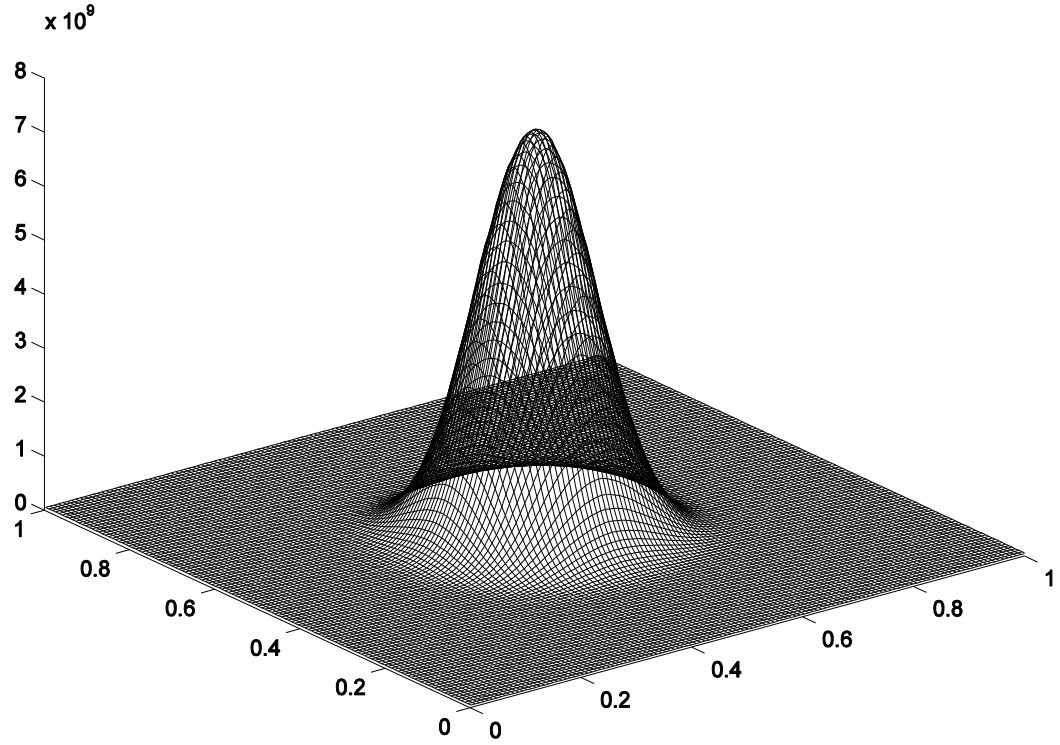
A Simple Moving Mesh Method for Blow-up Problems



by Lauren DeDieu

Advisor: George Chen

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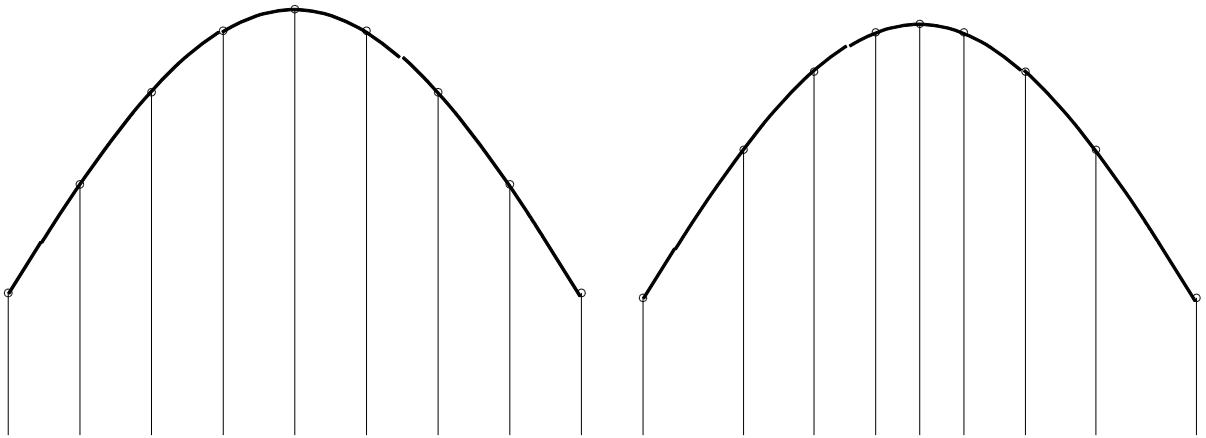
Moving Mesh Methods: Introduction

- Are one of the most **powerful methods** to numerically **solve** time dependent partial differential equations (**PDE**) with some kind of **singularity** (shock waves & blow-up problems).
- **Fixed number** of **mesh points**
- Moves mesh adaptively so **errors** are **distributed uniformly**.



Uniform Mesh

Moving Mesh

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Moving Mesh Methods: Existing Work

- **Winslow** (1967), **Thompson** (1985), and others proposed various elliptic equations and variational methods for defining adaptive mapping.

- **Budd, Huang**, and **Russel** (2009) in their paper *Adaptivity with Moving Grids* summarize traditional **hp-refinement** methods in which mesh is **added/deleted** based on **estimates** about the solution error.

- They conclude **r-adaptive** methods, which initially use **uniform mesh** which later becomes **concentrated** where the solution has **interesting behaviour**, have **enormous potential**, although they are currently in their **early stages** of development.

Moving Mesh Methods: Past Work

- **Ceniceros and Hou (2001)**:

–developed a simple yet efficient **dynamically adaptive** mesh generator for **time-dependent** problems.

–**Mesh** is **generated** directly from the **physical domain**, so the **mesh equations** become quite **simple.**

Moving Mesh Methods: Past Work

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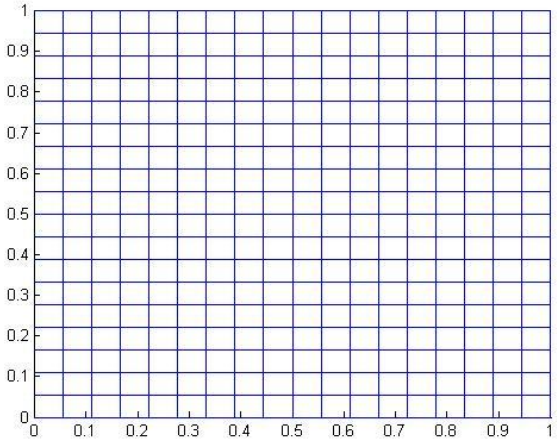
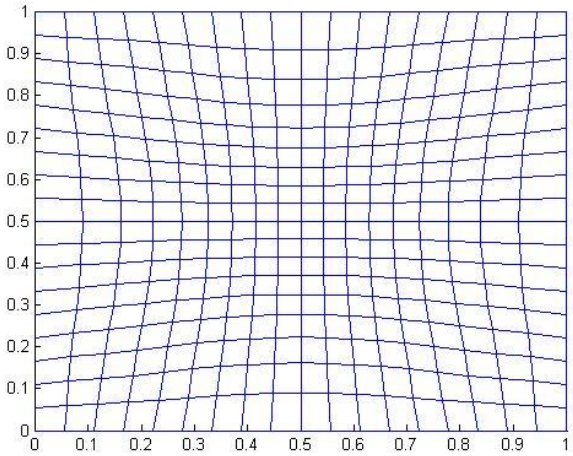


Computational Domain
(ξ,η)

$$x = x(\xi, \eta), \quad y = y(\xi, \eta)$$



Physical Domain
(x,y)

Moving Mesh Methods: Past Work

- **Ceniceros and Hou**:

  -developed a simple yet efficient **dynamically adaptive** mesh generator for **time-dependent** problems.

  -**Mesh** is **generated** directly from the **physical domain**, so the **mesh equations** become quite **simple.**

- We focus on **blow-up problems**, and **simplify** and **improve Ceniceros and Hou`s** results by adding an **additional term** to make the mesh more **orthogonal**.

