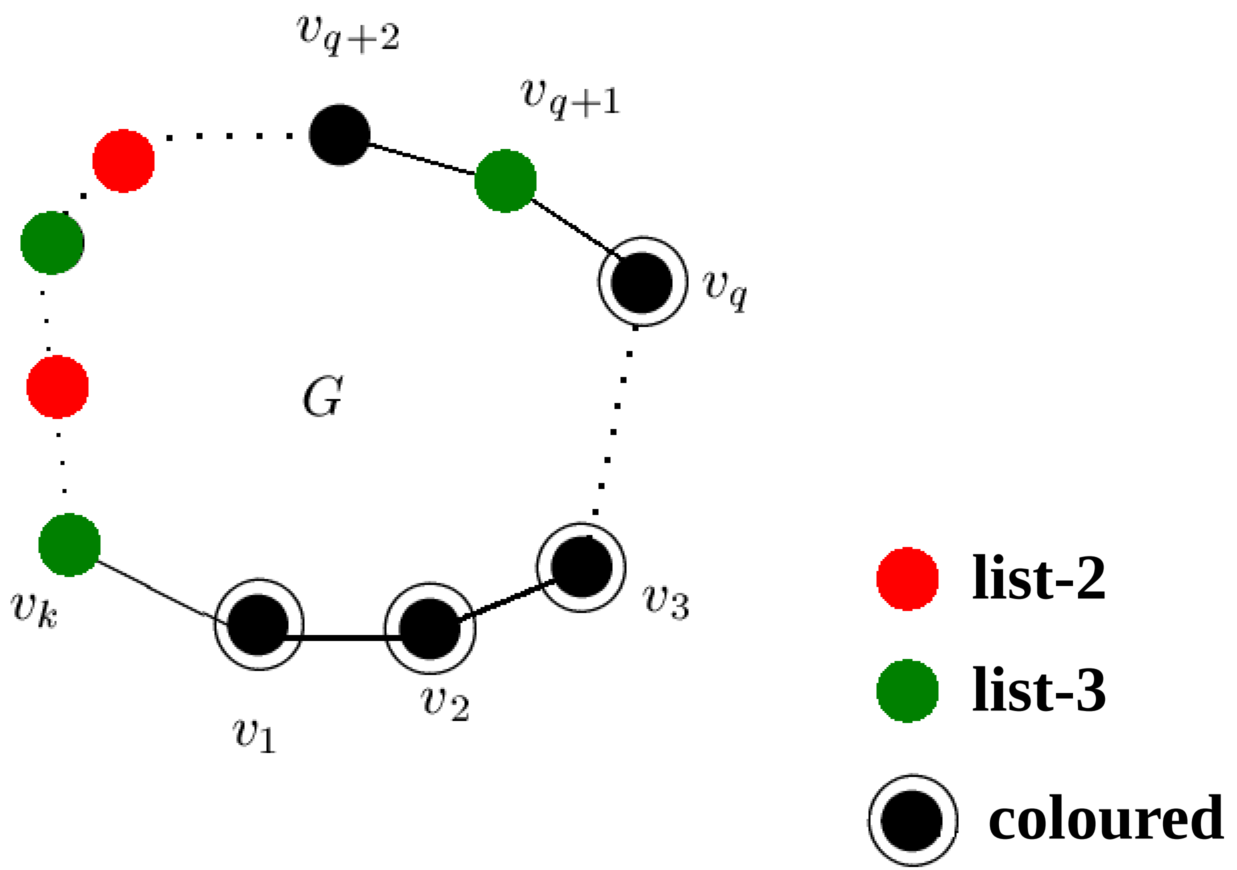
The graph G has the following properties:

- **Planar**
- **No triangles or 4-cycles.**
- **The only coloured part of G is a 3-colouring of a path P on the boundary C, where P has at most 6 vertices.**

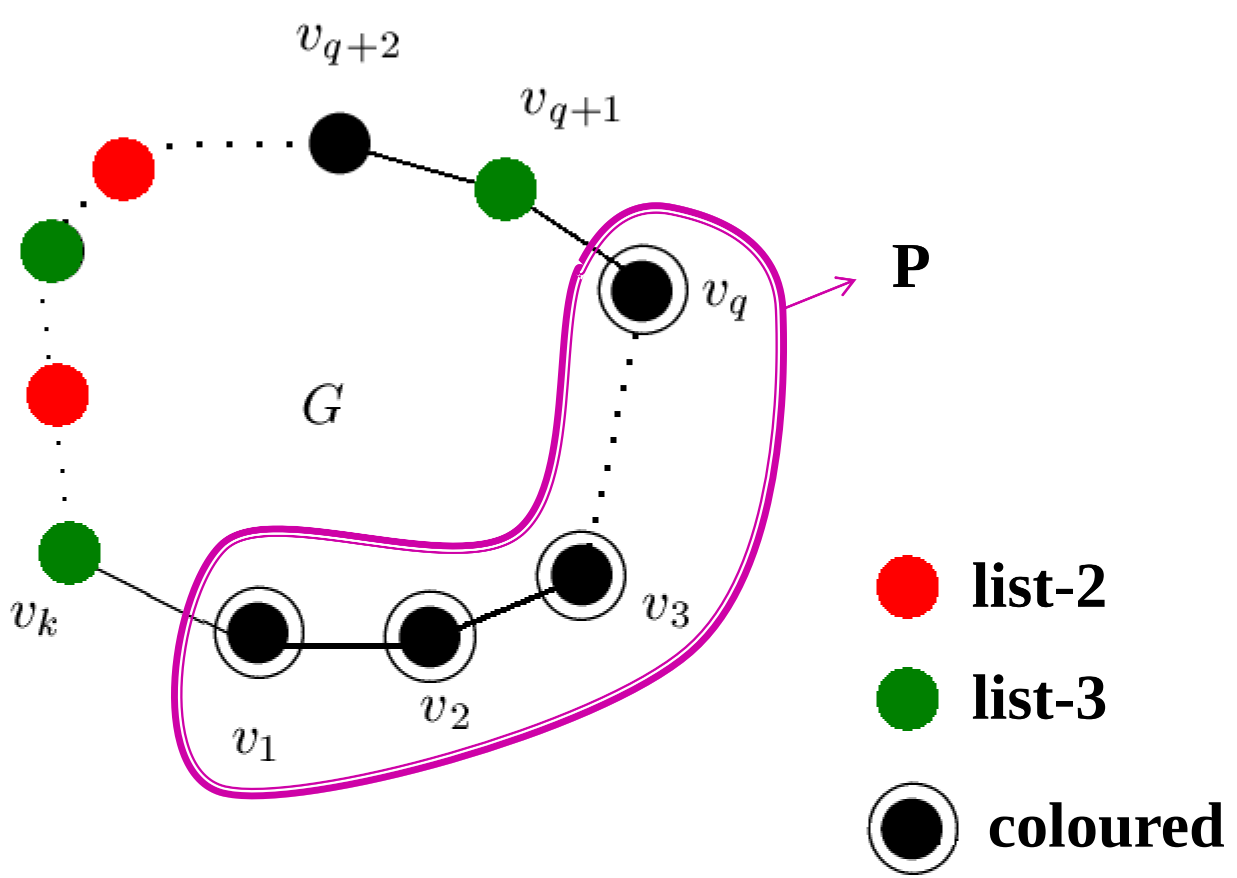
The graph G has the following properties:

- **All vertices not in C are list-3 vertices.**
- **All vertices in C are list-2 or list-3, except the coloured vertices of P (which are list-1).**
- **There is no edge joining vertices whose list have size less than 3 (except for the edges in P).**

The graph G has the following properties:

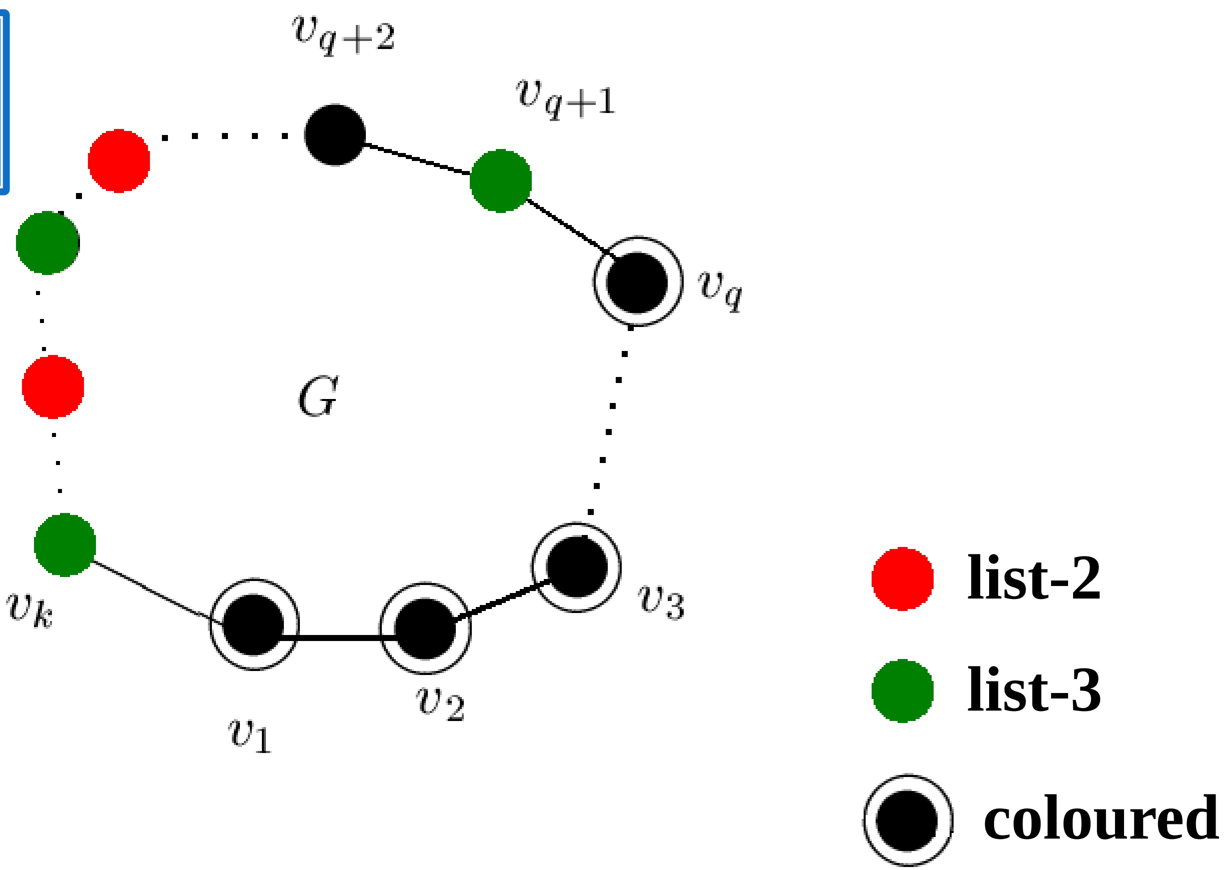


The graph G has the following properties:



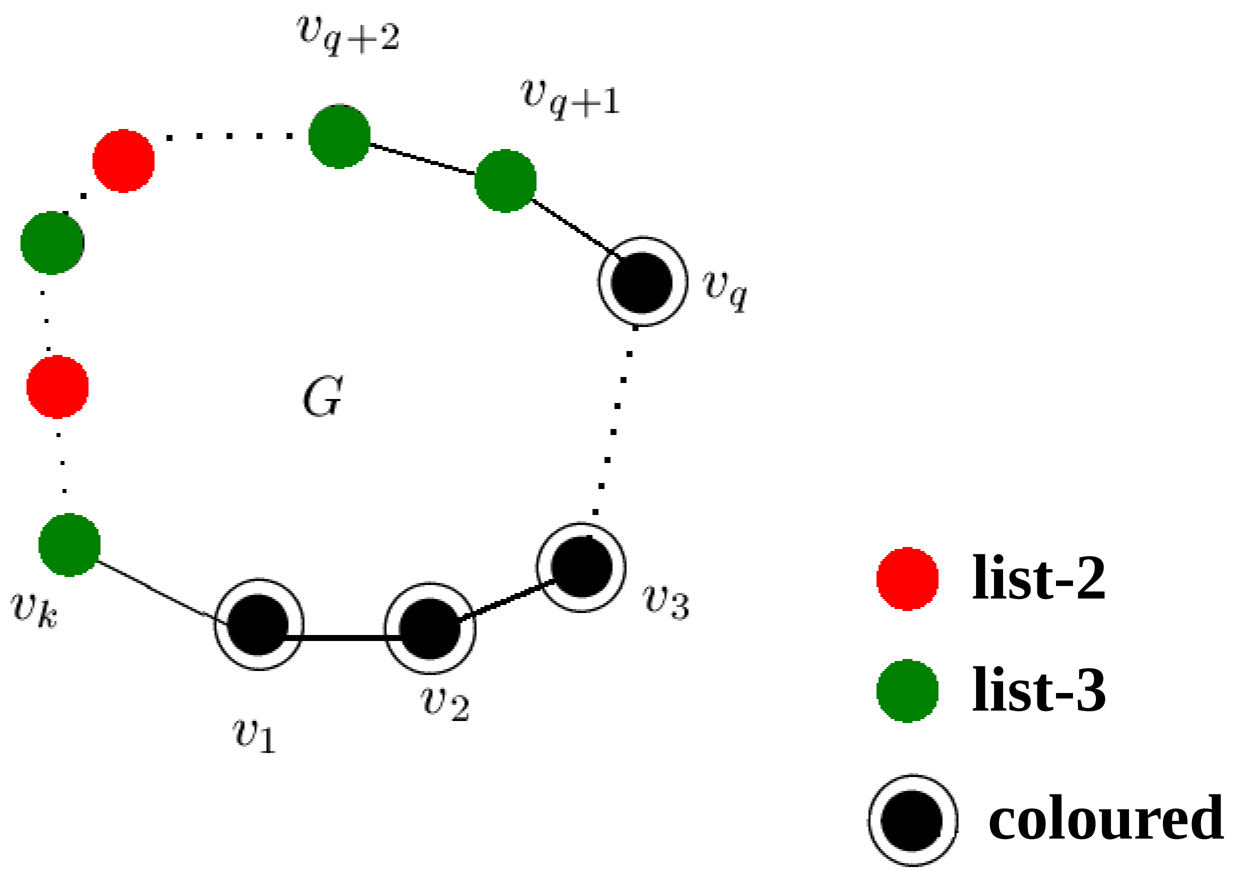
The graph G has the following properties:

v_{q+2} may be a list-2 or a list-3 vertex.



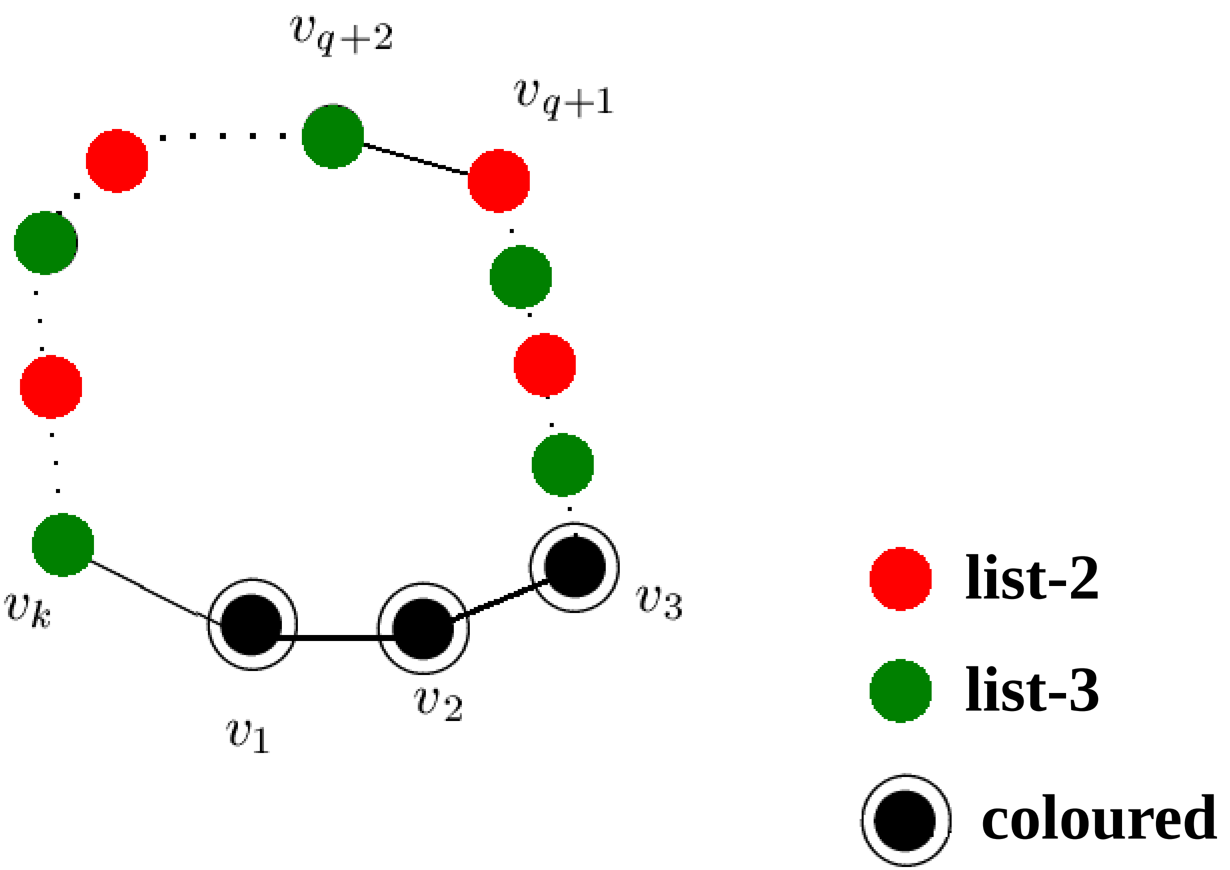
The graph G has the following properties:

Suppose v_{q+2} is a list-3 vertex.



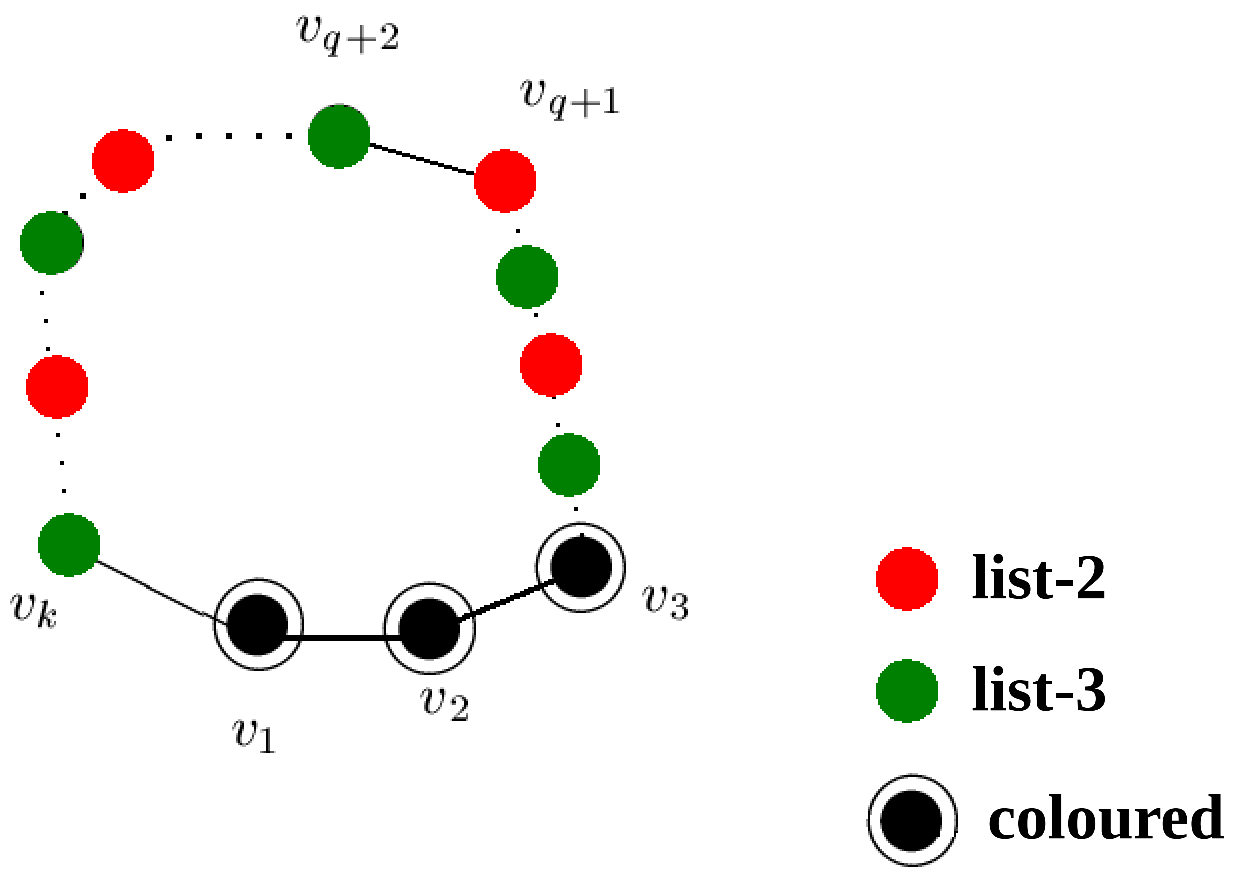
The graph G has the following properties:

Delete v_q, and delete its colour from its neighbours' lists.



The graph G has the following properties:

Check to make sure the properties of G still hold, then colour by induction.