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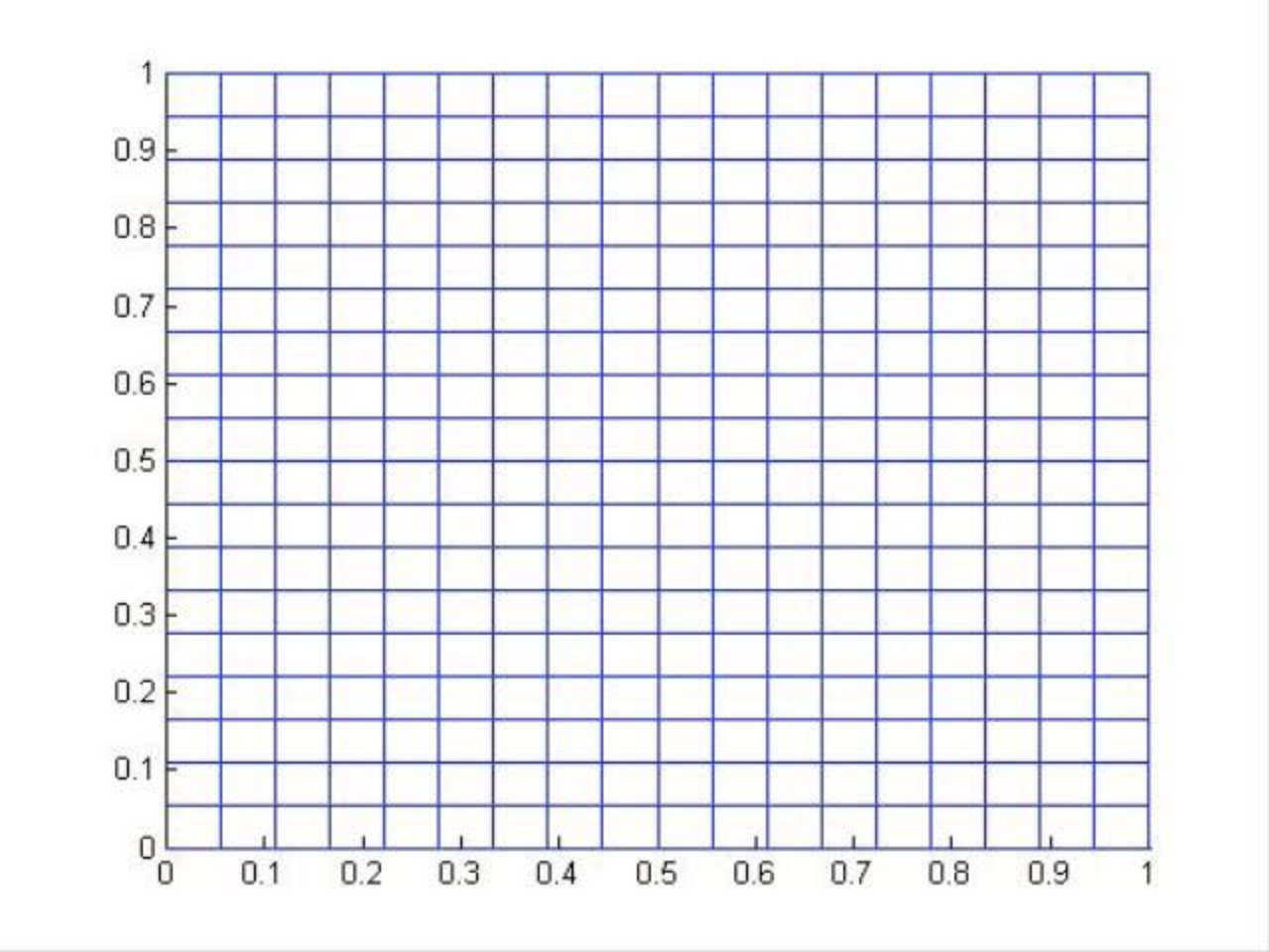
Moving the Mesh



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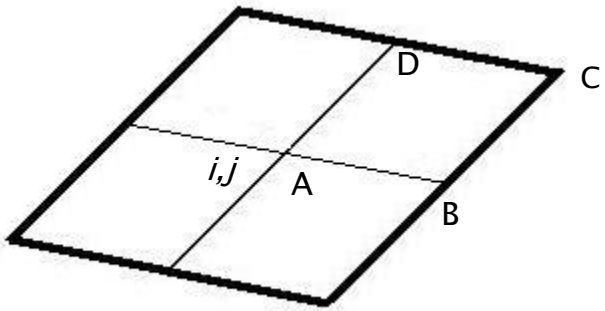
Minimizing Diagonal Lengths

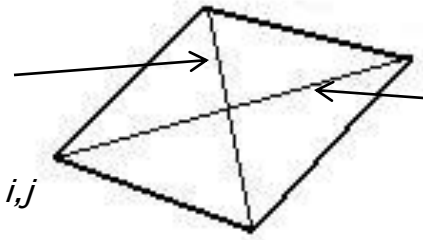
In two dimensional cases, a cell has four vertices:

$A(x_{i,j}, y_{i,j})$, $B(x_{i+1,j}, y_{i+1,j})$, $C(x_{i+1,j+1}, y_{i+1,j+1})$, $D(x_{i,j+1}, y_{i,j+1})$



The two diagonal lengths are:

$$\sqrt{(x_{i+1,j} - x_{i,j+1})^2 + (y_{i+1,j} - y_{i,j+1})^2}$$



$$\sqrt{(x_{i,j} - x_{i+1,j+1})^2 + (y_{i,j} - y_{i+1,j+1})^2}$$

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Minimizing Diagonal Lengths

The mesh can be generated by minimizing the following problem:

$$I[x,y]=\sum_{i,j=1}^{n-1}\left(\frac{M_{i+1,j+1}+M_{i,j}}{2}\right)\left[\left(x_{i+1,j+1}-x_{i,j}\right)^2+\left(y_{i+1,j+1}-y_{i,j}\right)^2\right]$$

$$+\left(\frac{M_{i,j+1}+M_{i+1,j}}{2}\right)\left[\left(x_{i,j+1}-x_{i+1,j}\right)^2+\left(y_{i,j+1}-y_{i+1,j}\right)^2\right]$$

where M is the monitor function $\mathrm{M}(u)=\sqrt{u^4+c_1\left|\nabla u\right|^2+c_2}$

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Minimizing Diagonal Lengths

Minimizing this problem is equivalent to solving the following linear algebra system:

$$\frac{1}{2}\frac{\partial I}{\partial x_{i,j}} = -(\mathrm{M}_{i+1,j+1} + \mathrm{M}_{i,j})(x_{i+1,j+1} - x_{i,j}) + (\mathrm{M}_{i,j} + \mathrm{M}_{i-1,j-1})(x_{i,j} - x_{i-1,j-1})$$

$$+ (\mathrm{M}_{i,j} + \mathrm{M}_{i+1,j-1})(x_{i,j} - x_{i+1,j-1}) - (\mathrm{M}_{i-1,j+1} + \mathrm{M}_{i,j})(x_{i-1,j+1} - x_{i,j})$$

$$= 0 \qquad (2)$$

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Minimizing Diagonal Lengths

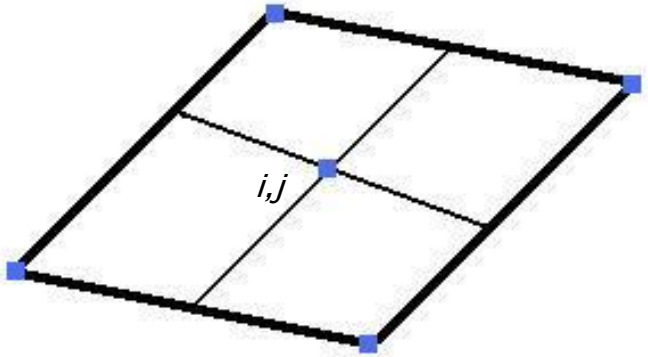
and

$$\frac{1}{2}\frac{\partial I}{\partial y_{i,j}} = -(\mathrm{M}_{i+1,j+1}+\mathrm{M}_{i,j})(y_{i+1,j+1}-y_{i,j})+(\mathrm{M}_{i,j}+\mathrm{M}_{i-1,j-1})(y_{i,j}-y_{i-1,j-1})$$

$$+(\mathrm{M}_{i,j}+\mathrm{M}_{i+1,j-1})(y_{i,j}-y_{i+1,j-1})-(\mathrm{M}_{i-1,j+1}+\mathrm{M}_{i,j})(y_{i-1,j+1}-y_{i,j})$$

$$=0$$

where $1<i,\ j<n.$



(3)

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Minimizing Diagonal Lengths

Note: When minimizing equations, if the coefficient is large, then the variable will be small, and vice versa.

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Minimizing Diagonal Lengths

Note: When minimizing equations, if the coefficient is large, then the variable will be small, and vice versa.

For example:

large small

$$\frac{1}{2}\frac{\partial I}{\partial y_{i,j}} = -(\mathrm{M}_{i+1,j+1} + \mathrm{M}_{i,j})(y_{i+1,j+1} - y_{i,j}) + (\mathrm{M}_{i,j} + \mathrm{M}_{i-1,j-1})(y_{i,j} - y_{i-1,j-1})$$
$$+ (\mathrm{M}_{i,j} + \mathrm{M}_{i+1,j-1})(y_{i,j} - y_{i+1,j-1}) - (\mathrm{M}_{i-1,j+1} + \mathrm{M}_{i,j})(y_{i-1,j+1} - y_{i,j})$$
$$= 0$$

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Minimizing Diagonal Lengths

Note: When minimizing equations, if the coefficient is large, then the variable will be small, and vice versa.

For example:

small large

$$\frac{1}{2}\frac{\partial I}{\partial y_{i,j}} = -(\mathrm{M}_{i+1,j+1} + \mathrm{M}_{i,j})(y_{i+1,j+1} - y_{i,j}) + (\mathrm{M}_{i,j} + \mathrm{M}_{i-1,j-1})(y_{i,j} - y_{i-1,j-1})$$
$$+ (\mathrm{M}_{i,j} + \mathrm{M}_{i+1,j-1})(y_{i,j} - y_{i+1,j-1}) - (\mathrm{M}_{i-1,j+1} + \mathrm{M}_{i,j})(y_{i-1,j+1} - y_{i,j})$$
$$= 0$$

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Minimizing Diagonal Lengths

Let's see a simple example:

Minimize $f(x, y, z) = ax^2 + by^2 + cz^2$ **where** $x, y, z \geq 0$, $x + y + z = 1$, **and** $a = c = 1, b = 3$.

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Minimizing Diagonal Lengths

Minimize $f(x,y,z) = ax^2 + by^2 + cz^2$ **where** $x, y, z \geq 0$, $x + y + z = 1$, **and** $a = c = 1, b = 3$.

$$f(x,y,z) = a(1-y-z)^2 + by^2 + cz^2$$

$$f'(y) = -2a(1-y-z) + 2by$$

$$-2a + 2ay + 2az + 2by = 0$$

$$-a + ay + az + by = 0$$

$$f'(z) = -2a(1-y-z) + 2cz$$

$$= -2a + 2ay + 2az + 2cz = 0$$

$$z = \frac{a - ay}{a + c}$$

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Minimizing Diagonal Lengths

Minimize $f(x,y,z) = ax^2 + by^2 + cz^2$ **where** $x, y, z \geq 0$, $x + y + z = 1$, **and** $a = c = 1, b = 3$.

$$x = 1 - y - z$$

$$z = \frac{a - ay}{a + c}$$

$$f'(y) = -a + ay + az + by = 0$$

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Minimizing Diagonal Lengths

Minimize $f(x,y,z) = ax^2 + by^2 + cz^2$ **where** $x, y, z \geq 0$, $x + y + z = 1$, **and** $a = c = 1, b = 3$.

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$$-a + ay + a\left(\frac{a - ay}{a + c}\right) + by = 0$$

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Minimizing Diagonal Lengths

Minimize $f(x,y,z) = ax^2 + by^2 + cz^2$ **where** $x, y, z \geq 0$, $x + y + z = 1$, **and** $a = c = 1, b = 3$.

$$x = 1 - y - z$$

$$z = \frac{a - ay}{a + c}$$

$$f'(y) = -a + ay + az + by = 0$$

$$-a + ay + a\left(\frac{a - ay}{a + c}\right) + by = 0$$

$$-1 + y + \left(\frac{1 - y}{1 + 1}\right) + 3y = 0$$

$$-1 + 4y + \frac{1}{2} - \frac{y}{2} = 0$$

$$\frac{7}{2}y = \frac{1}{2}$$

$$y = \frac{1}{7}$$

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Minimizing Diagonal Lengths

Minimize $f(x,y,z) = ax^2 + by^2 + cz^2$ **where** $x, y, z \geq 0$, $x + y + z = 1$, **and** $a = c = 1, b = 3$.

$$y = \frac{1}{7}$$

$$z = \frac{a - ay}{a + c} = \frac{1 - \frac{1}{7}}{1+1} = \frac{\frac{6}{7}}{2} = \frac{6}{14} = \frac{3}{7}$$

$$x = 1 - y - z = 1 - \frac{1}{7} - \frac{3}{7} = \frac{3}{7}$$

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Minimizing Diagonal Lengths

Minimize $f(x,y,z) = ax^2 + by^2 + cz^2$ **where** $x, y, z \geq 0$, $x + y + z = 1$, **and** $a = c = 1, b = 3$.

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Minimizing Diagonal Lengths

In order to make the mesh more smooth and orthogonal, we add higher order difference terms:

$$I[x,y]=\sum_{i,j=1}^{n-1}\left\{\left(\mathrm{M}_{i+1,j+1}+\mathrm{M}_{i,j}\right)\left[\left(x_{i+1,j+1}-x_{i,j}\right)^2+\left(y_{i+1,j+1}-y_{i,j}\right)^2\right]\right.$$

$$\left.+\left(\mathrm{M}_{i,j+1}+\mathrm{M}_{i+1,j}\right)\left[\left(x_{i,j+1}-x_{i+1,j}\right)^2+\left(y_{i,j+1}-y_{i+1,j}\right)^2\right]\right\}$$

$$+c\sum_{i,j=2}^{n-1}\left\{\mathrm{M}_{i,j}\left[\left(x_{i+1,j}-2x_{i,j}+x_{i-1,j}\right)^2+\left(x_{i,j+1}-2x_{i,j}+x_{i,j-1}\right)^2\right.\right.$$

$$\left.\left.+\left(y_{i+1,j}-2y_{i,j}+x_{i-1,j}\right)^2+\left(y_{i,j+1}-2y_{i,j}+x_{i,j-1}\right)^2\right]\right\}$$

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Minimizing Diagonal Lengths

We can use the following semi-discretized differential equations to replace (2) & (3):

$$\frac{\partial x_{i,j}}{\partial t} = \frac{1}{\tau}(\mathrm{M}_{i+1,j+1} + \mathrm{M}_{i,j})(x_{i+1,j+1} - x_{i,j}) - (\mathrm{M}_{i,j} + \mathrm{M}_{i-1,j-1})(x_{i,j} - x_{i-1,j-1})$$
$$-(\mathrm{M}_{i,j} + \mathrm{M}_{i+1,j-1})(x_{i,j} - x_{i+1,j-1}) + (\mathrm{M}_{i-1,j+1} + \mathrm{M}_{i,j})(x_{i-1,j+1} - x_{i,j})$$

(4)

* And similarly for y.*

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Minimizing Diagonal Lengths

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$$- (\mathrm{M}_{i,j} + \mathrm{M}_{i+1,j-1})(x_{i,j} - x_{i+1,j-1}) + (\mathrm{M}_{i-1,j+1} + \mathrm{M}_{i,j})(x_{i-1,j+1} - x_{i,j})$$

$$- c\mathrm{M}_{i,j}(-4x_{i,j} + x_{i+1,j} + x_{i-1,j} + x_{i,j+1} + x_{i,j-1})$$

$$+ c\mathrm{M}_{i-1,j}(x_{i,j} - 2x_{i-1,j} + x_{i-2,j})$$

$$+ c\mathrm{M}_{i+2,j}(x_{i+2,j} - 2x_{i+1,j} + x_{i,j})$$

$$+ c\mathrm{M}_{i,j-1}(x_{i,j} - 2x_{i,j-1} + x_{i,j-2})$$

$$+ c\mathrm{M}_{i,j+1}(x_{i,j+2} - 2x_{i,j+1} + x_{i,j})$$

2nd Order Difference

(4)

* And similarly for y.*

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Why Solving (4) is Better than Solving (2)-(3):

Disadvantages of (2)–(3):

1. Dependent on initial values.
2. The convergence rate is very slow, so takes a long time to compute.
3. When u is large it may be unstable (small oscillation).

Advantages of (4):

1. Always has a solution and is smooth.
2. Easier to control the movement of mesh.

Minimizing Diagonal Lengths

Minimizing this problem is equivalent to solving the following linear algebra system:

$$\frac{1}{2}\frac{\partial I}{\partial x_{i,j}} = -(\mathrm{M}_{i+1,j+1} + \mathrm{M}_{i,j})(x_{i+1,j+1} - x_{i,j}) + (\mathrm{M}_{i,j} + \mathrm{M}_{i-1,j-1})(x_{i,j} - x_{i-1,j-1})$$

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$$= 0 \qquad (2)$$

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Minimizing Diagonal Lengths

and

$$\frac{1}{2}\frac{\partial I}{\partial y_{i,j}} = -(\mathrm{M}_{i+1,j+1}+\mathrm{M}_{i,j})(y_{i+1,j+1}-y_{i,j})+(\mathrm{M}_{i,j}+\mathrm{M}_{i-1,j-1})(y_{i,j}-y_{i-1,j-1})$$

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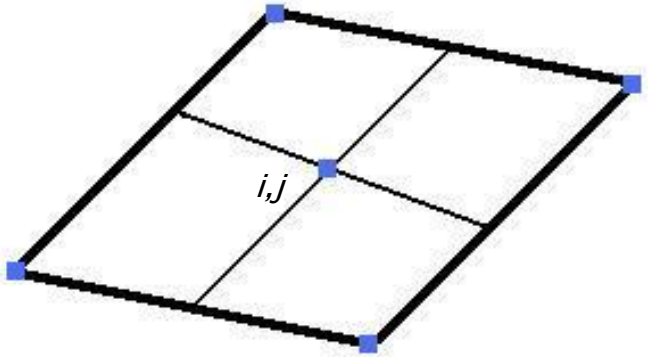
$$=0$$

where $1<i,\ j<n.$



(3)

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$$-(\mathrm{M}_{i,j} + \mathrm{M}_{i+1,j-1})(x_{i,j} - x_{i+1,j-1}) + (\mathrm{M}_{i-1,j+1} + \mathrm{M}_{i,j})(x_{i-1,j+1} - x_{i,j})$$

(4)

* And similarly for y.*

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Numerical Examples for Partial Differential Equations

Consider the heat equation:

$$\begin{cases} u_t = \Delta u + f(u) \\ u\big|_{\partial\Omega} = 0 \\ u(x, y, 0) = \phi(x, y) \end{cases}$$

where $\Omega = [a,b] \times [a,b]$.