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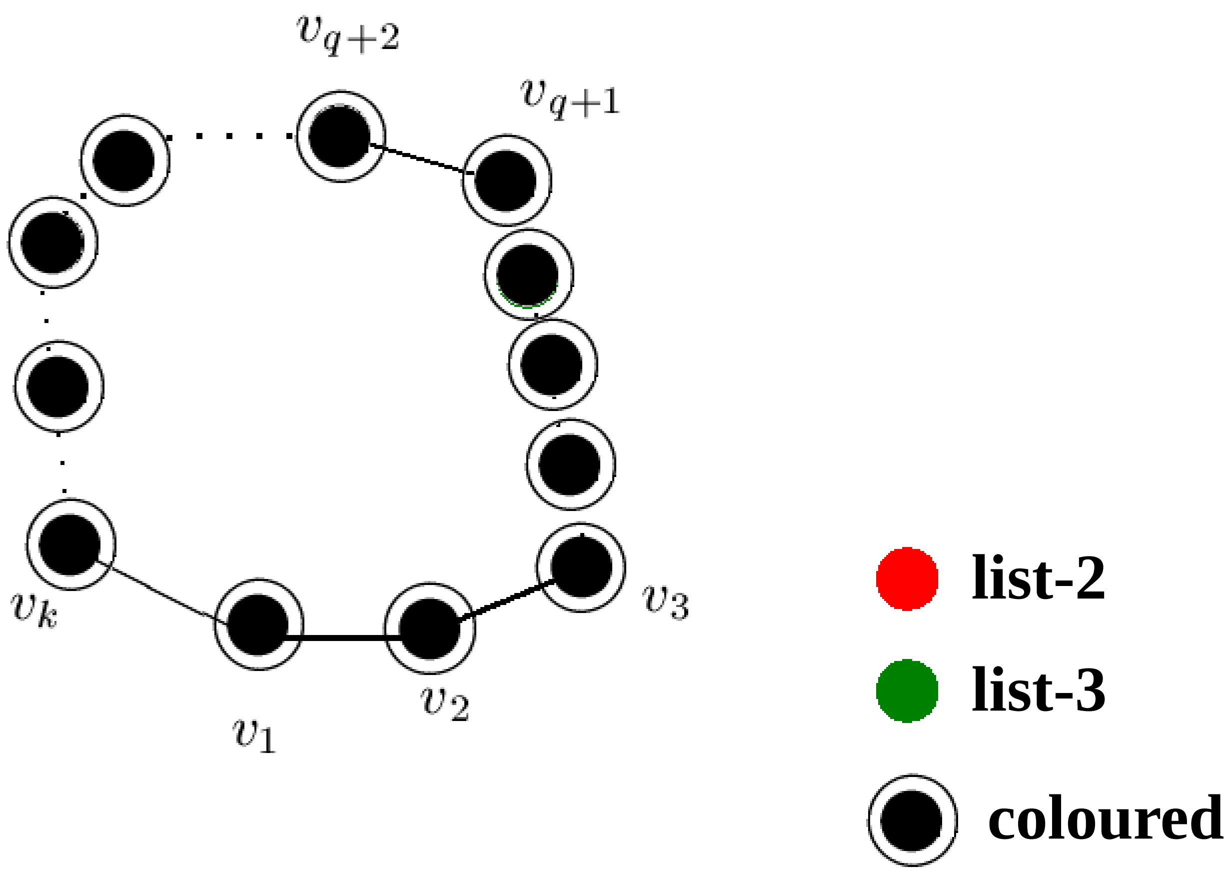
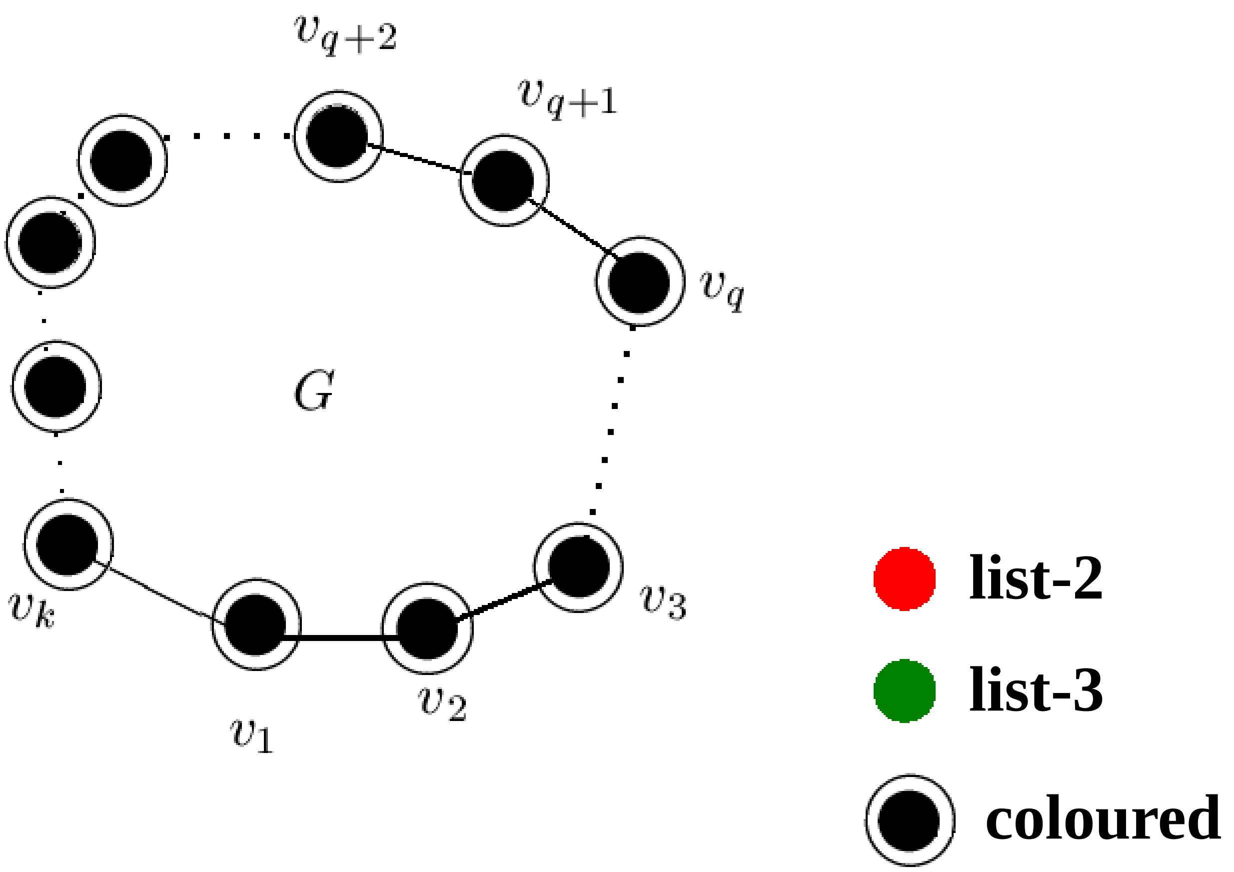