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Numerical Examples for Partial Differential Equations

In the computational domain we have:

$$\bar{u}(\xi,\eta,t) = u[x(\xi,\eta,t), y(\xi,\eta,t), t]$$

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Chain Rule

$$= u_x x_t + u_y y_t + u_t\Big|_{fixed\ x,y}$$

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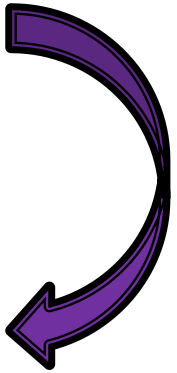
Numerical Examples for Partial Differential Equations

In the computational domain we have:

$$\bar{u}(\xi,\eta,t) = u[x(\xi,\eta,t), y(\xi,\eta,t), t]$$

$$\bar{u}_\xi = u_x x_\xi + u_y y_\xi$$

$$\bar{u}_\eta = u_x x_\eta + u_y y_\eta$$



Chain Rule

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Numerical Examples for Partial Differential Equations

$$\bar{u}_\xi = u_x x_\xi + u_y y_\xi$$

$$\bar{u}_\eta = u_x x_\eta + u_y y_\eta$$

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Numerical Examples for Partial Differential Equations

$$\bar{u}_\xi = u_x x_\xi + u_y y_\xi$$

$$\bar{u}_\eta = u_x x_\eta + u_y y_\eta$$

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Numerical Examples for Partial Differential Equations

$$\bar{u}_\xi = u_x x_\xi + u_y y_\xi$$

$$\bar{u}_\eta = u_x x_\eta + u_y y_\eta$$

$$\begin{pmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \bar{u}_\xi \\ \bar{u}_\eta \end{pmatrix}$$

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Numerical Examples for Partial Differential Equations

$$\begin{pmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \bar{u}_\xi \\ \bar{u}_\eta \end{pmatrix}$$

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Numerical Examples for Partial Differential Equations

$$\begin{pmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \bar{u}_\xi \\ \bar{u}_\eta \end{pmatrix}$$

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{pmatrix}^{-1} \begin{pmatrix} \bar{u}_\xi \\ \bar{u}_\eta \end{pmatrix}$$

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \frac{1}{x_\xi y_\eta - x_\eta y_\xi} \begin{pmatrix} y_\eta & -y_\xi \\ -x_\eta & x_\xi \end{pmatrix} \begin{pmatrix} \bar{u}_\xi \\ \bar{u}_\eta \end{pmatrix}$$

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Numerical Examples for Partial Differential Equations

Let $J = x_\xi y_\eta - y_\xi x\eta$

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \bar{u}_\xi y_\eta - \bar{u}_\eta y_\xi \\ -\bar{u}_\xi x_\eta + \bar{u}_\eta x_\xi \end{pmatrix}$$

$$u_x = \frac{1}{J}\left(\bar{u}_\xi y_\eta - \bar{u}_\eta y_\xi\right)$$

$$u_y = \frac{1}{J}\left(\bar{u}_\eta x_\xi - \bar{u}_\xi x_\eta\right)$$

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Numerical Examples for Partial Differential Equations

In the computational domain we have:

$$\bar{u}_t\Big|_{fixed\,\xi,\eta} = u_x x_t + u_y y_t + u_t\Big|_{fixed\ x,y}$$

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Numerical Examples for Partial Differential Equations

In the computational domain we have:

$$\bar{u}_t\Big|_{fixed\,\xi,\eta} = u_x x_t + u_y y_t + u_t\Big|_{fixed\,x,y}$$

$$= \frac{1}{J}\underbrace{\left(\bar{u}_\xi y_\eta - \bar{u}_\eta y_\xi\right)}_{u_x} x_t + \underbrace{\left(\bar{u}_\eta x_\xi - \bar{u}_\xi x_\eta\right)}_{u_y} y_t + \Delta u + f(u)$$

where $J = x_\xi y_\eta - y_\xi x_\eta$

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Numerical Examples for Partial Differential Equations

In the computational domain we have:

$$= \frac{1}{J}\left[\bar{u}_\xi y_\eta - \bar{u}_\eta y_\xi\right)\bar{x}_t + \left(\bar{u}_\eta x_\xi - \bar{u}_\xi x_\eta\right)\bar{y}_t\Big] + \Delta u + f(u)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Second Derivative

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Numerical Examples for Partial Differential Equations

In the computational domain we have:

$$= \frac{1}{J}\left[\left(\bar{u}_\xi y_\eta - \bar{u}_\eta y_\xi\right)\bar{x}_t + \left(\bar{u}_\eta x_\xi - \bar{u}_\xi x_\eta\right)\bar{y}_t\right] + \Delta u + f(u)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{1}{J}\underbrace{\left(\bar{u}_\xi y_\eta - \bar{u}_\eta y_\xi\right)}_{u_x}\right]$$

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Numerical Examples for Partial Differential Equations

In the computational domain we have:

$$= \frac{1}{J}\left[\left(\bar{u}_\xi y_\eta - \bar{u}_\eta y_\xi\right)\bar{x}_t + \left(\bar{u}_\eta x_\xi - \bar{u}_\xi x_\eta\right)\bar{y}_t\right] + \Delta u + f(u)$$

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Let w = u_x

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Numerical Examples for Partial Differential Equations

In the computational domain we have:

$$= \frac{1}{J}\left[\left(\bar{u}_\xi y_\eta - \bar{u}_\eta y_\xi\right)x_t + \left(\bar{u}_\eta x_\xi - \bar{u}_\xi x_\eta\right)y_t\right] + \Delta u + f(u)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{1}{J}\underbrace{\left(\bar{u}_\xi y_\eta - \bar{u}_\eta y_\xi\right)}_{u_x}\right]$$

Let $w = u_x$

$$\bar{w}_\xi = w_x x_\xi + w_y y_\xi$$

$$\bar{w}_\eta = w_x x_\eta + w_y y_\eta$$

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Numerical Examples for Partial Differential Equations

In the computational domain we have:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(\overline{w}) = \frac{1}{J}\left(\overline{w}_\xi y_\eta - \overline{w}_\eta y_\xi\right)$$

Found using matrix method like last time.

Sub u_x back in

$$= \frac{1}{J}\left(\left[\frac{1}{J}\left(\overline{u}_\xi y_\eta - \overline{u}_\eta y_\xi\right)\right]_\xi y_\eta - \left[\frac{1}{J}\left(\overline{u}_\xi y_\eta - \overline{u}_\eta y_\xi\right)\right]_\eta y_\xi\right)$$

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Numerical Examples for Partial Differential Equations

So

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{J}\left(\left[\frac{1}{J}\left(\bar{u}_\xi y_\eta - \bar{u}_\eta y_\xi\right)\right]_\xi y_\eta - \left[\frac{1}{J}\left(\bar{u}_\xi y_\eta - \bar{u}_\eta y_\xi\right)\right]_\eta y_\xi\right)$$

and similarly

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{J}\left(\left[\frac{1}{J}\left(\bar{u}_\eta x_\xi - \bar{u}_\xi x_\eta\right)\right]_\eta x_\xi - \left[\frac{1}{J}\left(\bar{u}_\eta x_\xi - \bar{u}_\xi x_\eta\right)\right]_\xi y_\eta\right)$$

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Heat Equation

$$\begin{cases} u_t = \Delta u + f(u) \\ u\big|_{\partial\Omega} = 0 \\ u(x, y, 0) = \phi(x, y) \end{cases}$$

where $\Omega=[a,b]\times[a,b]$.

Let

$$f(u) = u^3$$

$$\phi(x, y) = 10\sin(\pi x)\sin(\pi y)$$

where $\Omega=[0,1]\times[0,1]$.

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Monitor Function

$$\mathrm{M}(x,y,u) = \sqrt{u^4 + c_1 \left|\nabla u\right|^2 + c_2}$$

$$c_1 = \frac{1}{c_2} \qquad c_2 = 3 \left[\max(u)\right]^{3/2}$$

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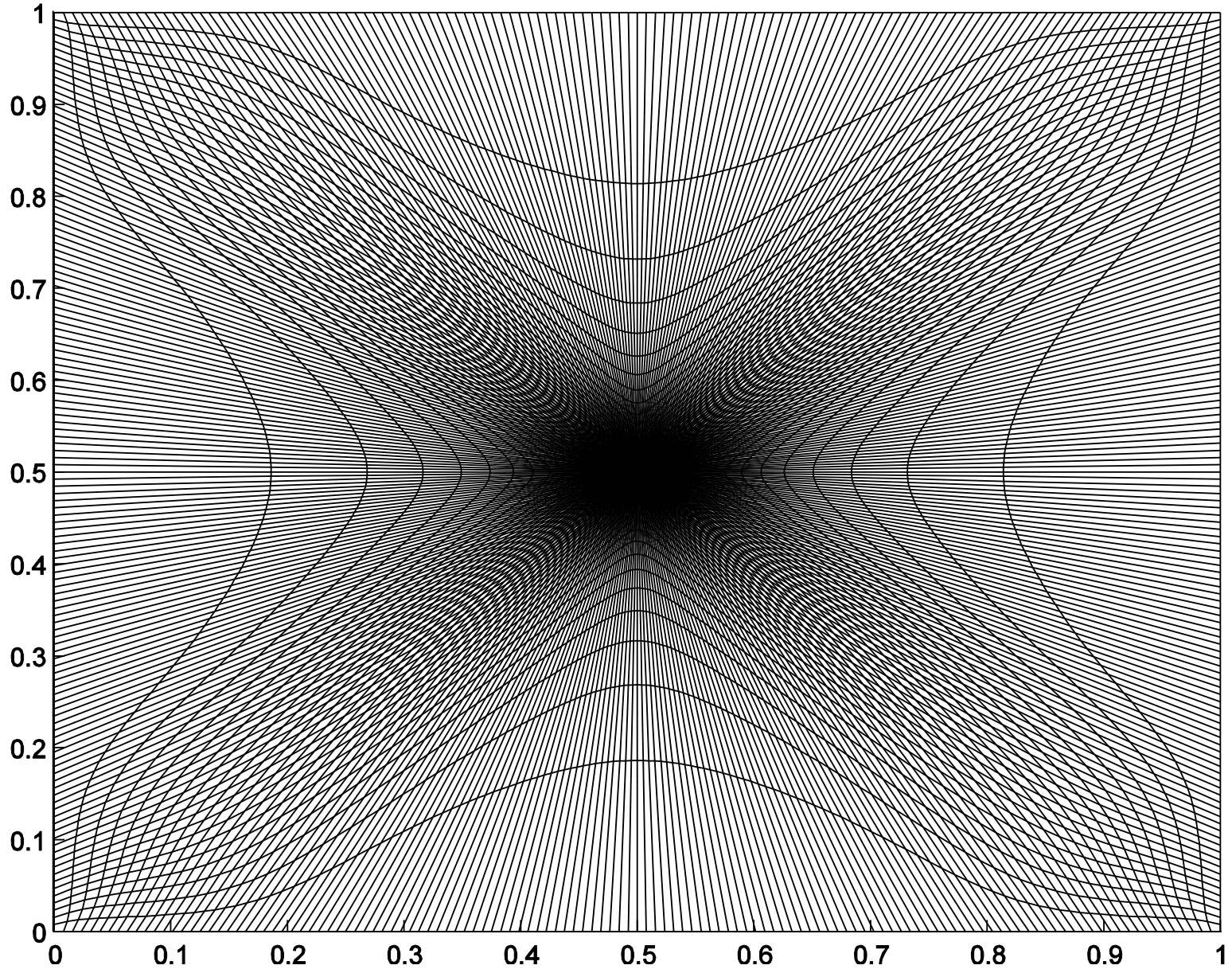

(a) Global View in Physical Domain

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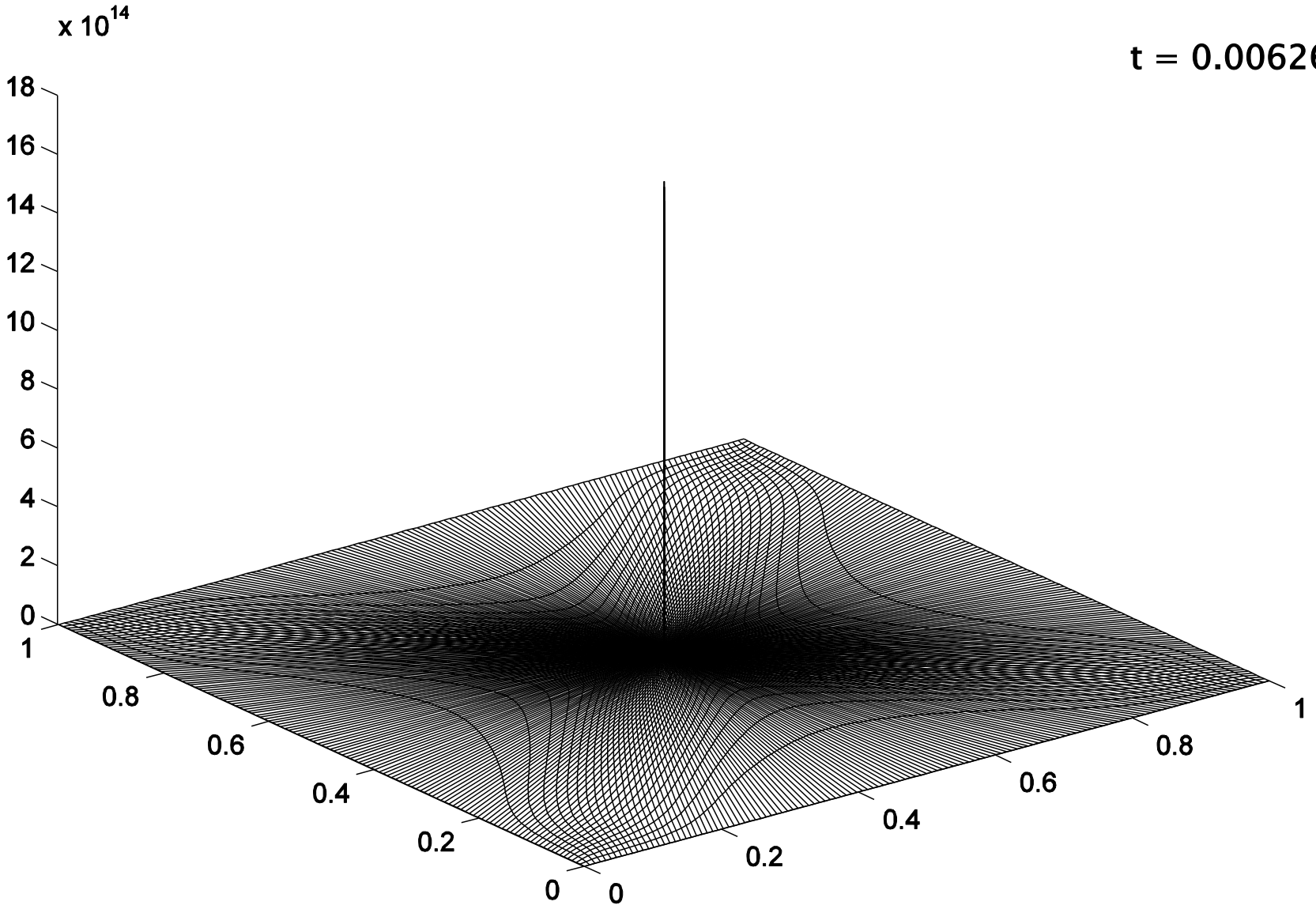