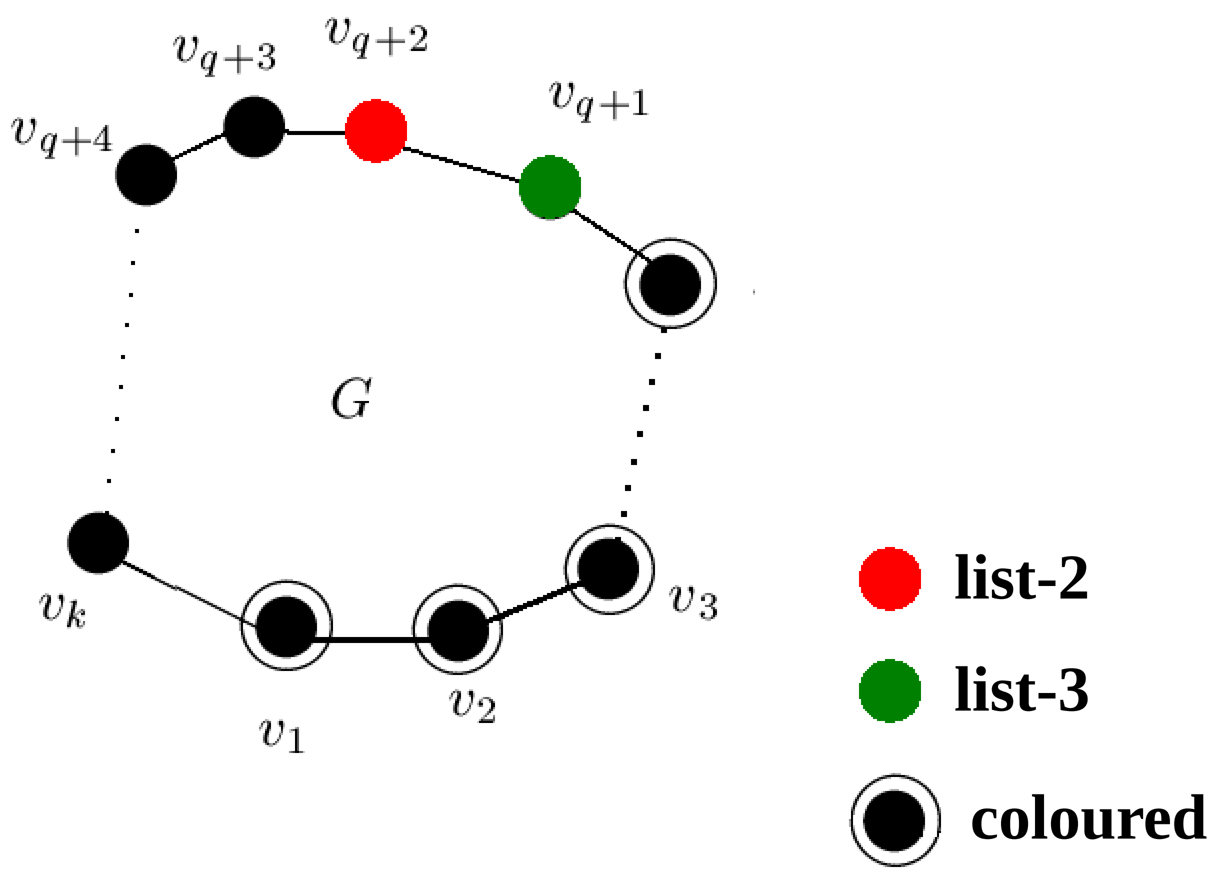
The graph G has the following properties:

Put v_q back into G.