(b) Solution u(x, y) in Physical Space

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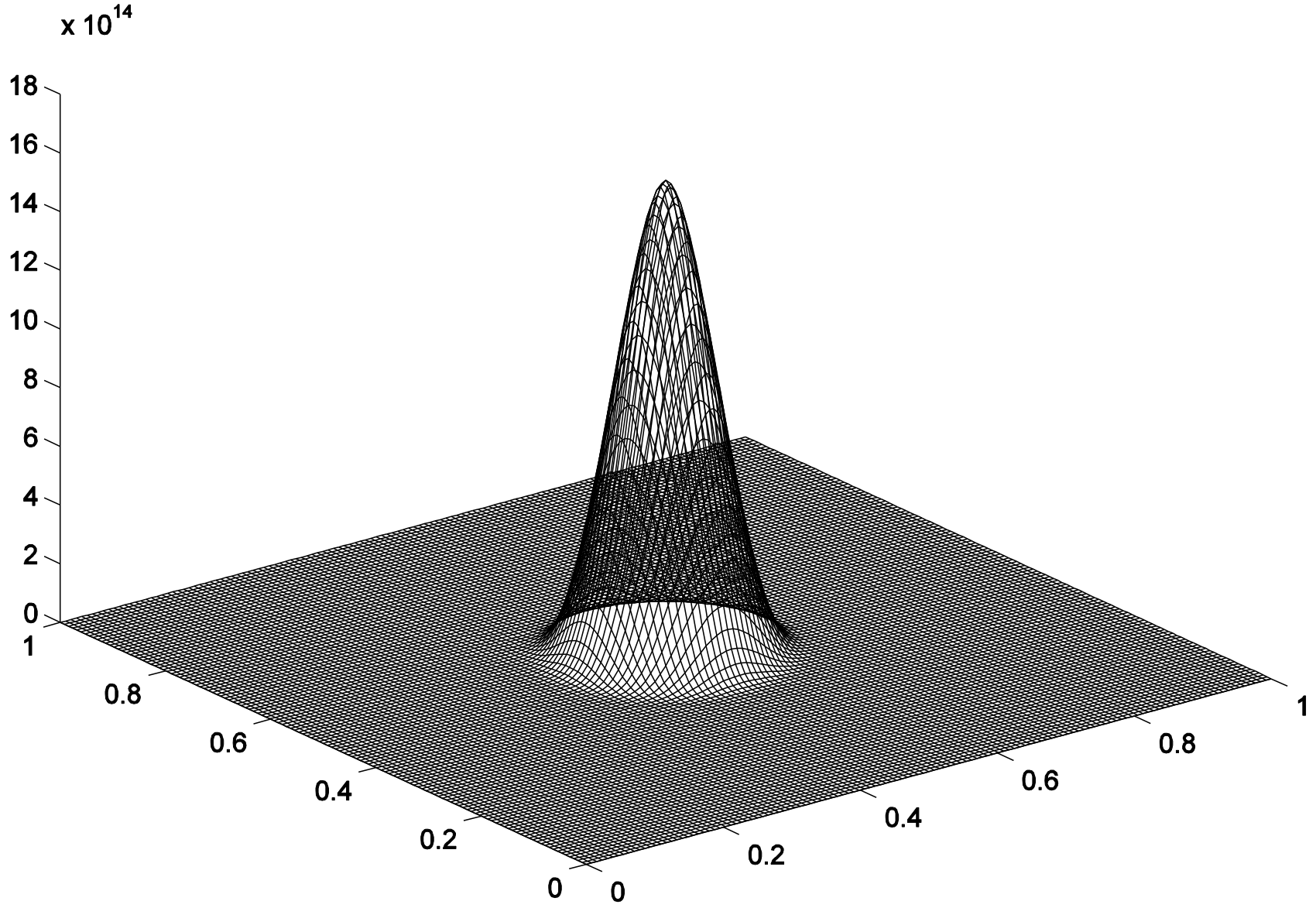

(c) Solution u(ξ, η) in Computational Space

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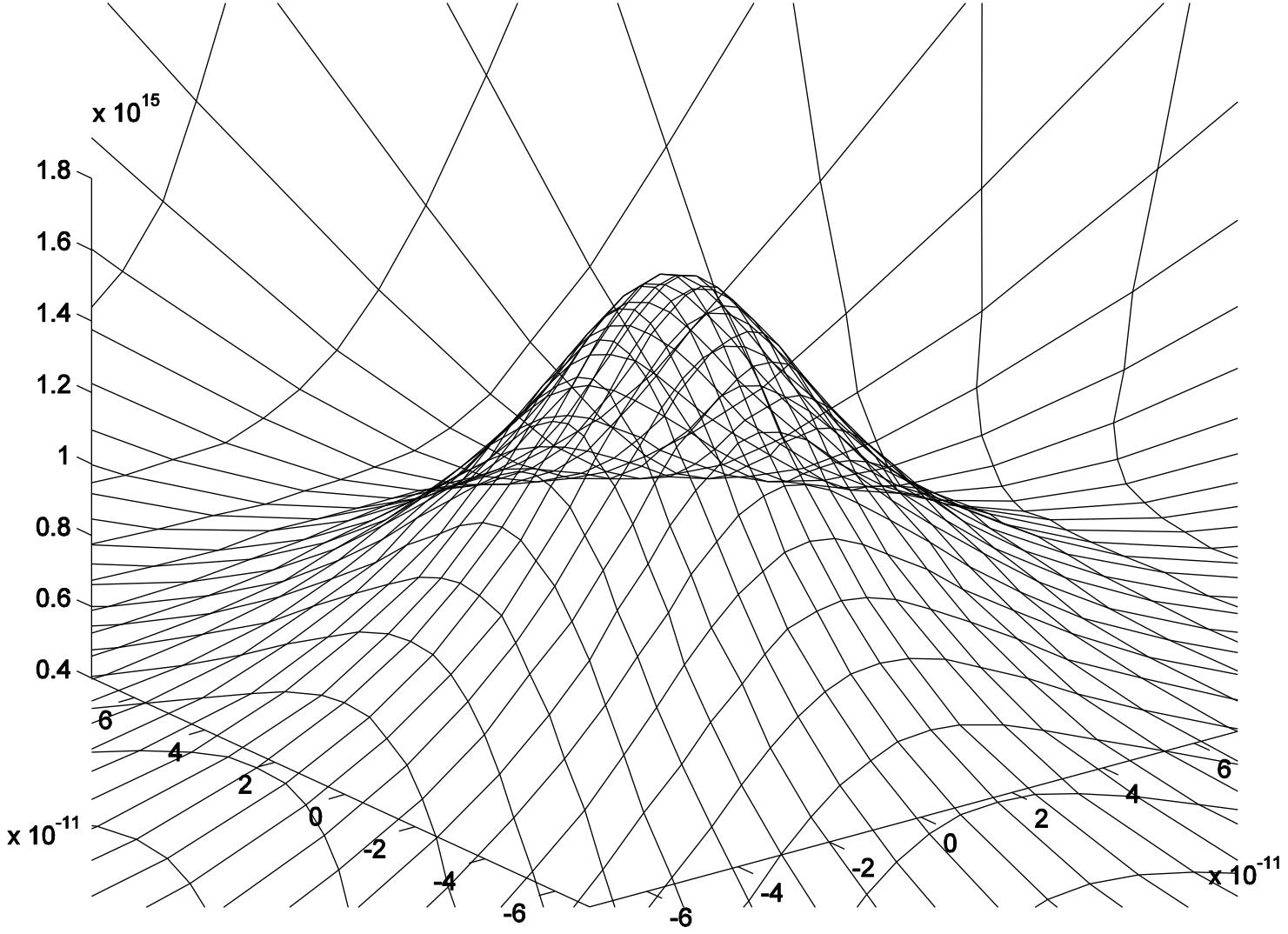

(d) Solution near one of the peaks in the physical domain in the range

$$\left[0.5-7\times10^{-15} \quad 0.5+7\times10^{-15}\right]^2.$$

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Heat Equation

$$\begin{cases} u_t = \Delta u + f(u) \\ u\big|_{\partial\Omega} = 0 \\ u(x, y, 0) = \phi(x, y) \end{cases}$$

where $\Omega=[a,b]\times[a,b]$.

Let

$$f(u) = 4\sqrt{1+u^5}$$

$$\phi(x, y) = 20\sin^2(2\pi x)\sin^2(\pi y)$$

where $\Omega=[0,1]\times[0,1]$.

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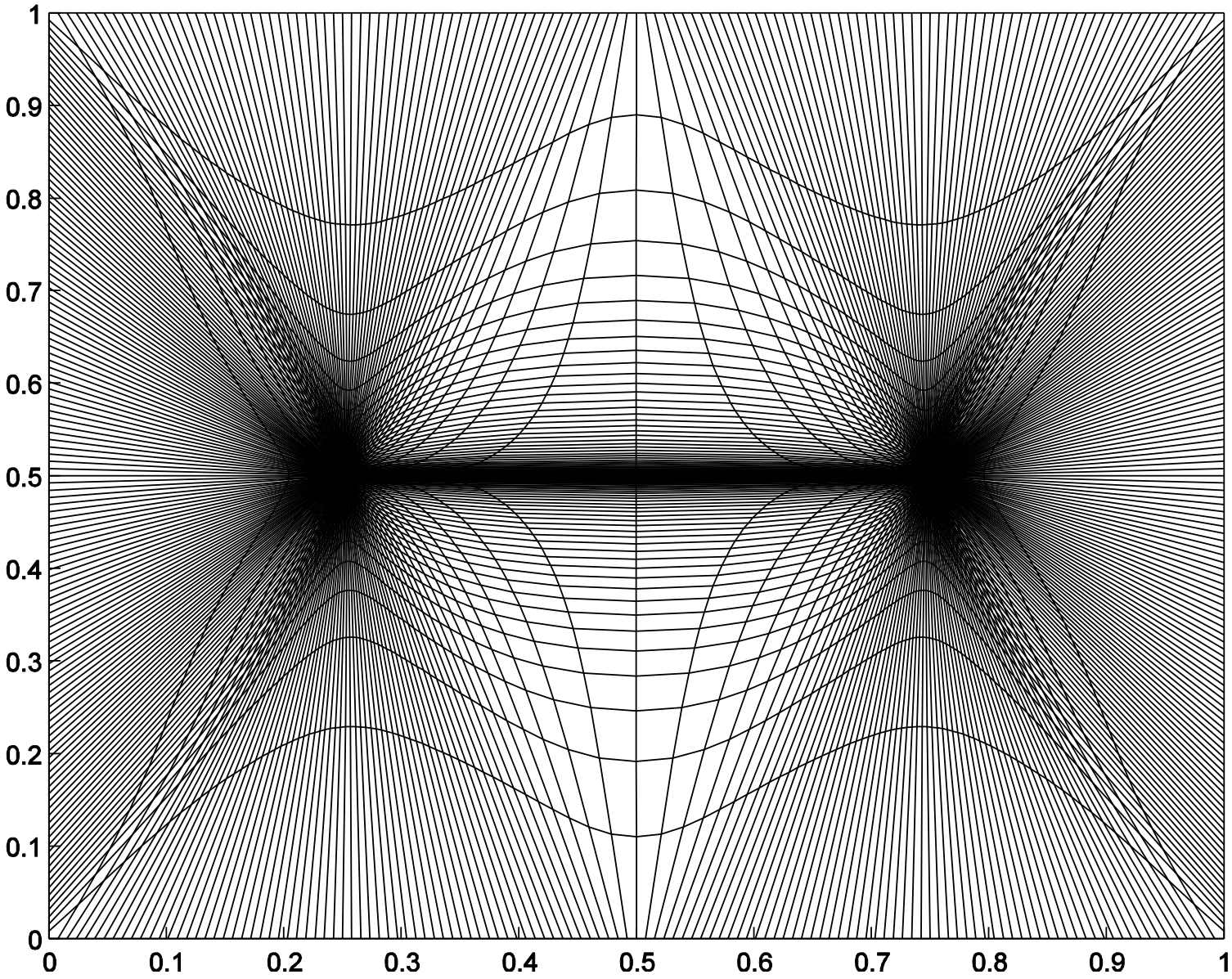

(a) Global View in Physical Domain

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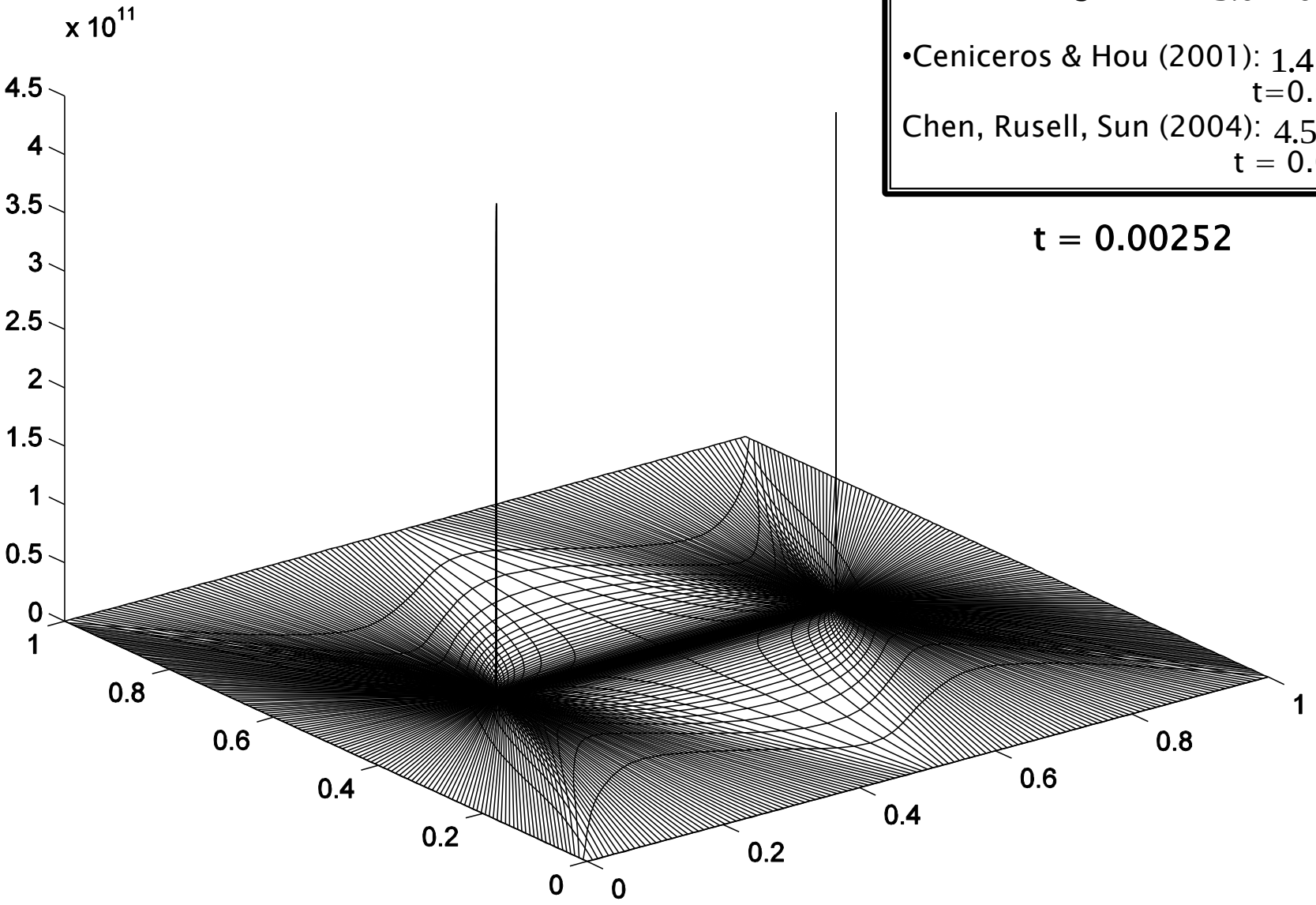
- Ren & Wang(2000): 3.0×10^{5}
- Ceniceros & Hou (2001): 1.4×10^{7} t=0.00258
- Chen, Rusell, Sun (2004): 4.5×10^{8} t = 0.00249

t = 0.00252



(b) Solution u(x, y) in Physical Space

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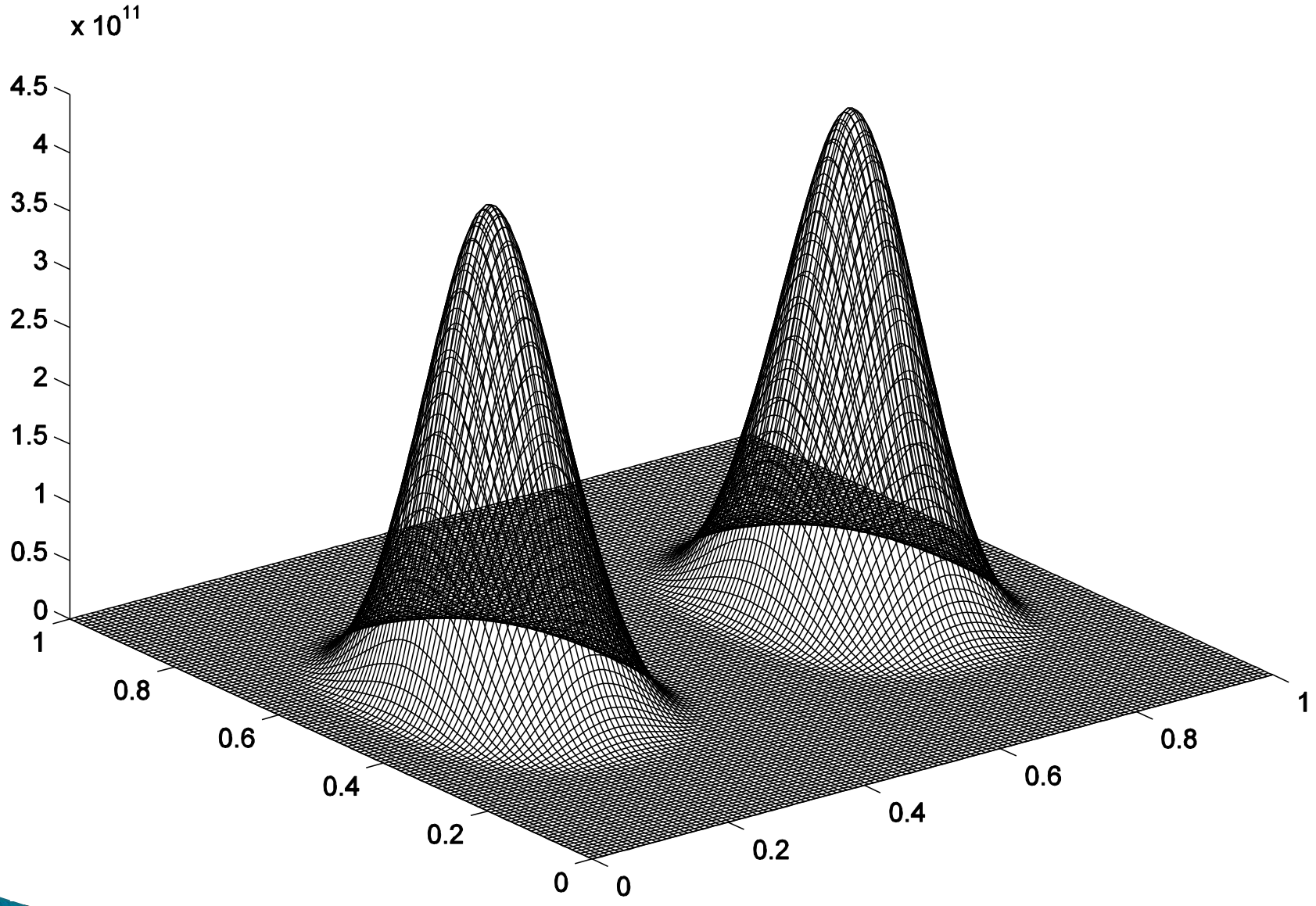