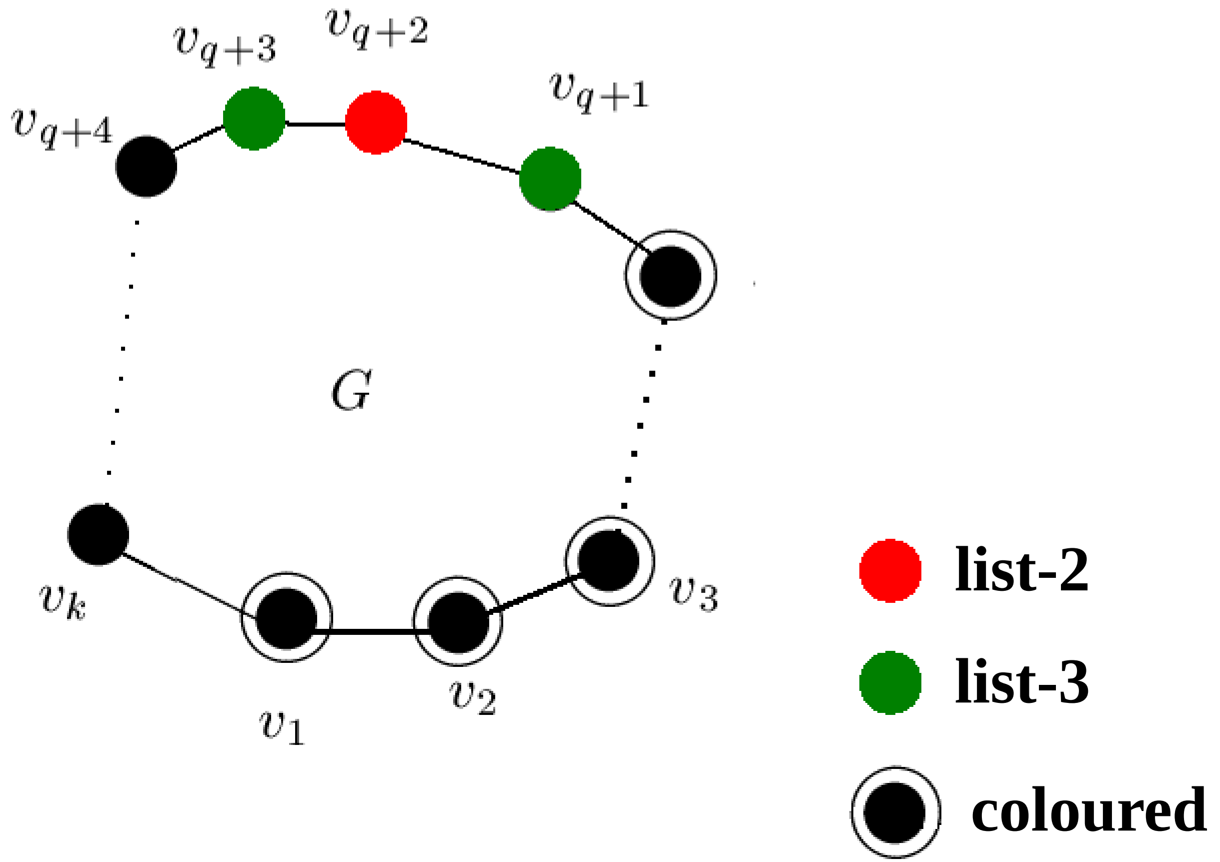
The graph G has the following properties:

Now suppose v_{q+2} is a list-2 vertex.



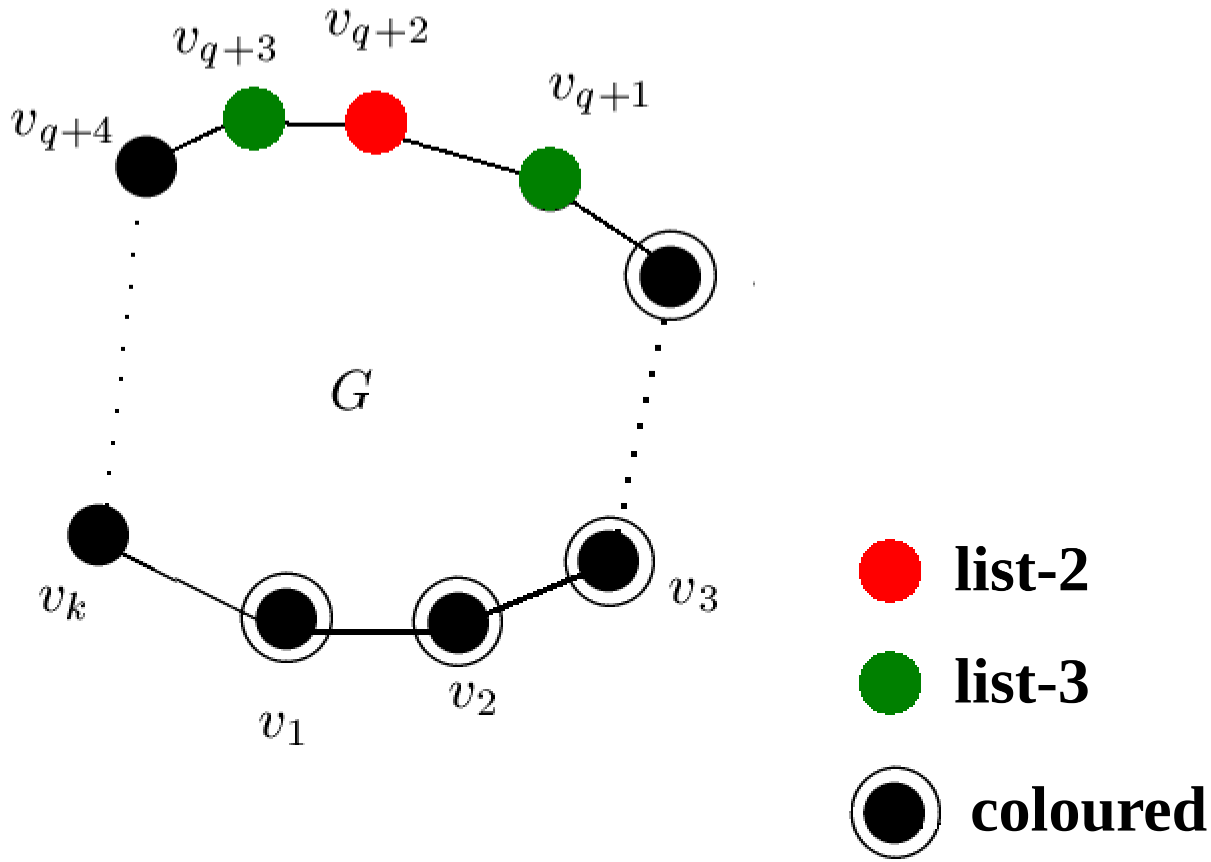
The graph G has the following properties:

v_{q+3} **must be a list-3 vertex.**



The graph G has the following properties:

Complete the proof by considering what happens when v_{q+4} is a list-3, and when it is a list-2.