(c) Solution u(ξ, η) in Computational Space

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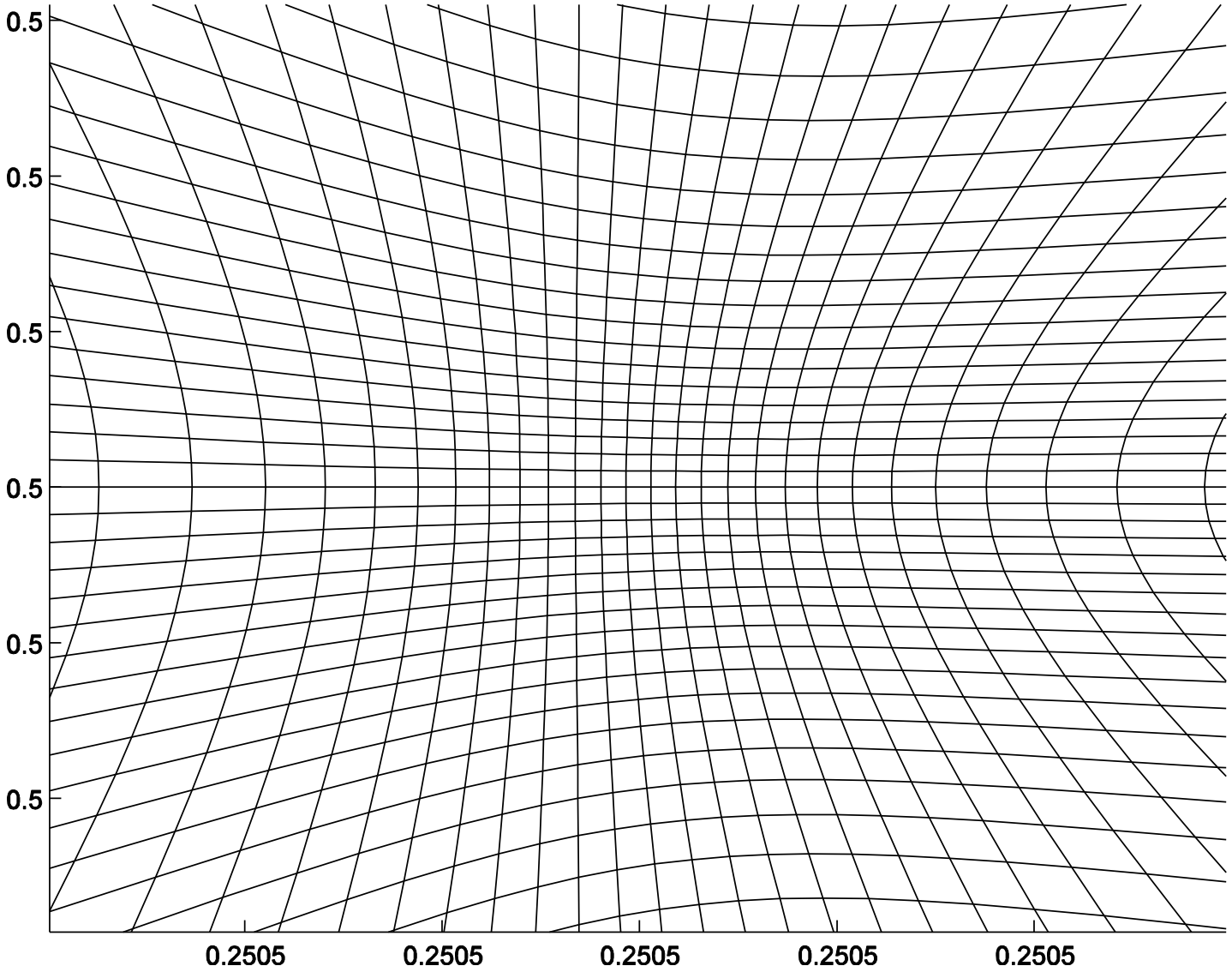

(d) Mesh near one of the peaks in the physical domain in the range [0.25046672 0.25046677] x [0.49999998 0.50000002] .

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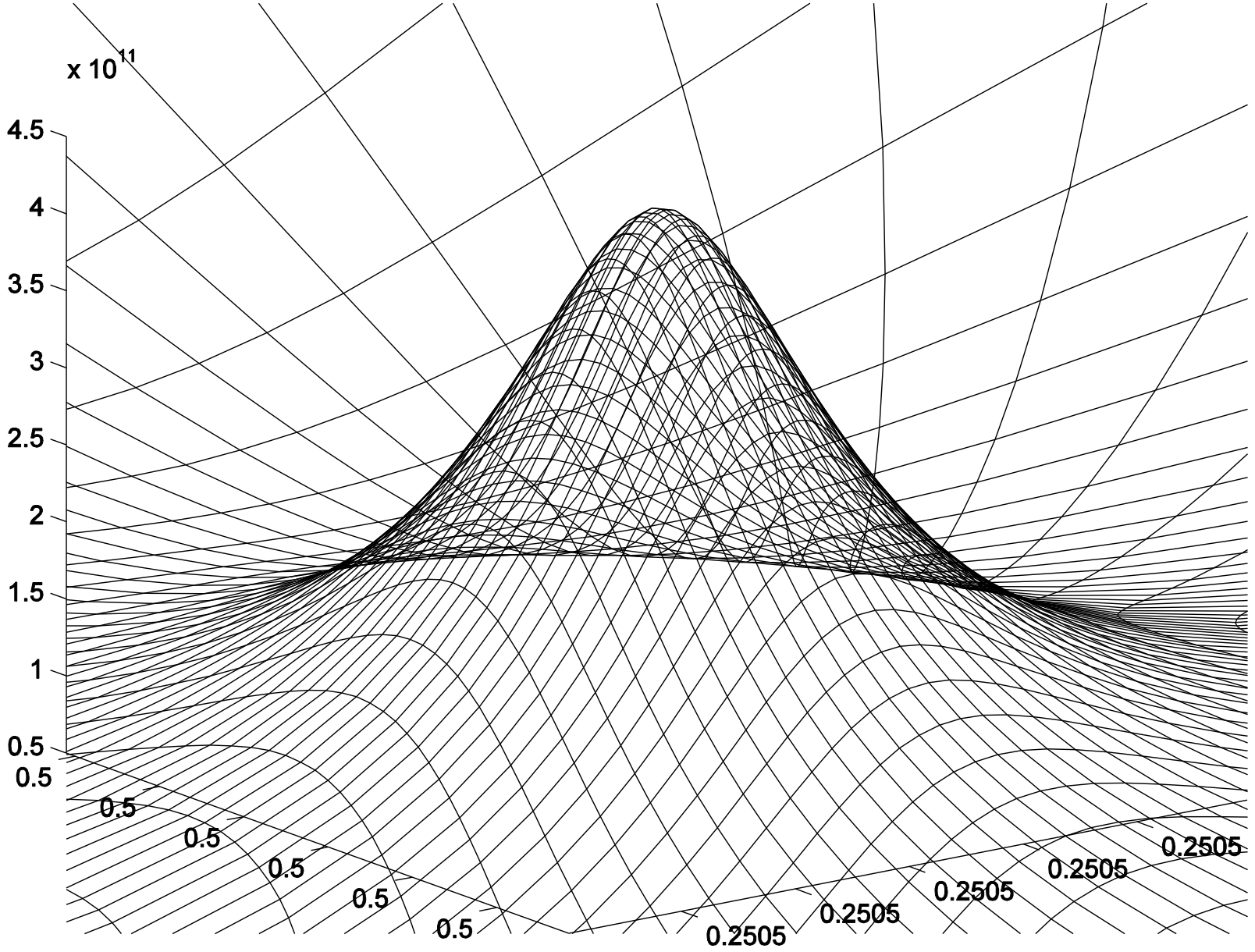

(e) Solution near one of the peaks in the physical domain

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Conclusion

- The simple moving mesh method is very good at solving parabolic equations with blowup properties.
- Numerical computations show that this method is much faster than traditional methods. (It takes only half a day to compute on a normal PC.)

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Further Work

- We will select more efficient monitor functions to improve current results.
- We will try to apply this method to other real problems.
- In particular, we will apply our method to a nonlinear damped p-system with an unbounded domain.