Proof of Grötzsch's Theorem

↳ C. Thomassen

Any triangle-free planar graph G is 3-colourable.

Rules out :

- **G has no 4-cycles.** **(Theorem 1)**
- **Separating 4-cycles.**
- **Interior facial 4-cycles.**
- **C is a 4-cycle.** **(Theorem 1)**

Extensions of Grötzsch's Theorem

A planar graph G is 3-colourable if there are:

- **At most 3 triangles.** B. Grünbaum (1963)

Extensions of Grötzsch's Theorem

A planar graph G is 3-colourable if there are:

- **At most 3 triangles.** B. Grünbaum (1963)
- **No 5-cycles and all triangles are at least 2 vertices away from each other.** Borodin, Raspaud (2000)

Extensions of Grötzsch's Theorem

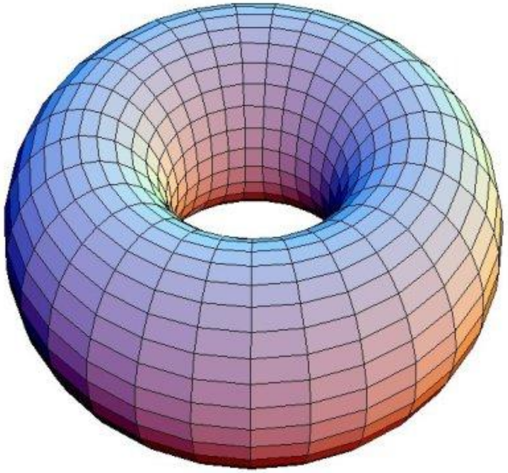
A planar graph G is 3-colourable if there are:

- **At most 3 triangles.** B. Grünbaum (1963)
- **No 5-cycles and all triangles are at least 2 vertices away from each other.** Borodin, Glebov (2010)
- **No 5-cycles, no 7-cycles, and no triangles share a common vertex.** Baogang Xu (2006)

Future Work

Generalize Grötzsch's Theorem for more complicated surfaces:

<u>Torus</u>:



Future Work

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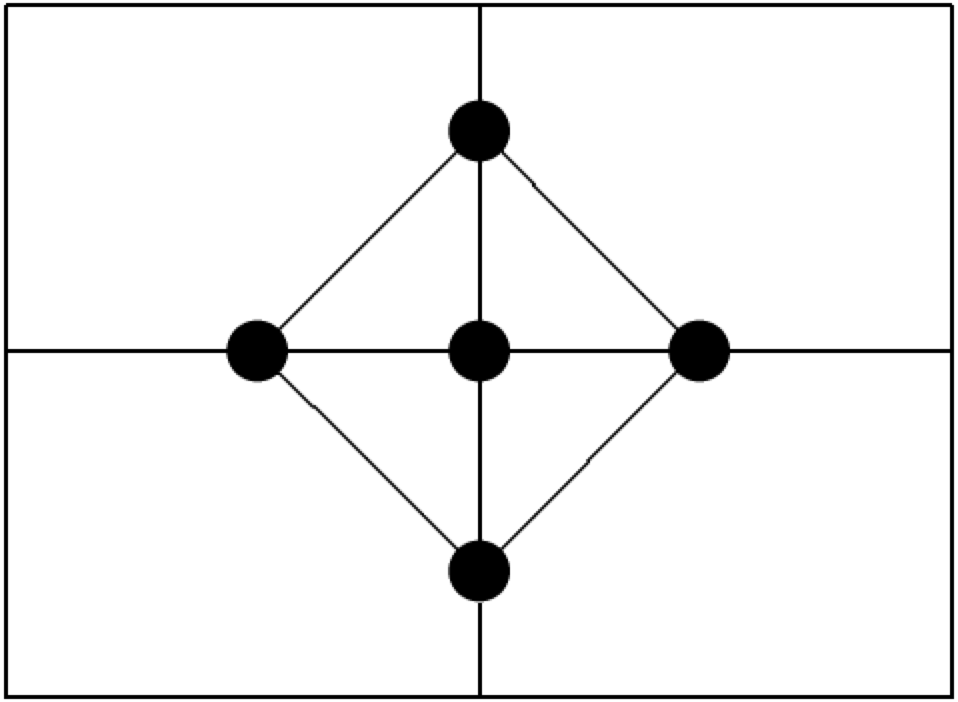

Future Work

<u>Torus</u>: 3-colourable in G if it has no triangles and no quadrilaterals. C. Thomassen (1994)

Future Work

Torus: 3-colourable in G if it has no triangles and no quadrilaterals. C. Thomassen (1994)

- **Generalize this to perhaps allow for triangles at minimum distance k.**

By using a result recently obtained by Thomassen and Kawarabayashi (2009): every planar graph can be decomposed into an independent set and a forest.

Thank You

Discharging

Euler's Formula: V + F – M = 2

V: number of vertices
F: number of faces
M: number of edges

$$2M = \sum_{\substack{vertices\\ v_i}} \deg(v_i) = \sum_{\substack{faces\\ f_i}} |f_i|$$